ON k-FACILE PERFECT NUMBERS

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Abstract. For a positive integer n, let $\sigma(n)$ denote the sum of all positive divisors of n. Then n is said to be a k-facile perfect number if $\sigma(n) = 2n + d_1 d_2 \cdots d_k$, where $1 < d_1, d_2, \ldots, d_k < n$ are distinct divisors of n. This paper characterizes k-facile perfect numbers and establishes their relationships with other special numbers.

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1. Introduction

Let $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ be the canonical representation of a positive integer n, where $p_1, p_2, \ldots, p_m$ are distinct prime numbers with $p_1 < p_2 < \cdots < p_m$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are non-negative integers. The sigma function $\sigma(n)$ gives the sum of all the positive divisors of n, i.e

$$\sigma(n) = \sum_{d|n} d = \prod_{i=1}^{m} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

A number is said to be perfect, abundant, or deficient number if $\sigma(n) = 2n, \sigma(n) > 2n$ or $\sigma(n) < 2n$ respectively. The numbers $6, 28, 496, 8128, 33550336, \ldots$ (see the OESIS sequence A000396 [1]), $12, 18, 20, 24, 30, 36, 40, 42, \ldots$ (see the OESIS sequence A005101 [1]), and $1, 2, 3, 4, 5, 7, 8, 9, 10, 11, \ldots$ (see the sequence A005100 [1]) are respectively the first few perfect, abundant, and deficient numbers. Mathematicians have been interested in exploring perfect numbers since antiquity. It is well known from the work of Euclid and Euler that an integer n is perfect if and only if $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both primes. The prime numbers of the form $2^p - 1$ are known as Mersenne primes. Proving the existence of an odd perfect number is still an open question that many researchers are working on. In search of odd perfect numbers, mathematicians have been trying to generalize the concept of existing numbers. They have also defined various numbers, which are related to perfect numbers in a very close manner (for instance, the last author's joint work with Laugier and Sarmah [2], with Dutta [3] and with Mahanta and Yaqubi [4]). Near-perfect numbers are one such generalization that has garnered a lot of attention. In 2012, Pollack and Shevelev [5] introduced k-near-perfect numbers. A number n is called a k-near-perfect number if n is the sum of all of its proper divisors with at most k exceptions (called redundant divisors) for $k \geq 1$. When $k = 1$, we get near-perfect number with exactly one redundant divisor. A number n is said to be a near-perfect if

$$\sigma(n) = 2n + d,$$

where the redundant divisor d, is a proper divisor of n. They [5] presented an upper bound on the count of near-perfect numbers and further proved that there are infinitely many k-near-perfect numbers with exactly k redundant divisors for all large k. Several generalizations of perfect numbers (including near-perfect numbers) are 'additive' in nature, that is, they focus on generalizing the basic equation $\sigma(n) = 2n$ to equations of the type

$$\sigma(n) = \ell n + \sum_{i=1}^{k} a_i d_i,$$

where ℓ, k are natural numbers, a_i's are positive integers and d_i are divisors of n. Here, we will focus on a new kind of generalization called facile perfect numbers.

Definition 1. *For $k > 1$, a natural number n is said to be k-facile perfect number if*

$$\sigma(n) = 2n + \prod_{i=1}^{k} d_i, \tag{1}$$

where $d_i > 1$ for $(1 \leq i \leq k)$ are distinct proper divisors of n, known as facile divisors of n.

For example, we consider $n = 40$. The set of divisors of 40 is $\{1, 2, 4, 5, 8, 10, 20, 40\}$ and we can see that

$$\sigma(n) = \sigma(40) = 90 = 2 \times 40 + 2 \times 5.$$

Therefore, 40 is a $2-$facile perfect number with facile divisors 2 and 5.

Remark 1. *Clearly, 1-facile perfect numbers are near-perfect.*

Proposition 1. *A k-facile perfect number is abundant.*

We have the following elementary inequality, which we use in the sequel

$$2 < \frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}. \tag{2}$$

This paper is structured as follows: Section 2 contains the characterization of k-facile perfect numbers depending on the number of distinct prime factors; Section 3 contains the connections between k-facile perfect numbers and other classes of special numbers; and finally, section 4 contains certain natural questions which arise from our work.

2. Characterization of k-facile perfect numbers

Theorem 1. *There are no k-facile perfect numbers with one prime factor.*

Proof. Let $n = p^\alpha$ be a k-facile perfect number with facile divisors $d_i = p^{\alpha_i}$ $(1 \leq i \leq k)$, for a prime number p and positive integers α, α_i with $0 < \alpha_i < \alpha$. Then $\sigma(p^\alpha) = 2 \times p^\alpha + \prod_{i=1}^{k} p^{\alpha_i}$, which gives

$$p^\alpha(2 - p) = 1 + (p - 1) \prod_{i=1}^{k} p^{\alpha_i}. \tag{3}$$

If $p = 2$, we have $\prod_{i=1}^{k} p^{\alpha_i} = -1$, this is a contradiction. When $p > 2$, the right side of (3) is positive, but the left side of (3) is less than zero, which is a contradiction. This completes the proof. $\square$

Remark 2. *If we take $k = 1$ in the above theorem 1, then the proof works without change. This also shows that there are no near-perfect numbers with one prime factor. This is already known by the work of Ren and Chen [6].*

Theorem 2. *There exists no k-facile perfect number with only two linear prime factors.*

Proof. For any two prime numbers $p_1, p_2 (p_1 < p_2)$, let us assume that $n = p_1 p_2$ is a k-facile perfect number with facile divisors d_i's $(1 \leq i \leq k)$. Then $\sigma(n) = (p_1 + 1)(p_2 + 1)$ and thus it follows from (1) that

$$\prod_{i=1}^{k} d_i = p_1 + p_2 - p_1 p_2 + 1.$$

For all $p_1 \geq 2, p_1 + p_2 - p_1 p_2 + 1 \leq 0$. This implies that $\prod_{i=1}^{k} d_i \leq 0$, which is a contradiction. $\square$

Remark 3. *We notice again that the proof works even if $k = 1$, so this proves the non-existence of such near-perfect numbers, which is already known.*

Theorem 3. *There are no odd k-facile perfect numbers of the form $n = p_1^{\alpha_1} p_2^{\alpha_2}$ for any prime numbers p_1, p_2 and natural numbers $\alpha_1, \alpha_2 \geq 2$.*

Proof. Let us assume that $n = p_1^{\alpha_1} p_2^{\alpha_2}$ is an odd k-facile perfect number, where p_1, p_2 are two prime numbers and α_1, α_2 are natural numbers. It follows from Theorem 2 that α_1, α_2 both are not equal to 1. Now we get

$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} < \frac{p_1^{\alpha_1+1}}{p_1 - 1} \cdot \frac{p_2^{\alpha_2+1}}{p_2 - 1} = n \cdot \frac{p_1 p_2}{(p_1 - 1)(p_2 - 1)}. \tag{4}$$

We will use the inequality $\dfrac{p}{p-1} < \dfrac{q}{q-1}$, whenever $p > q$, without commentary from hereon. Since n is odd, $p_1 \geq 3$. Then from equation (4), we obtain

$$\sigma(n) < n \cdot \frac{3 \times 5}{2 \times 4} < 2n.$$

This contradicts Proposition 1 and hence proves the theorem. □

Remark 4. *We see that the proof works if we set $k = 1$ as well. This proves the non-existence of such near-perfect numbers, as is already known from the work of Tang et.al. [7].*

Theorem 4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2}$ be the prime factorization of a 2-facile perfect number n, where p_1, p_2 are distinct prime numbers and α_1, α_2 are natural numbers . Then*

(a) *If $\alpha_1 = 2$ and $\alpha_2 = 1$, then there exists no 2-facile perfect number.*
(b) *If $\alpha_1 = 3$ and $\alpha_2 = 1$, then $3 \leq p_2 \leq 7$.*
(c) *If $\alpha_1 = 4$ and $\alpha_2 = 1$, then 368 is the only 2-facile perfect number.*
(d) *If $\alpha_1 = 5$ and $\alpha_2 = 1$, then 224, 992 and 1504 are the only three 2-facile perfect numbers.*
(e) *If $\alpha_1 = 3$ and $\alpha_2 = 2$, then there exists no 2-facile perfect number.*

Proof. Let us consider a 2-facile perfect number $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where α_1, α_2 are natural numbers and p_1, p_2 are distinct prime numbers with $p_1 < p_2$. Theorem 3 implies that n is even therefore $p_1 = 2$.

(a) For $\alpha_1 = 2$ and $\alpha_2 = 1$, we have $n = 2^2 p_2$ and $\sigma(n) = 7(p_2 + 1)$. By using equation (2), we get

$$2 < \frac{\sigma(n)}{n} = \frac{7(p_2 + 1)}{2^2 p_2} \Rightarrow 8p_2 < 7p_2 + 7 \Rightarrow p_2 < 7.$$

Therefore, $p_2 = 3$ or 5. When $p_2 = 3, n = 2^2 \times 3 = 12, \sigma(n) = 28 = 2 \times 12 + 4$ and when $p_2 = 5, n = 2^2 \times 5 = 20, \sigma(n) = 42 = 2 \times 20 + 2$. It is clear that both 12 and 20 are not 2-facile perfect numbers. Thus, we conclude that there exist no 2-facile perfect numbers when $\alpha_1 = 2$ and $\alpha_2 = 1$.

(b) For $\alpha_1 = 3$ and $\alpha_2 = 1$, we have $n = 2^3 p_2$ and $\sigma(n) = 15(p_2 + 1)$. By using equation (2), we get

$$2 < \frac{15(p_2 + 1)}{2^3 p_2} \Rightarrow 16p_2 < 15p_2 + 15 \Rightarrow p_2 < 15.$$

Therefore, $3 \leq p_2 \leq 13$. But for $p_2 = 11, n = 2^3 \times 11 = 88, \sigma(n) = 180 = 2 \times 88 + 4$ and for $p_2 = 13, n = 2^2 \times 13 = 104, \sigma(n) = 210 = 2 \times 104 + 2$. It is clear that both 11 and 13 are not 2-facile perfect numbers. Thus, we conclude that when $\alpha_1 = 3$ and $\alpha_2 = 1$, we have $3 \leq p_2 \leq 7$.

(c) For $\alpha_1 = 4$ and $\alpha_2 = 1$, we have $368 = 2^4 \times p_2 \Rightarrow p_2 = 23$ and clearly 368 is a 2-facile perfect number with facile divisors 2 and 4. From equation (2), we have $p_2 < 31$. Therefore, when $3 \leq p_2 \leq 29$, numerical computations can easily verify that all numbers of the form $n = 2^4 p_2$ are not $2-$facile perfect numbers except for $p_2 = 23$. This proves that 368 is the only 2-facile perfect number when $\alpha_1 = 4$ and $\alpha_2 = 1$.

(d) For $\alpha_1 = 5$ and $\alpha_2 = 1$, we have $n = 2^5 p_2$ and $\sigma(n) = 63(p_2 + 1)$. By using equation (2), we get $p_2 < 63$. Therefore, $3 \leq p_2 \leq 61$. When $p_2 = 7$, we have $n = 2^5 \times 7 = 224$ and $\sigma(n) = 504 = 2 \times 224 + 56$. When $p_2 = 31$, we have $n = 2^5 \times 31 = 992$ and $\sigma(n) = 2016 = 2 \times 992 + 32$. When $p_2 = 47$, we have $n = 2^5 \times 47 = 1504$ and $\sigma(n) = 3024 = 2 \times 1504 + 16$. Numerical computations can easily verify that all numbers of the form $n = 2^5 p_2$ are not $2-$facile perfect numbers except for $p_2 = 7, 31$ and 47. This proves that $224, 992, 1504$ are the only 2-facile perfect number when $\alpha_1 = 5$ and $\alpha_2 = 1$.

(e) For $\alpha_1 = 3$ and $\alpha_2 = 2$, we have $n = 2^3 p_2^2$ and $\sigma(n) = 15 \times \dfrac{p_2^3 - 1}{p_2 - 1}$. By using equation (2) we get

$$2 < \frac{15(p_2^3 - 1)}{2^3 p_2^2 (p_2 - 1)} \Rightarrow p_2^2 - 15p_2 - 15 < 0 \Rightarrow p_2 < 16.$$

Therefore, $3 \leq p_2 \leq 13$. It can be easily checked by numerical computation that all the numbers of the form $n = 2^3 p_2^2$ are not 2-facile perfect numbers for $3 \leq p \leq 13$. Thus, we conclude that there exist no 2-facile perfect numbers when $\alpha_1 = 3, \alpha_2 = 2$.

□

Theorem 5. *For any three odd primes p_1, p_2 and p_3, there are no odd k-facile perfect number of the form $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ if α_i's are odd natural numbers.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ be an odd k-facile perfect number, where p_1, p_2 and p_3 are any three distinct odd primes with $p_1 < p_2 < p_3$. Then all the facile divisors being odd, we observe that the right side of equation (1) is odd. If all the natural numbers α_i's are odd, then $\sum_{j=1}^{\alpha_i} p_i^j, i = 1, 2, 3$ is odd and consequently, $\sigma(n) = \prod_{i=1}^{3} \dfrac{p_i^{\alpha_i + 1} - 1}{p_i - 1} = \prod_{i=1}^{3} (1 + p_i + p_i^2 + \cdots + p_i^{\alpha_i})$ is even. This shows that the left side of equation (1) is even, which is a contradiction. □

Theorem 6. *For any prime $p \geq 5$, there exists a 2-facile perfect number n of the form $n = 2 \times 3p$ if and only if the product of the facile divisors is $2^2 \times 3$.*

Proof. For any prime $p \geq 5$, let n be a 2-facile perfect number of the form $n = 2 \times 3p$. If d_1, d_2 are the facile divisors of n, then $1 < d_1, d_2 < n$ and

$$d_1 d_2 = \sigma(2 \times 3p) - 2 \times 2 \times 3p = 2^2 \times 3.$$

Now, conversely let $n = 2 \times 3p$ be a positive integer with divisors d_1 and d_2 such that $1 < d_1, d_2 < n$ and $d_1 d_2 = 2^2 \times 3$. Then $\sigma(n) = 12(p + 1) = 12p + 12 = 2(2 \times 3p) + 2^2 \times 3$. This proves that n is a 2-facile perfect number. □

It is possible to calculate the bounds for the primes of k-facile perfect numbers. The following result gives the bound for few facile perfect numbers.

Theorem 7. *For distinct primes p_i's and non-negative integers α_i's, let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_\ell^{\alpha_\ell}$ be a k-facile perfect number with the facile divisor d_i's. Then*

(a) *If $\ell = 3$ and $p_1 = 3$, then $p_2 = 5$ and $7 \leq p_3 \leq 13$.*
(b) *If $\ell = 4$ and $p_1 = 3$, then $p_2 = 5$ or 7.*
  (b.1) *If $p_2 = 5$, then $p_3 \leq 31$.*
  (b.2) *If $p_2 = 7$, then $p_3 = 11$ or 13.*
(c) *If $\ell = 5$ and $p_1 = 3$, then $p_2 \leq 11$.*
  (c.1) *If $p_2 = 5$, then $p_3 \leq 41$.*
  (c.2) *If $p_2 = 7$, then $p_3 \leq 19$.*
  (c.3) *If $p_2 = 11$, then $p_3 = 13$.*

(d) *If $\ell = 6$ and $p_1 = 3$, then $p_2 \leq 11$.*
   (d.1) *If $p_2 = 5$, then $p_3 \leq 53$.*
   (d.2) *If $p_2 = 7$, then $p_3 \leq 23$.*
   (d.3) *If $p_2 = 11$, then $p_3 \leq 17$.*

(e) *If $\ell = 7$ and $p_1 = 3$, then $p_2 \leq 13$.*
   (e.1) *If $p_2 = 5$, then $p_3 \leq 67$.*
   (e.2) *If $p_2 = 7$ or $p_2 = 11$, then $p_3 \leq 19$.*
   (e.3) *If $p_2 = 13$, then $p_3 = 17$.*

Proof. Since the proofs of the different cases are similar, we only prove (a) here. We have

$$\sigma(n) = \prod_{i=1}^{\ell} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} < \prod_{i=1}^{\ell} \frac{p_i^{\alpha_i+1}}{p_i - 1} = n \prod_{i=1}^{\ell} \frac{p_i}{p_i - 1}.$$

When $\ell = 3$, for $p_1 = 3$, we consider $p_2 \geq 7$. Then

$$\sigma(n) < n \times \frac{3}{2} \times \frac{7}{6} \times \frac{p_3}{p_3 - 1} = n \times \frac{7}{4} \times \frac{p_3}{p_3 - 1}.$$

But for all $p_3 \geq 11$, we have $\sigma(n) < 2n$, which contradicts Proposition 1. So, for $p_1 = 3$, we have $p_2 = 5$. Next, we assume that for $p_3 \geq 17$. Then

$$\sigma(n) < n \times \frac{3}{2} \times \frac{5}{4} \times \frac{17}{16} \Rightarrow \sigma(n) < 2n.$$

This is impossible. Thus, $7 \leq p_3 \leq 13$. And similarly, we can prove the other bounds. □

3. Relationship between facile perfect numbers and other special numbers

In this section, we relate facile perfect numbers with other unique numbers. Even perfect numbers are strongly related to the Mersenne primes; therefore, in the first part of this section, we study the relationship between facile perfect numbers and Mersenne primes. We denote Mersenne primes by $M_p = 2^p - 1$, where p is a prime.

Theorem 8. *For any prime $p > 2$, $n = 2^p M_p$ is a k-facile perfect number, where 2^p is the product of the facile divisors, and M_p is a Mersenne prime.*

Proof. The proof follows immediately from equation (1); therefore, we omit the details here. □

The following theorem establishes a relation between k-facile numbers and perfect numbers.

Theorem 9. *For any prime p and natural number ℓ, an integer $n = 2^\ell P$ is a k-facile perfect number with $2^p(2^\ell - 1)$ as the product of the facile divisors, where $P = 2^{p-1}(2^p - 1)$ is an even perfect number.*

Proof. We consider $n = 2^\ell P = 2^{l+p-1}(2^p - 1)$, where ℓ is a natural number. Then $2n = 2^{l+p}(2^p - 1)$ and $\sigma(n) = (2^{l+p} - 1)2^p$. Therefore,

$$\sigma(n) = 2^{l+p}2^p - 2^p = 2^{l+p}(2^p - 1) + 2^{l+p} - 2^p = 2^{l+p}(2^p - 1) + 2^p(2^l - 1) = 2n + 2^p(2^l - 1).$$

It follows from equation (1) that n is a k-facile perfect number and $2^p(2^l - 1)$ is the product of its facile divisors. □

The construction given by Pollack and Shevelev [5] of the three types of even near-perfect numbers are as follows:

- Type 1: For the positive integers ℓ, s, with $s \geq \ell + 1$, $n = 2^{\ell-1}(2^{\ell} - 2^{s} - 1)$ is a near-perfect number with 2^s as the redundant divisor, where $(2^{\ell} - 2^{s} - 1)$ a prime. The first few near-perfect numbers of this type of construction are $12, 20, 56, 88, \ldots$ [1].
- Type 2: For the Mersenne prime $2^p - 1$, $n = 2^{2p-1}(2^p - 1)$ is a near-perfect number with $2^p(2^p - 1)$ as the redundant divisor. The three near-perfect numbers of this type are $24, 224$ and 15872 [6].
- Type 3: For the Mersenne prime $2^p - 1$, $n = 2^{p-1}(2^p - 1)^2$ is a near-perfect number with $2^p - 1$ as the redundant divisor. The three near-perfect numbers of this type are $18, 196$ and 15376 [6].

In addition to these three types of construction of near-perfect numbers, the number 40 is also near-perfect. In 2013, Ren and Chen [6] improved this result and determined all near-perfect numbers with two distinct prime factors. Li and Liao [8] provided two equivalent conditions of all even near-perfect numbers of the forms $2^{\alpha}p_1p_2$ and $2^{\alpha}p_1^2p_2$ in the year 2015. Tang et al. [7] determined all deficient perfect numbers with at most two distinct prime factors and proved that there are no odd near-perfect numbers with three distinct prime divisors. A number n is called a deficient-perfect number if

$$\sigma(n) = 2n - d,$$

where the deficiency divisor d, is a proper divisor of n. We can see that the facile perfect numbers and near-perfect numbers are closely related. Even near-perfect numbers of type 1 are 2-facile perfect numbers except those near-perfect numbers having redundant divisors 2 and 4. For example, 12, whose redundant divisor is 4, is not a 2-facile perfect number. Similarly, 20 having redundant divisor 2 is not a 2-facile perfect number. The number 40 is a 2-facile perfect number with facile divisors 2 and 5. The following results establish some relationships between near-perfect numbers and facile perfect numbers.

Theorem 10. *A near-perfect number n of type 1 is a 2-facile perfect number if the redundant divisor d of n is greater than or equal to 2^3.*

Proof. Let $n = 2^{m-1}(2^m - 2^{\ell} - 1)$ be a near-perfect number of type 1, where $\ell \geq 3, m \geq l+1$. Let d_1, d_2 be any two proper divisors of n such that $d_1 = 2^{t_1}, d_2 = 2^{t_2} (t_1 \neq t_2)$. Clearly, $t_1 + t_2 \geq 3$, since $t_1 + t_2 < 3$ is not possible. Then $2n = 2^m(2^m - 2^l - 1)$. And,

$$\sigma(n) = (2^m - 1)(2^m - 2^{\ell}) = 2^m(2^m - 2^{\ell} - 1) + 2^{\ell}.$$

This shows that

$$\sigma(n) = 2n + 2^{\ell}. \tag{5}$$

Since $t_1 < \ell$, $2^{t_1} | 2^{\ell}$ and $m \geq \ell + 1$, it implies $2^{\ell} | 2^{m-1}$. Thus, when $2^{t_1} | 2^{m-1}$, it implies $2^{t_1} | n$. Similarly, we can show that $2^{t_2} | n$. Therefore, it follows from equation (5) that $\sigma(n) = 2n + d_1d_2$. This proves that n is a 2-facile perfect number. □

Theorem 11. *All near-perfect numbers of type 2 are k-facile perfect numbers.*

Proof. A near-perfect number $n = 2^{2p-1}(2^p - 1)$ of type 2 can be written as $n = 2^p 2^{p-1}(2^p - 1)$. Therefore, by Theorem 9, n is a k-facile perfect number. □

Remark 5. *Redundant divisor of a near-perfect number $n = 2^{p-1}(2^p - 1)^2$ of type 3 is $2^p - 1$ (see [5]). Since $2^p - 1$ is a prime number it is irreducible. For this reason, a near-perfect number of type 3 is not a k-facile perfect number.*

Another well-studied generalization of perfect numbers is the k-perfect numbers. A natural number n is called k-perfect number, where $k \in \mathbb{N}$ if

$$\sigma(n) = kn + n = (k+1)n. \tag{6}$$

We can see that 1-perfect numbers are perfect numbers. The numbers $120, 972, 523776$ and 459818240 are $2-$perfect, while the numbers $30240, 32760, 142990848$ and 66433720320 are 3-perfect numbers. The following results describe the relation between k-perfect and k-facile perfect numbers.

Theorem 12. *A 2-facile perfect number is a 2-perfect number if and only if the product of the facile divisors is the number itself.*

Proof. Let us consider a 2-facile perfect number n, whose facile divisors are d_1 and d_2. If n is a 2-perfect number, then from equation (6), we have

$$\sigma(n) = (2+1)n = 2n + n. \tag{7}$$

Then from equations (1) and (7), it follows that $d_1 d_2 = n$. Conversely, let the facile divisors $d_1 d_2 = n$. It follows from (1) that

$$\sigma(n) = 2n + n = (2+1)n.$$

This shows that n is a 2-perfect number. □

We now extend Theorem 12 to the following result.

Theorem 13. *A k-facile perfect number is a k-perfect number if and only if the product of the facile divisors is $(k-1)$ times of the number itself.*

Proof. The proof is similar to the proof of Theorem 12, hence we omit the explanation. □

Finally, we establish the connections between facile perfect numbers and Fermat primes. Numbers of the form $F_n = 2^{2^n} + 1$, where n is a non-negative integer, are known as Fermat numbers and prime numbers of this form are called Fermat primes. For $n = 0, 1, 2, 3, 4$, respective Fermat primes are $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$. Fermat conjectured that all the numbers F_n are prime, which was later disproved by Leonhard Euler in the year 1732 who showed that 641 is a factor of $F_5 = 4294967297$. So far, there are only five Fermat primes discovered. The following result establishes a relationship between 2-facile perfect numbers and Fermat primes.

Theorem 14. *For any Fermat prime F_p, if $\ell \leq p$, then a number n of the form $n = 2^\ell F_p$ is not a 2-facile perfect number, where $\ell \in \mathbb{N}$.*

Proof. Let us consider $n = 2^\ell F_p = 2^\ell(2^{2^p} + 1), 0 \leq p \leq 4$ and let n be a 2-facile perfect number with facile divisors $d_1$and d_2. Then, it follows that

$$d_1 d_2 = \sigma(n) - 2n = (2^{\ell+1} - 1)(2^{2^p} + 2) - 2^{\ell+1}(2^{2^p} + 1) = 2^{\ell+1} - 2^{2^p} - 2.$$

When $\ell = p$, we have, for all $p \geq 0, p + 1 \leq 2^p$; Therefore

$$d_1 d_2 = 2^{p+1} - 2^{2^p} - 2 < 0. \tag{8}$$

This is a contradiction. Again when $\ell < p$, we have, for all, $p \geq 0, \ell + 1 < p + 1 \leq 2^p$. Therefore,

$$d_1 d_2 = 2^{\ell+1} - 2^{2^p} - 2 < 0. \tag{9}$$

This is a contradiction. Thus, the number $n = 2^\ell F_p$ is not a 2-facile perfect number. □

Remark 6. *For $\ell > p$, the number $n = 2^\ell F_p$ can occasionally be a 2-facile perfect number like $2^3 F_0 = 24$ and $2^3 F_1 = 40$. However, it remains an open problem to find the bounds on ℓ to make n a 2-facile perfect number.*

4. Concluding Remarks

Several natural questions arise from our research. Here is a list of a few of them.

(1) In this paper, we have focused on characterizing arithmetic properties of k-facile perfect numbers. It would be interesting to analyze analytical aspects of such numbers as well, such as was done by Pollack and Shevelev [5] for near-perfect numbers.
(2) An easy generalization of the concept of deficient perfect numbers would be to study the following equation

$$\sigma(n) = 2n - \prod_{i=1}^{k} d_i,$$

where d_i's are proper divisors of n and $k \geq 1$. We expect similar results for this type of generalization also to be true.
(3) We can further generalize equation (1) to

$$\sigma(n) = \ell n + \prod_{i=1}^{k} d_i,$$

and look at analogous properties of n for this generalization as well.
(4) It would appear that some of the techniques used by other authors (cf. [3], [7], etc.) would be possible to apply in our case as well.

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