JEE Mathematics:

Dr. Manjil P. Saikia
https://manjilsaikia.in/olympiads
manjil@iiitmanipur.ac.in.

Problem 1: Let $P = (a_{ij})$ and $Q = (b_{ij})$ be two 3×3 matrices with $b_{ij} = 2^{i+j} a_{ij}$ for all $1 \leq i, j \leq 3$. If $\det P = 2$, then what is $\det B$? (JEE 2012)

Sol^n: One way would be to just write

$$Q = \begin{pmatrix} 4a_{11} & 8a_{12} & 16a_{13} \\ 8a_{21} & 16a_{22} & 32a_{23} \\ 16a_{31} & 32a_{32} & 64a_{33} \end{pmatrix}$$

$$\Rightarrow \det Q = 4\times 8\times 16 \det \begin{pmatrix} a_{11} & 2a_{12} & 4a_{13} \\ a_{21} & 2a_{22} & 4a_{23} \\ a_{31} & 2a_{32} & 4a_{33} \end{pmatrix}.$$

$$= 4\times 8\times 16\times 2\times 4\times \det P$$

$$= 2^{13}. //$$

(There is an easier way!)

Clearly, $\frac{\det Q}{\det P}$ = constant. Then we can just identify this constant.

Just take $P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then $Q = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 64 \end{pmatrix}$.

So, $\det Q = 2^{13}$. //

Problem 2: Let z be a complex number s.t. the imaginary part of z is non-zero and $a = z^2 + z + 1$ is real. Then a cannot take the value:

(a) -1, (b) $1/3$, (c) $1/2$, (d) $3/4$. (JEE 2014).

Sol^n! Let $z = x + iy$, then $a = (x+iy)^2 + (x+iy) + 1$

$\Rightarrow a = (x^2 - y^2 + x + 1) + i(2xy + y)$.

Since $a \in \mathbb{R}$ so, $2xy + y = 0 \Rightarrow 2x + 1 = 0$ since $y \neq 0$

$\Rightarrow x = -1/2$.

Thus, $a = 3/4 - y^2$.

As $y \in \mathbb{R}$, and $y \neq 0$, so $y^2 > 0$. Hence, $a < 3/4$, so $a \neq 3/4$.

//.

Problem 3: The fn $f : [0,3] \to [1,29]$ defined by

$$f(x) = 2x^3 - 15x^2 + 36x + 1 \quad \text{is,}$$

(a) 1-1 & onto,

(b) onto but not 1-1,

(c) 1-1 but not onto,

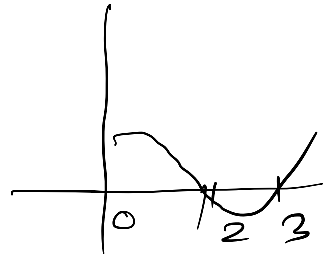
(d) neither 1-1 nor onto.

Sol^n: $f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$.

Both roots of $f'(x)$ are simple and lie in $[0,3]$.

So, f' is positive on $[0,2)$ and negative on $(2,3]$

So, f is neither strictly increasing nor decreasing on $[0,3]$ so not 1-1.



f is increasing on $[0,2]$, the image of $[0,2]$ is the interval $[f(0), f(2)] = [1, 29]$.

Thus f is onto.

Answer is (b). //

Problem 4: Let $a_1, a_2, \ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. What is the least positive integer n for which $a_n < 0$? (JEE 2012).

Sol^n: Let $b_i = 1/a_i$, then b_i's are in an A.P.

Let the common diff. be d. Here $b_1 = 1/a_1 = 1/5$

$b_{20} = 1/a_{20} = 1/25$.

Also, $b_{20} = b_1 + 19d \Rightarrow d = -4/475$.

So, $b_n = \dfrac{-4(n-1)}{475} + 1/5 = \dfrac{96-4n}{475}$.

Clearly a_n & b_n have the same sign.

$\therefore b_n < 0$ iff $96 < 4n$

The least value of n for which this is true is $n = 25$. //

<u>Problem 5</u>: Let X and Y be two events s.t. $P(X|Y) = 1/2$, $P(Y|X) = 1/3$ and $P(X \cap Y) = 1/6$. What is $P(X \cup Y)$? Are X and Y independent?

(JEE 2012)

<u>Soln:</u> $P(X|Y) = \dfrac{P(X\cap Y)}{P(Y)} = 1/2$

$P(Y|X) = \dfrac{P(X\cap Y)}{P(X)} = 1/3$

Also $P(X\cap Y) = 1/6$.

So, $P(X) = 1/2$ and $P(Y) = 1/3$.

$P(X\cup Y) = P(X) + P(Y) - P(X\cap Y) = 1/2 + 1/3 - 1/6 = 2/3$.

Also, $P(X\cap Y) = P(X)\,P(Y)$ so, X and Y are independent. //.

Problem 6: What is the centre of the circle inscribed in the square determined by the two pairs of lines

$$x^2 - 8x + 12 = 0$$

and, $y^2 - 14y + 45 = 0$? (JEE 2003)

Soln: Solving we get, $x = 2, 6$
$y = 5, 9$

The centre of the circle is the mid-pt. of the diagonal. Pick the points $(2,5)$ and $(6,9)$. The mid pt is,

$$\left(\frac{6+2}{2}, \frac{9+5}{2}\right) = (4,7) \,.//.$$

Problem 7: What is the co-effecient of t^{24} in the $\exp^n$ of $(1+t^2)^{12}(1+t^{12})(1+t^{24})$? (JEE 2003)

<u>Sol^n:</u> $(1+t^{12})(1+t^{24}) = (1+t^{12}+t^{24}+t^{36})$.

So, $[t^{24}] = \left([t^{24}] + [t^{12}] + [t^{0}]\right)(1+t^{2})^{12}$

$= 1+\binom{12}{6}+1. = 2+\binom{12}{6}$. //.

Problem 8: What is the natural domain of

$$f(x) = \sqrt{\sin^{-1}(2x) + \pi/6} \; ?$$ (JEE 2003)

<u>Sol^n:</u> For $\sin^{-1}(2x)$ to be defined we need $2x \in [-1, 1]$,

and hence $x \in [-1/2, 1/2]$.

For $\sqrt{\sin^{-1}(2x) + \pi/6}$ to be defined we need,

$\sin^{-1}(2x) \geq -\pi/6$ <u>ie</u> $2x \geq \sin(-\pi/6)$

$\iff 2x \geq -1/2$

$\iff x \geq -1/4$.

So, the answer is $x \in [-1/4, 1/2]$. //

Problem 9: Let n and k be positive integers s.t. $n \geq k$.

Prove that $$2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + \cdots + (-1)\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}.$$

(JEE 2003).

Sol^n:

Let $S = 2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + \cdots + (-1)^k\binom{n}{k}\binom{n-k}{0}$

$$= \binom{n}{k}\sum_{r=0}^{k} \underbrace{(-1)^r\, 2^{k-r}\binom{k}{r}}_{r^{th}\text{ term of } (2-1)^k}$$

$$= \binom{n}{k}. \quad //$$

Problem 10: Let α and β be the roots of $x^2 - 6x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, what is the value of $\dfrac{a_{10} - 2a_8}{2a_9}$?

(JEE 2011)

Solⁿ: we have, $\alpha^2 = 6\alpha + 2$

$$\Rightarrow \alpha^n = 6\alpha^{n-1} + 2\alpha^{n-2} \quad \text{--- (1)}$$

and, $\beta^n = 6\beta^{n-1} + 2\beta^{n-2}$ --- (2)

(1) − (2) $\Rightarrow$ $a_n = 6a_{n-1} + 2a_{n-2}$.

$$\Rightarrow a_{10} = 6a_9 + 2a_8 \Rightarrow \frac{a_{10} - 2a_8}{2a_9} = 3.$$

Problem 11: Let (x_0, y_0) be the sol^n of the following eq^{n}s:

$$(2x)^{\ln 2} = (3y)^{\ln 3}$$

and $$3^{\ln x} = 2^{\ln y}.$$

What is the value of x_0? (JEE 2011)

Sol^n: Take logarithm on both sides to get,

$$(2x)^{\ln 2} = (3y)^{\ln 3}$$

$$\Rightarrow \ln(2x)^{\ln 2} = \ln(3y)^{\ln 3}$$

$$\Rightarrow (\ln 2)^2 + (\ln x)(\ln 2) = (\ln 3)^2 + (\ln y)(\ln 3)$$

& $(\ln 3)(\ln x) = (\ln 2)(\ln y)$

Solving for $\ln x$ & $\ln y$ we get, $\ln x = -\ln 2 \Rightarrow x = 1/2$ //

Problem 12: Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and

$$Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}.$$

What is the relationship bet^n P and Q? (JEE 2011)

Sol^n:

$\sin\theta - \cos\theta = \sqrt{2}\cos\theta \Rightarrow (\sqrt{2}+1)\cos\theta = \sin\theta$

$\sin\theta + \cos\theta = \sqrt{2}\sin\theta \Rightarrow (\sqrt{2}-1)\sin\theta = \cos\theta.$

$$\Rightarrow \cos\theta = \frac{\sin\theta}{\sqrt{2}+1}$$

$$\Rightarrow (\sqrt{2}+1)\cos\theta = \sin\theta.$$

So, $P = Q$. //

Thank You!