universität wien

DISSERTATION / DOCTORAL THESIS

Titel der Dissertation /Title of the Doctoral Thesis

„Topics on Alternating Sign Matrices and Aztec Rectangles“

verfasst von / submitted by

Manjil Pratim Saikia

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, 2019 / Vienna 2019

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 796 605 405
Dissertationsgebiet lt. Studienblatt / field of study as it appears on the student record sheet:	Mathematik
Betreut von / Supervisor:	Univ.-Prof. Mag. Dr. Ilse Fischer, Privatdoz.

To my parents

Loknath and Subarna

SUMMARY

Mathematicians, specially combinatorialists are always on the lookout for beautiful structures and formulas. Two such beautiful formulas are respectively the number of alternating sign matrices (ASMs) of a given order and the number of domino tilings of Aztec diamonds. Each is given by a simple expression, in the first case it is a quotients of factorials, while in the second case it is a power of 2. The existence of formulas of such compelling simplicity and beauty have encouraged mathematicians to look for further refinements in these objects and enumerate them. This thesis deals with two such problems which are inspired by the lookout for formulas of some degree of simplicity and beauty.

In the first part of this thesis, we study refined enumeration of ASMs with respect to boundary statistics. An ASM of order n is an $n \times n$ matrix with entries in the set $\{0, \pm 1\}$ with all row and column sums equal to 1 and where non-zero entries alternate in sign. These matrices were introduced by Robbins and Rumsey, who together with Mills conjectured a simple product formula for the number of ASMs. It was noticed already in the 1980s (by Robbins) soon after these type of matrices were defined that the symmetry classes of ASMs also have very simple product formula for their numbers. This led Robbins to conjecture several formulas for these numbers, and the program of proving these formulas was recently completed in 2017.

A moment's thought yields several simple properties of any ASM. One of these is that the first row (or any boundary row or column) can contain only one 1, otherwise the alternating condition is violated. This motivated refined enumeration of ASMs; that is, the enumeration of a fixed order ASM with the position of the 1 in the first row (or any other boundary row or column) fixed. This was done by Zeilberger using techniques that arose from statistical physics models. In the first part of the thesis we prove such refined enumeration formulas for several symmetry classes of ASMs (vertically symmetric, vertically and horizontally symmetric, quarter-turn symmetric, off-diagonally and off-antidiagonally symmetric and vertically and off-diagonally symmetric), as well as for some closely related classes of matrices (vertically and horizontally perverse ASMs and quasi quarter-turn symmetric ASMs). Our results prove conjectures of Robbins, Fischer and Duchon.

In the second part of this thesis, we study domino tilings of Aztec rectangles, which is a natural extension of an Aztec diamond. The union of all unit squares inside the contour $|x| + |y| = n + 1$ is called an Aztec diamond of order n. If we tile such an Aztec diamond using dominoes (which are union of two adjacent unit squares), then we will get $2^{\binom{n+1}{2}}$ many such tilings which completely cover the Aztec diamond with no overlapping dominoes or empty spaces in the contour. An easy generalization of these diamonds is the Aztec rectangle where we extend the south-east and north-west sides. The resulting figure is not tilable by dominoes. However, if we remove some squares from one of these extended boundaries then

we can tile the resulting region using dominoes. The number of such tilings was already counted by Mills, Robbins and Rumsey.

We look at the more general problem of removing arbitrary many squares from not one side of such an Aztec rectangle, but from all of the boundary sides. We are able to prove a Pfaffian formula for the number of such tilings. As corollaries, we also get such a formula for the number of tilings of Aztec diamonds with arbitrary boundary squares missing from all sides. The entries of the Pfaffian in both these cases are given by number of domino tilings of either Aztec rectangles or Aztec diamonds with specific boundary squares missing (either on two adjacent sides or two opposite sides). The technique that is used for proving these results is called Kuo condensation. We also present a generalization of Kuo's result, which in itself is a generalization of a result of Ciucu.

ZUSAMMENFASSUNG

Mathematiker, im besonderen Kombinatoriker, sind immer auf der Suche nach schönen Strukturen und Formeln. Die Abzählformeln für alternierende Vorzeichenmatrizen (ASMs) einer gewissen Größe sowie für Parkettierungen eines Azteken-Diamants mittels Dominosteinen sind zwei schöne Beispiele dafür. Beide sind durch einfache Ausdrücke gegeben: im ersten Fall durch Quotienten von Fakultäten, im Zweiten durch eine Potenz von 2. Die Existenz von solch einfachen und besonders schönen Formeln hat Mathematiker ermutigt, Verfeinerungen der ursprünglichen Objekte zu betrachten und diese abzuzählen. Diese Dissertation behandelt zwei Problembereiche, welche durch die Suche nach Formeln von besonderer Einfachheit und Schönheit inspiriert wurden.

Im ersten Teil dieser Dissertation betrachten wir verfeinerte Abzählungen von ASMs bezüglich Statistiken auf deren Rändern. Eine alternierende Vorzeichenmatrizen der Ordnung n ist eine $n \times n$ Matrix mit Einträgen aus der Menge $\{0, \pm 1\}$, sodass alle Spalten- und Zeilensummen gleich 1 sind und die Einträge ungleich 0 in ihrem Vorzeichen alternieren. Diese Matrizen wurden durch Robbins und Ramsey eingeführt, die gemeinsam mit Mills eine einfache Produktformel für die Anzahl der ASM vermuteten. Bereits in den 1980ern, kurz nach der Definition dieser Matrizen, bemerkte Robbins, dass auch die Abzählung der meisten Symmetrieklassen von ASMs ausgesprochen einfache Produktformeln haben. In weiterer Folge stellte Robbins Vermutungen für explizite Produktformeln der meisten Symmetrieklassen auf, deren Beweis erst in 2017 komplettiert wurde.

Durch kurze Überlegungen lassen sich einige einfache Eigenschaften von ASMs erkennen. Eine solche Eingenschaft ist, dass eine ASM genau einen Eintrag gleich 1 in ihrer ersten Reihe hat; das Gleiche gilt für die letzte Reihe bzw. für die erste oder letzte Spalte. Dies motivierte eine verfeinerte Abzählung von ASMs, bei welcher die Position der 1 in der obersten Reihe fixiert ist. Die dazugehörigen Abzählformeln wurden von Zeilberger unter der Verwendung von Techniken aus der statistischen Physik bewiesen. In dem ersten Teil dieser Arbeit beweisen wir Abzählformeln für solche Verfeinerungen einiger Symmetrieklassen von ASMs (vertikal symmetrisch, vertikal und horizontal symmetrisch, Vierteldrehung symmetrisch, antidiagonal und abseits-antidiagonal symmetrisch, vertikal und antidiagonal symmetrisch) sowie für eng verwandte Klassen von Matrizen (vertikal und horizontal symmetrische perverse ASMs, quasi Vierteldrehung symmetrische ASMs). Unsere Resultate beweisen Vermutungen von Robbins, Fischer und Duchon.

Im zweiten Teil dieser Dissertation betrachten wir Parkettierungen von Azteken-Rechtecken durch Dominosteine. Ein Azteken-Diamant der Größe n ist die Vereinigung von allen Quadraten innerhalb der Kontur $|x| + |y| = n + 1$. Man kann beweisen, dass die Anzahl der vollständigen Parkettierungen ohne Überlappungen und Löcher eines Azteken-Diamanten mit Dominosteinen, dies sind Vereinigung von zwei benachbarten Quadraten, gleich $2^{\binom{n+1}{2}}$ ist. Eine einfache Verallgemeinerung dieser Diamanten sind Azteken-Rechtecke, welche durch Erweitern auf der Südost und Nordwest Seite erhalten werden. Die resultierende Figur ist nicht mehr mittels Dominosteinen parkettierbar, jedoch wird sie es durch das Entfernen von einigen Quadraten an den erweiterten Rändern. Diese Objekte wurden bereits durch Mills, Robbins und Rumsey abgezählt.

Wir beschäftigen uns in dieser Arbeit mit einem allgemeineren Problem indem wir erlauben, dass die Quadrate nicht nur von einer Seite sondern von allen Rändern des Azteken-Rechtecks entfernt werden dürfen. Für diese Abzählung konnten wir eine Pfaffsche Formel beweisen. Als Konsequenz davon erhalten wir eine solche Formel für die Anzahl der Azteken-Diamanten, wobei von allen Seiten Quadrate entfernt werden dürfen. Die Einträge der Pfaffschen Form zugehörigen Matrix sind in beiden Fällen durch die Anzahl von Parkettierungen von Azteken-Rechtecken bzw. Azteken-Diamanten mit spezielle Randbedingungen, die Quadrate dürfen entweder auf zwei benachbarten oder gegenüberliegenden Seiten entfernt werden, gegeben. Um dieses Resulatat zu beweisen benützen wir Kuo-Kondensation. Weiters präsentieren wir eine Verallgemeinerung von Kuos Resultat, welches auch eine Verallgemeinerung von einem Resultat von Ciucu darstellt.

ACKNOWLEDGMENTS

I would like to place my sincere thanks to my supervisor Ilse Fischer for her untiring help, support and guidance throughout my PhD studies. Thank you for your several acts of kindness towards me, for reading and correcting my drafts, and for giving me freedom throughout the duration of my time in Vienna.

I also express my gratitude to the Austrian Science Fund (FWF) START grant Y463 and SFB grant F50 for funding me during my studies as well as for allowing me to travel to various places for conferences and summer schools. A special word of gratitude goes to Ms. Julia Hui-chun Cheng who assisted me with all the office work after each conference trip.

All the amazing people with whom I had the privilege to share office space and conference experiences: Alex, Cesar, Chen, Florian, Gaurav, Hans, Josef, Kim, Koushik, Meesue, Myrto, Nishad, Olga, Robin, Roger, Sabine, Tomack, Ting, Vivien and Yvonne; thank you so much for making this memorable. I am especially thankful to Florian Aigner and Josef Küstner for helping me with the German translation of the summary of this thesis. Thank you to Prof. Michael Schlosser for several discussions about mathematics and exchanging a variety of stories over beer, along with Dr. Gaurav Bhatnagar.

I am thankful to my friends - both near and far: Angshuman, Bishal, Dhritishna, Himan, Kamal, Kantadorshi, Madhurrya, Pankaj, Parama, Rajdeep, Salik and Saurav for keeping me in their thoughts and encouraging me constantly.

Thank you mentors Prof. Nayandeep Deka Baruah and Prof. Mangesh B. Rege; without your encouragement and kindness this would not have been possible.

I am forever grateful to my parents - Loknath and Subarna, and my sister - Dhanya for their love, support and understanding in every step of my life. Thank you also to my extended family: uncles Chetana and Guna, cousins Bhabani and Gitanjali for their constant encouragement and love.

My final, and most heartfelt thanks goes to my wife - Khimpi for being a constant source of happiness in my life.

CONTENTS

LIST OF FIGURES

INTRODUCTION

1 INTRODUCTION

The aim of this introduction is to give a brief overview of the objects that are studied in this thesis and how they relate to each other. In Part i we study *alternating sign matrices* (ASMs) and derive refined enumeration formulas for symmetry classes of these matrices. In Part ii we study domino tilings of *Aztec rectangles* and enumerate them when the boundary sides of the Aztec rectangle has arbitrary defects. These rectangles are obtained by generalizing regions on the square lattice called *Aztec diamonds*. It would appear at first glance that the topics of the thesis are not related, however there is a connection between ASMs and Aztec diamonds which we will specify shortly.

It is possible to read the individual parts separately after reading this introduction as they are self-contained in themselves.

1.1 ALTERNATING SIGN MATRICES

An *alternating sign matrix* (ASM) of order n is an $n \times n$ matrix with entries in the set $\{\pm 1, 0\}$ such that all row and column sums are equal to 1 and the non-zero entries alternate in each row and column. An example of an ASM of order 7 is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices, first introduced by David P. Robbins and Howard Rumsey, Jr. in the 1980s, have given rise to a lot of nice enumerative conjectures and results. Robbins, Rumsey and W. H. Mills [30] conjectured that the number of ASMs of order n is given by

$$A_n := \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

This conjecture was later proved by Doron Zeilberger [43] and shortly after also by Greg Kuperberg [27], using vastly different techniques. A very detailed description of this conjecture and Kuperberg's proof can be found in a book by David Bressoud [8]. Another approach towards proving this result was developed by Ilse Fischer [16, 19].

Richard Stanley [40] suggested the study of symmetry classes of ASMs shortly after these objects were introduced. This led Robbins [37, 38] to conjecture various product formulas for the different symmetry classes of ASMs. The program of proving these formulas was accomplished by work of Kuperberg [28], Soichi Okada [31], A. V. Razumov and Yu. G. Stroganov [33, 35, 36], and recent work of Roger E. Behrend, Fischer and Matjaž Konvalinka [7].

It is easy to see that there is precisely one occurrence of 1 in the first row of any ASM. This suggests the study of some refined enumerations of ASMs and symmetry classes thereof with respect to the position of this 1. The refined enumeration of all ASMs with the position of the unique 1 in the first row fixed was conjectured by Mills, Robbins and Rumsey [30] to be

$$\frac{\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}}{\binom{3n-2}{n-1}}A_n.$$

This was proven by Zeilberger [44]. In case of the symmetry classes of ASMs, such type of refinement has been studied for vertically symmetric ASMs and half-turn symmetric ASMs, both by Razumov and Stroganov [34, 35]. The objective of Part i of this thesis is to study refined enumeration results of similar kind for several symmetry classes of ASMs and related classes of matrices.

By reflection, an ASM also has a unique 1 in the last row as well as in the first and last column. Various mathematicians have studied related refinements, considering restrictions on a combination of two or more of the boundaries of an ASMs. These works include but are not limited to those by Arvind Ayyer and Dan Romik [5], Behrend [6], Fischer [18], Fischer and Romik [20], Razumov and Stroganov [33, 34], Romik and Matan Karklinsky [23], Stroganov [41], etc. using a variety of tools, but predominantly techniques that arise in statistical physics, which we will explain in brief later.

1.2 AZTEC DIAMONDS

Noam Elkies, Kuperberg, Michael Larsen and James Propp [14] introduced a new class of objects which they called Aztec diamonds. The *Aztec diamond* of order n (denoted by $AD(n)$) is the union of all unit squares inside the contour $|x| + |y| = n + 1$ (see Figure 1.1 for an Aztec diamond of order 3). A domino is the union of any two unit squares sharing an edge, and a domino tiling of a region is a covering of the region by dominoes so that there are no gaps or overlaps. They [14, 15] considered the problem of counting domino tilings the Aztec diamond with dominoes and presented four different proofs of the following result.

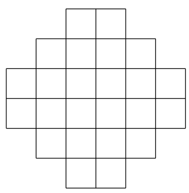


Figure 1.1: AD(3), the Aztec diamond of order 3.

Theorem 1.1. *The number of domino tilings of the Aztec diamond of order* n *is* $2^{n(n+1)/2}$.

This work subsequently inspired lot of follow ups, including the natural extension of the Aztec diamond to the Aztec rectangle (see Figure 1.2). We denote by $\mathcal{AR}_{a,b}$ the *Aztec rectangle* which has a unit squares on the southwestern side and b unit squares on the northwestern side. In the remainder of this thesis, we assume $a \leqslant b$ unless otherwise mentioned. For $a < b$, $\mathcal{AR}_{a,b}$ does not have any tiling by dominoes. The non-tileability of the region $\mathcal{AR}_{a,b}$ becomes evident if we look at the checkerboard representation of $\mathcal{AR}_{a,b}$ (see Figure 1.2). However, if we remove $b - a$ unit squares from the southeastern side then we have a simple product formula found by Mills, Robbins and Rumsey [30].

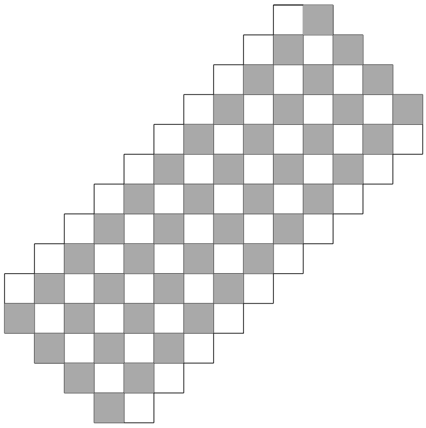


Figure 1.2: Checkerboard representation of an Aztec rectangle with $a = 4, b = 10$.

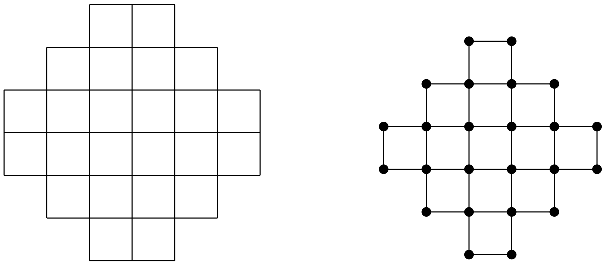
Theorem 1.2. *Let* $a < b$ *be positive integers and* $1 \leqslant s_1 < s_2 < \cdots < s_a \leqslant b$. *Then the number of domino tilings of* $\mathcal{AR}_{a,b}$ *where all unit squares from the southeastern side are removed except for those in positions* $s_1, s_2, \ldots, s_a$ *is*

$$2^{a(a+1)/2} \prod_{1 \leqslant i < j \leqslant a} \frac{s_j - s_i}{j - i}.$$

A natural question to ask now would be: what about enumeration formulas for domino tilings of $\mathcal{AR}_{a,b}$ with unit squares removed from more than one boundary side? We answer this question in Part ii of this thesis, where we consider the most general case of defects (removed unit squares) on all boundary sides.

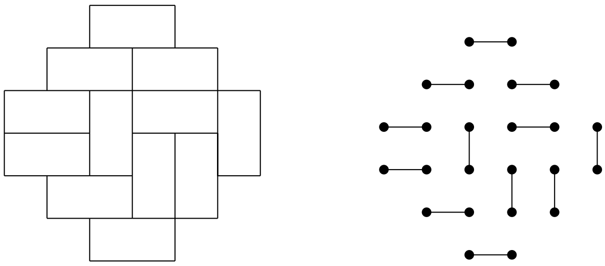
1.3 CONNECTION BETWEEN ASMS AND AZTEC DIAMONDS

We can recast the problem of enumerating domino tilings into a perfect matching problem. A perfect matching of a graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching. It is easy to see that domino tilings of a region can be identified with perfect matchings of its *planar dual graph*, the graph that is obtained if we identify each unit square with a vertex and unit squares sharing an edge with each other is identified with an edge (see Figure 1.3). So for any region R on the square lattice we denote by M(R) the number of domino tilings of R, equivalently the number of perfect matchings of the planar dual graph of the region R. For instance, Figure 1.4 shows an example of this equivalence for AD(3).



(a) AD(3). (b) Planar dual graph of AD(3).

Figure 1.3: Equivalence of tilings and matchings.



(a) Domino tiling. (b) Perfect matching.

Figure 1.4: Example of the equivalence of tilings and matchings.

If we now rotate the planar dual graph of an Aztec Diamond by 45°, we see that this graph is made of n rows of n diamond shaped cells. If we assign an entry 1, 0 or -1 to each such cell in the perfect matching where a cell is covered by 2, 1 or 0 edge(s) then we get a correspondence between a perfect matching and an ASM. This is illustrated in Figure 1.5 for the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here the heavier lines are the edges that count towards the perfect matching.

This gives us the following result which connects domino tilings of Aztec diamonds and enumeration of ASMs (see the work of Elkies, Kuperberg, Larson and Propp [14, 15], as well as of Mihai Ciucu [9])

$$\#\text{ Domino tilings of } AD(n) = \sum_{A \in \mathcal{A}_{n+1}} 2^{N_-(A)},$$

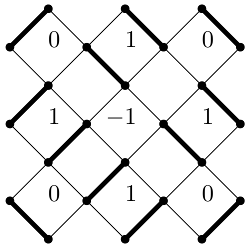


Figure 1.5: ASMs and Aztec diamonds.

where $\mathcal{A}_n$ is the set of all $n \times n$ ASMs and $N_-(A)$ is the number of -1's in A. The sum in the right-hand side is also called 2-enumeration of ASMs.

1.4 OTHER RELATED OBJECTS

There are several combinatorial objects which are intimately related to ASMs as well as Aztec diamonds. In fact, several of these objects are counted by the same sequence of numbers as that of ASMs: $1, 2, 7, 42, 429, \ldots$. We mention a few below (the list is non-exhaustive).

1. **Monotone Triangles** A *monotone triangle* of size n is a triangular array of integers $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ of the form

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & a_{3,1} & & a_{3,2} & & a_{3,3} & \\ \iddots & & & & & & \ddots \\ a_{n,1} & & \cdots & & \cdots & & a_{n,n} \end{array}$$

   such that all row entries are strictly increasing and each entry is weakly between its two bottom entries.

   The set of monotone triangles of size n with bottom row equal to $(1, 2, \ldots, n)$ is equinumerous with order n ASMs [37].

2. **Descending Plane Partitions** A *descending plane partition* is an array of positive integers $(d_{i,j})_{1 \leqslant i \leqslant r, i \leqslant j \leqslant \lambda_i + i - 1}$ of the form

$$\begin{array}{cccccc} d_{1,1} & d_{1,2} & d_{1,3} & \cdots & & d_{1,\lambda_1} \\ & d_{2,2} & d_{2,3} & \cdots & & d_{2,\lambda_2+1} \\ & & \ddots & & \iddots & \\ & & & d_{r,r} \quad \cdots \quad d_{r,\lambda_r+r-1} & & \end{array}$$

   such that

   - all row entries are weakly decreasing,

- all column entries are strictly decreasing,
- the number of entries in each row is strictly less than the first entry in the same row and is at least as large as the first entry in the following row.

It is known that the number of descending plane partitions with entries at most n are equinumerous with order n ASMs.

3. **Totally Symmetric Self-Complementary Plane Partitions** A *plane partition* in an $a \times b \times c$ box is a subset

$$PP \subset \{1, 2, \cdots, a\} \times \{1, 2, \cdots, b\} \times \{1, 2, \cdots, c\}$$

with $(i', j', k') \in PP$ if $(i, j, k) \in PP$ and $(i', j', k') \leqslant (i, j, k)$. If a plane partition has all the symmetries (that is, $(i, j, k) \in PP$ if and only if all six permutations of (i, j, k) are also in PP) and is its own complement (that is, if $(i, j, k) \in PP$ then $(2n+1-i, 2n+1-j, 2n+1-k) \notin PP$), then it is called *totally symmetric self-complementary plane partitions* (TSSCPP). An example of such a TSSCPP is given in Figure 1.6. The class of TSSCPPs inside a $2n \times 2n \times 2n$ box are

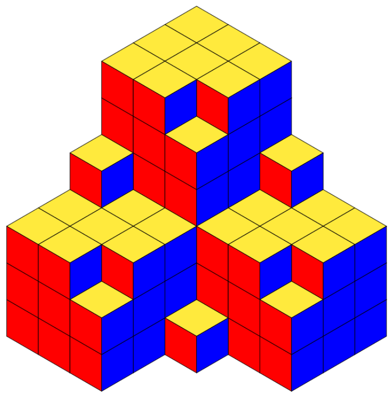


Figure 1.6: A totally symmetric self-complementary plane partition.

equinumerous with $n \times n$ ASMs.

4. **Alternating Sign Triangles** An *alternating sign triangle* (AST) of size n is a triangular array

$$\begin{array}{ccccc} a_{1,1} & a_{1,2} & \cdots & a_{1,2n-2} & a_{1,2n-1} \\ & a_{2,2} & \cdots & a_{2,2n-2} & \\ & & \vdots & & \\ & & a_{n,n} & & \end{array}$$

such that

- the entries are either $1, -1$ or 0,
- along the columns and rows the non-zero entries alternate,
- in each column the first non-zero entry from the top is a 1 and the rowsums are equal to 1.

It was shown by Ayyer, Behrend and Fischer [3] that the number of size n ASTs is equal to order n ASMs.

Apart from a simple bijection between monotone triangles of size n with bottom row $(1, 2, \dots, n)$ and order n ASMs, no explicit bijections are known between the other objects mentioned above. One of the most outstanding open problems in enumerative combinatorics is to find such a bijection.

1.5 STRUCTURE OF THE THESIS

The remainder of the thesis is structured as follows:

- Chapter 2 describes in brief the connection between ASMs and the six-vertex model which we use in the subsequent chapters to deduce our results.
- Chapter 3 deals with the refined enumeration of ASMs with vertical symmetry. In particular, one result proves a conjecture of Fischer [17].
- Chapter 4 deals with the refined enumeration of ASMs with off-diagonal symmetry.
- Chapter 5 deals with the refined enumeration of ASMs with quarter-turn symmetry. In particular the results prove conjectures of Robbins [38] and Duchon [13].
- Chapter 6 explains one of the major techniques used in enumerating tilings of regions in a finite lattice, called Kuo condensation. This chapter also presents a generalization of the method.
- Chapter 7 deals with enumerating domino tilings of Aztec rectangles with defects on one or two boundary sides.
- Chapter 8 presents the most general results possible for enumerating domino tilings of Aztec rectangles with arbitrary defects on all boundaries.
- Appendix A contains a result which is used frequently in Part i, as well as enumerates a weighted rhombus tiling problem for certain quartered hexagons.

Part I

REFINED ENUMERATION OF ALTERNATING SIGN MATRICES

In this part we prove refined enumeration results on several symmetry classes as well as related classes of alternating sign matrices with respect to classical boundary statistics, using the six-vertex model of statistical physics. More precisely, we study vertically symmetric, vertically and horizontally symmetric, vertically and horizontally perverse, off-diagonally and off-antidiagonally symmetric, vertically and off-diagonally symmetric, quarter-turn symmetric as well as quasi quarter-turn symmetric alternating sign matrices. Our results prove conjectures of Fischer, Duchon and Robbins. This part corresponds to joint work with Ilse Fischer [21].

2 ASMS AND THE SIX VERTEX MODEL

Kuperberg's proof of the ASM conjecture [27] was by exploiting a bijection between ASMs and a model in statistical physics, called the six-vertex model. In this chapter, we will explain this connection in brief.

We consider a quadratic sub-region of the square grid such that the boundary vertices are of degree 1, see Figure 2.1. A *configuration* of a corresponding statistical physics model is an orientation on the edges of this graph such that both the in-degree and the out-degree of each degree 4 vertex is 2. If the orientations of the edges incident with vertices of degree 1 are prescribed to be oriented inwards for horizontal edges and outwards for vertices edges (see Figure 2.1) then we say such a configuration satisfies the *domain wall boundary condition*. Such a setup is in bijection with ASMs, and is called the *six vertex model*, due to the six possibilities of assigning orientations to the edges incident with degree 4 vertices to make the in-degree and out-degree equal to 2. An example of such a configuration in given in Figure 2.1.

If we associate with each of the degree 4 vertices in a six-vertex configuration a number as given in Figure 2.2, then we obtain a matrix with entries in the set $\{\pm 1, 0\}$. It is not difficult to see that such a matrix will be an ASM, and we actually get a bijection between ASMs and configurations of the six-vertex model. For example, the matrix associated with the configuration in Figure 2.1 is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

With such a model, there is an associated *partition function* (say Z): it is the sum of the weights of all possible configurations (say $w(C)$) of the model. The *weight of a configuration* C is the product of all the weights of degree 4 vertices (say w_v for vertex v in the graph), which will be defined now. We associate with each horizontal line in the grid a spectral parameter x_i and with each vertical line a spectral parameter y_j as marked in Figure 2.1. The *label* of a vertex which is intersected by lines associated with the parameters x_i and y_j is x_i/y_j. Each of the vertices will be assigned a weight as given in Figure 2.2, where u has to be replaced by the label and q is a global parameter which we will specialize in the coming sections. We use the abbreviations

$$\bar{x} = x^{-1} \quad \text{and} \quad \sigma(x) = x - \bar{x}$$

throughout this thesis.

More formally, the above description says that the partition function is defined as

$$Z(n; \vec{x}; \vec{y}) = \sum_{C \in \mathcal{C}} w(C) = \sum_{C \in \mathcal{C}} \prod_{\substack{v \in C \\ v \text{ vertex of degree 4}}} w_v,$$

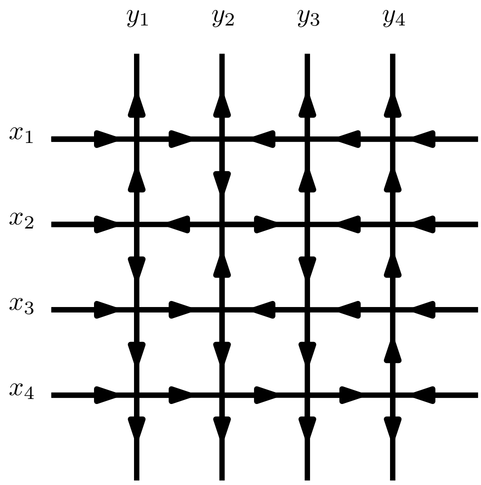


Figure 2.1: Six Vertex Model with Domain Wall Boundary Condition.

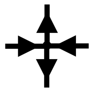
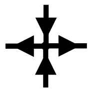
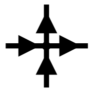
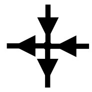
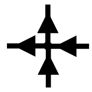
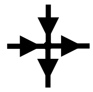
ASM	1	−1	0	0	0	0
six-vertex configuration						
weight	1	1	$\frac{\sigma(qu)}{\sigma(q^2)}$	$\frac{\sigma(qu)}{\sigma(q^2)}$	$\frac{\sigma(q\bar{u})}{\sigma(q^2)}$	$\frac{\sigma(q\bar{u})}{\sigma(q^2)}$

Figure 2.2: Correspondence between ASMs and six-vertex configurations.

where $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{y} = (y_1, y_2, \ldots, y_n)$ and $\mathcal{C}$ is the set of all possible configurations of the model we are considering. It is clear now that, if all the weights can be made equal to 1 in all configurations of the six-vertex model by specializing $\vec{x}, \vec{y}$ as well as q, then the partition function will just count the number of ASMs of a given size. The choices of weights in Figure 2.2 are appropriate to make this happen. This was essentially the approach used by Kuperberg [27] to prove the ASM conjecture.

We note that, rotating any of these vertices would result in changing the weights of the vertices in a way that the spectral parameters appearing in the weights would become their inverses. This will be used in the subsequent chapters without commentary.

3 ASMS WITH VERTICAL SYMMETRY

This chapter deals with ASMs which are vertically symmetric; in particular we look at ASMs with only vertical symmetry (VSASMs), ASMs with both vertical and horizontal symmetry (VHSASMs) and a related class of matrices called vertically and horizontally perverse ASMs (VHPASMs).

ASMs that are invariant under the reflection in the vertical symmetry axis only exist for odd order: this follows because an ASM has a unique 1 in the top row which then has to be situated in the central column of an ASM with vertical symmetry. This implies in particular that the top row of a such an ASM is fixed. It is also not hard to see that the central column of such an ASM is always $(1, -1, 1, -1, \ldots, 1)^T$. The second row of an ASM with vertical symmetry contains precisely two 1's, which are symmetrically arranged left and right of the central -1. Thus, the second row of such an ASM is determined by the position of the first 1 in the second row. The aim of this chapter is to prove refined enumeration results for VSASMs, VHSASMs and VHPASMs with respect to the position of the first 1 in the second row.

3.1 VERTICALLY SYMMETRIC ASMS

This section deals with the case of ASMs with only vertical symmetry. An example of such an ASM is the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

As already mentioned such ASMs occur for odd order and were enumerated by Kuperberg [28], who proved that the number of $(2n+1) \times (2n+1)$ VSASMs equals

$$\frac{1}{2^n} \prod_{j=1}^{n} \frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!}.$$

Later, Razumov and Stroganov [34] proved a refined enumeration result for VSASMs. They proved that the number of $(2n+1) \times (2n+1)$ VSASMs with the position of the unique 1 in the first column fixed at the i-th row is given by

$$\frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!} \sum_{k=1}^{i-1} (-1)^{i+k-1} \frac{(2n+k-2)!(4n-k-1)!}{(4n-2)!(k-1)!(2n-k)!}.$$

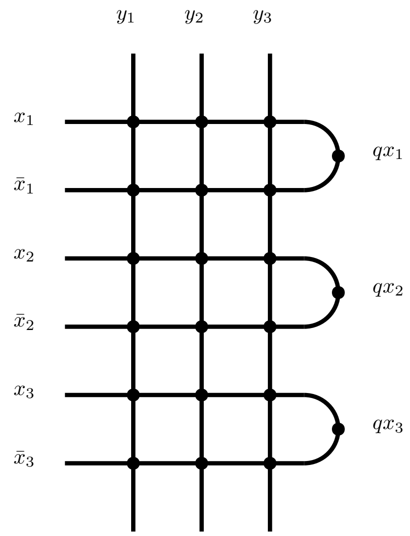


Figure 3.1: The grid corresponding to VSASMs.

Subsequently, Fischer [17] conjectured the following formula for the number of $(2n+1) \times (2n+1)$ VSASMs that have the first 1 in the second row in position i:

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!} \prod_{j=1}^{n-1} \frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!} =: A_V(2n+1, i).$$

In this section, we prove this conjecture.

From the discussion in the beginning of this chapter it is clear that $(2n+1) \times (2n+1)$ VSASMs correspond to $2n \times (n+1)$ matrices with entries in $\{\pm 1, 0\}$ that have the following properties.

1. The non-zero entries alternate in each row and column.
2. All column sums are 1 except for the last column which is always equal to $(1, -1, 1, \ldots, -1)^T$.
3. The first non-zero entry of each row and column is 1.

The 6×4 matrix with these properties that corresponds to the VSASM from above is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

(We deleted the bottom row as well as the last n columns.)

Next we use the well-known correspondence between ASMs and the six-vertex model as explained in Chapter 2 to translate the $2n \times (n+1)$ matrices into orientations of graphs as indicated in Figure 3.1. In our example we obtain Figure 3.2.

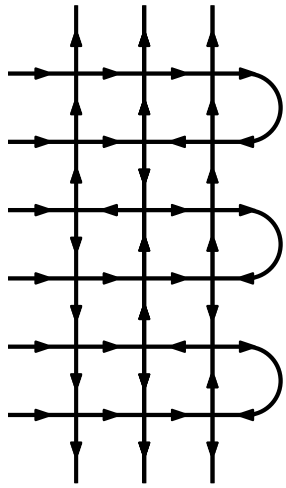


Figure 3.2: The six-vertex configuration of our example.

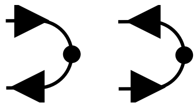


Figure 3.3: Weights of the U-turns.

Note that the right boundary, i.e., the fixed column $(1,-1,1,\ldots,-1)^T$, is modeled via U-turns with down-pointing orientation.

As for the related partition function $Z_U(n;x_1,\ldots,x_n;y_1,\ldots,y_n)$, we allow for the moment both up-pointing and down-pointing U-turns, and the labels and weights are as indicated in Figures 3.1 and 3.3, involving now an additional global parameter b. Later we will specialize b so that the weight of a configuration that has at least one up-pointing U-turn other than the top U-turn is 0. Osamu Tsuchiya [42] was the first who derived a formula for this partition function. Here we use Kuperberg's [28, Theorem 10] version (up to some normalization factor).

Theorem 3.1. *The U-turn partition function of order* n *is*

$$Z_U(n;x_1,\ldots,x_n;y_1,\ldots,y_n)$$
$$= \frac{\sigma(q^2)^{n-2n^2}\prod_{i=1}^{n}\sigma(b\bar{y}_i)\sigma(q^2x_i^2)\prod_{i,j=1}^{n}\alpha(x_i\bar{y}_j)\alpha(x_iy_j)}{\prod_{1\leqslant i<j\leqslant n}\sigma(\bar{x}_ix_j)\sigma(y_i\bar{y}_j)\prod_{1\leqslant i\leqslant j\leqslant n}\sigma(\bar{x}_i\bar{x}_j)\sigma(y_iy_j)}$$
$$\times \det_{1\leqslant i,j\leqslant n}\left(\frac{1}{\alpha(x_i\bar{y}_j)}-\frac{1}{\alpha(x_iy_j)}\right),$$

where $\alpha(x)=\sigma(qx)\sigma(q\bar{x})$.

In the following, we will specialize

$$(x_1,\ldots,x_n)=(x,1,\ldots,1) \quad \text{and} \quad (y_1,\ldots,y_n)=(1,\ldots,1),$$

as well as

$$b = q \quad \text{and} \quad q + \bar{q} = 1, \tag{3.1}$$

in the partition function. Next we explore how this specialization can be expressed in terms of the numbers $A_V(2n+1, i)$.

First of all, we note that $b = q$ and $x_i = 1$ for $i > 1$ implies that the configurations that have at least one up-pointing U-turn in positions $2, 3, \ldots, n$ have weight 0 and can therefore be omitted. For the remaining configurations we can distinguish between the cases where the topmost U-turn is down-pointing (Case 1) or not (Case 2).

Case 1. If the topmost U-turn is down-pointing, then the top row is forced and all vertex configurations are of type . In the second row, there is precisely one configuration of type , say in position i counted from the left, and the configurations right of it are all of type , while the configurations left of it are of type . Such configurations correspond to $(2n+1) \times (2n+1)$ VSASMs that have the first 1 in the second row in the i-th column. The top U-turn contributes $\sigma(q^2 x)$, while all other $n-1$ U-turns contribute $\sigma(q^2)$. In total such a configuration has the following weight

$$\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2n-i} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{i-1} \sigma(q^2 x)\sigma(q^2)^{n-1}.$$

This case contributes the following term towards the partition function

$$\sum_{i=1}^{n} A_V(2n+1, i) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{i} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2n} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-1} \sigma(q^2 x)\sigma(q^2)^{n-1}.$$

Case 2. If the topmost U-turn is up-pointing, there is a unique occurrence of  in the top row, say in position i. There is either one occurrence of  in the second row, say in position j with $1 \leqslant j < i$, or no such occurrence.

In the first case, the weight is

$$\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-j-2} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-2i+j-1} \sigma(\bar{x})\sigma(q^2)^{n-1}.$$

We notice that for fixed i, these configurations give rise to all the configurations counted by $A_V(2n+1, j)$. So, this case contributes the following term towards the partition function

$$\sum_{j=1}^{n} A_V(2n+1, j) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{j} \sum_{i=j+1}^{n} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-2i-2} \sigma(\bar{x})\sigma(q^2)^{n-1}.$$

In the second case the weight is

$$\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{i} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-i-1} \sigma(\bar{x})\sigma(q^2)^{n-1}.$$

We notice that such configurations are essentially the same as the ones counted by $A_V(2n+1,i)$ with just the first U-turn reversed, so this contributes the following term towards the partition function

$$\sum_{i=1}^{n} A_V(2n+1,i) \left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^{i} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-1} \sigma(\bar{x})\sigma(q^2)^{n-1}.$$

From this it follows that

$$\begin{aligned}
&Z_U(n; x, \underbrace{1,\dots,1}_{n-1}; \underbrace{1,\dots,1}_{n}) \\
&= \sum_{i=1}^{n} A_V(2n+1,i) \left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^{i} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-1} \sigma(\bar{x})\sigma(q^2)^{n-1} \\
&+ \sum_{i=1}^{n} A_V(2n+1,i) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{i} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2n} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-1} \sigma(q^2x)\sigma(q^2)^{n-1} \\
&+ \sum_{j=1}^{n} A_V(2n+1,j) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{j} \sum_{i=j+1}^{n} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-2i-2} \\
&\qquad \times \sigma(\bar{x})\sigma(q^2)^{n-1}, \qquad (3.2)
\end{aligned}$$

provided equation (3.1) is fulfilled.

In the end, we want to perform the following transformation of variable

$$z = \frac{\sigma(q\bar{x})}{\sigma(qx)}$$

and eliminate x. A tedious but straight forward computation shows that

$$\begin{aligned}
&-\sigma(q^2)^n \sigma(q\bar{x})^{-2n} \frac{1+z}{1-2z} Z_U(n; x, \underbrace{1,\dots,1}_{n-1}; \underbrace{1,\dots,1}_{n}) \\
&\qquad = \sum_{i=1}^{n} A_V(2n+1,i) \left(z^{i-2n-1} + z^{-i}\right). \qquad (3.3)
\end{aligned}$$

We define

$$W^-(\alpha_1,\dots,\alpha_n; x_1,\dots,x_n) = \det_{1\leqslant i,j\leqslant n} \left(x_i^{\alpha_j} - x_i^{-\alpha_j}\right)$$

and

$$\mathrm{Sp}_{2n}(\lambda_1,\dots,\lambda_n; x_1,\dots,x_n) = \frac{W^-(\lambda_1+n, \lambda_2+n-1,\dots,\lambda_n+1; x_1,\dots,x_n)}{W^-(n,n-1,\dots,1; x_1,\dots,x_n)}.$$

Then $\mathrm{Sp}_{2n}(\lambda_1,\dots,\lambda_n; x_1,\dots,x_n)$ is the character of the irreducible representation of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ corresponding to the partition $(\lambda_1,\dots,\lambda_n)$. Okada [31, Theorem 2.4] showed that

$$\begin{aligned}
&\frac{\sigma(q)^{-2n^2+2n}\sigma(q^2)^{2n^2-n}}{\prod_{i=1}^{n}\sigma(b\bar{y}_i)\sigma(q^2x_i^2)} Z_U(n; x_1,\dots,x_n; y_1,\dots,y_n) \\
&= 3^{-n(n-1)} \mathrm{Sp}_{4n}(n-1,n-1,n-2,n-2,\dots,0,0; x_1^2,\dots,x_n^2, y_1^2,\dots,y_n^2),
\end{aligned}$$

provided $q + \bar{q} = 1$. From this, we get the following for our special case

$$Z_U(n; x, 1, \dots, 1; 1, \dots, 1) = \sigma(q^2x^2)\sigma(q^2)^{n-1}3^{-n(n-1)} \times Sp_{4n}(n-1, n-1, \dots, 0, 0; x^2, 1, \dots, 1).$$

Combining this with equation (3.3), we obtain

$$-\frac{\sigma(q^2)^{2n-1}}{\sigma(q\bar{x})^{2n}}\frac{(1+z)}{(1-2z)}\sigma(q^2x^2)3^{-n(n-1)}Sp_{4n}(n-1, n-1, \dots, 0, 0; x^2, 1, \dots, 1) = \sum_{i=1}^{n} A_V(2n+1, i)\left(z^{i-2n-1} + z^{-i}\right). \quad (3.4)$$

On the other hand, Okada [31, Theorem 2.5] also showed that the partition function for off-diagonally symmetric ASMs (ASMs which are diagonally symmetric with a null diagonal) satisfy

$$\sigma(q)^{-2n^2+2n}\sigma(q^2)^{2n^2-2n}Z_O(n; x_1, \dots, x_{2n}) = 3^{-n(n-1)}Sp_{4n}(n-1, n-1, n-2, n-2, \dots, 0, 0; x_1^2, \dots, x_{2n}^2),$$

provided $q + \bar{q} = 1$, and where $Z_O(n; x_1, \dots, x_{2n})$ is the partition function for off-diagonally symmetric ASMs (OSASMs), which was explicitly calculated by Kuperberg [28, Theorem 10] (up to some normalization factor) as

$$Z_O(n; x_1, x_2, \dots, x_{2n}) = \frac{\sigma(q^2)^{2n-2n^2}\prod_{i,j=1}^{2n}\alpha(x_ix_j)}{\prod_{i,j=1}^{2n}\sigma(\bar{x}_ix_j)}\operatorname{Pf}_{1\leqslant i,j\leqslant 2n}\left(\frac{\sigma(\bar{x}_ix_j)}{\alpha(x_ix_j)}\right) \quad (3.5)$$

for general q and used by Okada to obtain the specialization. Here the *Pfaffian* of a triangular array $(a_{i,j})_{1\leqslant i<j\leqslant 2n}$ is defined as

$$\operatorname{Pf}_{1\leqslant i,j\leqslant 2n}(a_{i,j}) = \sum_{\substack{\pi=\{(i_1,j_1),\dots,(i_n,j_n)\} \\ \pi \text{ a perfect matching of} K_{2n}, i_k<j_k}} \operatorname{sgn}\pi \prod_{k=1}^{n} a_{i_k,j_k},$$

where $\operatorname{sgn}\pi$ is the sign of the permutation $i_1j_1\dots i_nj_n$.

Thus, it follows that

$$3^{-n(n-1)}Sp_{4n}(n-1, n-1, n-2, n-2, \dots, 0, 0; x^2, 1, \dots, 1) = Z_O(n; x, 1, \dots, 1), \quad (3.6)$$

provided $q + \bar{q} = 1$ holds. From the work of Razumov and Stroganov [33, Equation (24)], we know that

$$Z_O(n; x, 1, \dots, 1) = \sum_{i=2}^{2n} A_O(2n, i)z^{-i+2}\sigma(q\bar{x})^{2n-2}\sigma(q^2)^{-2n+2}, \quad (3.7)$$

where $A_O(2n, i)$ is the number of $2n \times 2n$ off-diagonally symmetric ASMs with the unique 1 in the first row in the i-th position, and for $i \geqslant 2$ these numbers are given by

$$A_O(2n, i) = \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!} \times \sum_{k=1}^{i-1} (-1)^{i+k-1} \frac{(2n+k-2)!(4n-k-1)!}{(4n-2)!(k-1)!(2n-k)!}, \quad (3.8)$$

whereas they are 0 when $i < 2$. Using equations (3.6) and (3.7), we obtain

$$Sp_{4n}(n-1, n-1, n-2, n-2, \ldots, 0, 0; x^2, 1, \ldots, 1) = 3^{n(n-1)} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-2} \sum_{i=2}^{2n} A_O(2n, i) z^{-i+2}. \quad (3.9)$$

Combining this with equation (3.4), we obtain

$$-\sigma(q^2)\sigma(q\bar{x})^{-2} \frac{1+z}{1-2z} \sigma(q^2x^2) \sum_{i=2}^{2n} A_O(2n, i) z^{-i+2} = \sum_{i=1}^{n} A_V(2n+1, i) \left(z^{i-2n-1} + z^{-i}\right).$$

Assuming $q + \bar{q} = 1$, we have $-\dfrac{\sigma(q^2)\sigma(q^2x^2)}{\sigma(q\bar{x})^2} = \dfrac{1-2z}{z^2}$, so the above becomes

$$\sum_{i=2}^{2n} A_O(2n, i) \left(z^{-i} + z^{-i+1}\right) = \sum_{i=1}^{n} A_V(2n+1, i) \left(z^{i-2n-1} + z^{-i}\right). \quad (3.10)$$

Now, we compare the coefficients of z^{-i} for $1 \leqslant i \leqslant n$ from equation (3.10) to conclude

$$A_V(2n+1, i) = A_O(2n, i) + A_O(2n, i+1). \quad (3.11)$$

Combining equations (3.11) and (3.8), we get the following theorem, which proves Fischer's conjecture.

Theorem 3.2. *The number of $(2n+1) \times (2n+1)$ VSASMs with the first 1 in its second row at position i is given by*

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!} \prod_{k=1}^{n-1} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!}.$$

Remark 1. *We notice that the refined enumeration numbers for VSASMs with respect to the position of the unique 1 in the first column coincides with the refined enumeration of OSASMs. This also gives us the following relation between the different refined enumeration numbers for VSASMs*

$$A_V(2n+1, i) = A_{VC}(2n+1, i) + A_{VC}(2n+1, i+1),$$

where $A_{VC}(2n+1, i)$ is the number of $(2n+1) \times (2n+1)$ VSASMs with the position of the unique 1 in the first column in position i.

3.2 VERTICALLY AND HORIZONTALLY SYMMETRIC ASMS

ASMs that are invariant under the reflection in the vertical symmetry axis as well as the horizontal symmetry axis also exist only for odd order. In this section we focus on such matrices. The enumeration formulas for VHSASMs were proved by Okada [31], who showed that the number of $(4n+1) \times (4n+1)$ VHSASMs is

$$\prod_{i=0}^{n-1} \frac{(3i+2)!(3n+3i)!}{(2n+i)!(3n+i)!}$$

and the number of $(4n+3) \times (4n+3)$ VHSASMs is

$$\prod_{i=0}^{n} \frac{(3i-1)!(3n+3i)!}{(2n+i)!(3n+i+1)!}.$$

No refined enumeration formulas for VHSASMs have been conjectured or proven so far in the literature.

Due to a slight difference between how the matrices are enumerated for order $4n+3$ and order $4n+1$, we deal with the refined enumeration of both the cases in separate subsections below. We assume $n \geqslant 1$, unless otherwise mentioned. The only VHSASM of order 1 is the single entry matrix (1) and for order 3, the following matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

3.2.1 *VHSASMs of order* $4n+3$

First, we consider the case for VHSASMs of order $4n+3$. It is clear that VHSASMs also have two 1's in its second row, and the second row is determined by the position of the first 1 in this row. The middle row of a VHSASMs is $(1, -1, 1, \ldots, -1, 1)$ by the horizontal symmetry. The aim of this subsection is to give a generating function result for the refined enumeration of order $4n+3$ VHSASMs with respect to the position of the first 1 in the second row.

An example of such a VHSASM of order 15 is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

From the preceding paragraph, it is clear that $(4n+3) \times (4n+3)$ VHSASMs correspond to $(2n+1) \times (2n+1)$ matrices with entries in $\{\pm 1, 0\}$ that have the following properties.

1. The non-zero entries alternate in each row and column.
2. The topmost non-zero entry of each column is 1, except the last column which is always $(-1, 1, \dots, 1, -1)^T$.
3. The first non-zero entry of each row is 1, except for the last row which is always $(-1, 1, \dots, 1, -1)$.

The 7×7 matrix with these properties that corresponds to the above VHSASM is

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1
\end{pmatrix}.$$

(We deleted the last $(2n+1)$ columns and rows as well as the first row and first column.)

Such a matrix has a unique 1 in its first row and let $A_{VH}(4n+3, i+1)$ be the number of such matrices that have this unique 1 in column i (the index i runs from

1 to $2n$, because the last position has a fixed -1). Note that $A_{VH}(4n+3,i)$ is now equal to the number of VHSASMs of order $4n+3$ where the first 1 in the second row is in position i. Due to the horizontal symmetry $A_{VH}(2n+1,1)=0$ for all n. Now, if we delete the top two rows from the $(2n+1)\times(2n+1)$ matrix we obtain a $(2n-1)\times(2n+1)$ matrix with properties 1. and 3., but property 2. replaced by the following,

2′. The topmost non-zero entry of each column is 1, except for the last column which is always $(-1,1,\dots,1,-1)^T$, and one other column whose topmost non-zero entry is -1 (if a non-zero entry exists at all in this column).

In our example, we obtain

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

We let $B_{VH}(4n+3,j)$ denote the number of such $(2n-1)\times(2n+1)$ matrices with the special column in 2′. being column j. When passing from the $(2n+1)\times(2n+1)$ matrix to the $(2n-1)\times(2n+1)$ matrix, the column j is the position of the first 1 in the second row of the $(2n+1)\times(2n+1)$ matrix if the second row contains two 1's and otherwise it is the position of the unique 1 in the top row. We can deduce the following simple relation between $A_{VH}(4n+3,i)$ and $B_{VH}(4n+3,j)$:

$$A_{VH}(4n+3,i+1)=\sum_{j=1}^{i} B_{VH}(4n+3,j)$$
$$\Leftrightarrow B_{VH}(4n+3,i)=A_{VH}(4n+3,i+1)-A_{VH}(4n+3,i). \quad (3.12)$$

Hence, in order to compute $A_{VH}(4n+3,i)$, it suffices to compute $B_{VH}(4n+3,i)$.

Next we use the correspondence between ASMs and the six-vertex model as explained in Chapter 2 to translate the $(2n+1)\times(2n+1)$ matrices into directed graphs (see Figure 3.4). In our example we obtain Figure 3.5. We note that the right boundary, i.e., the fixed column $(-1,1,\dots,-1)^T$, is modeled via U-turns with up-pointing orientation and the bottom boundary, i.e., the fixed row $(-1,1,-1,\dots,-1)$ is modeled via right pointing U-turns. For the partition function

$$Z_{UU}(n;x_1,\dots,x_n;y_1,\dots,y_n),$$

we allow both up-pointing and down-pointing U-turns for the right boundary as well as both right-pointing and left-pointing U-turns for the bottom boundary, and the weights are as indicated in Figure 3.6, involving now another global parameter c.

We shall use the following formula for this partition function that was derived by Kuperberg [28, Theorem 10] (up to some normalization factor).

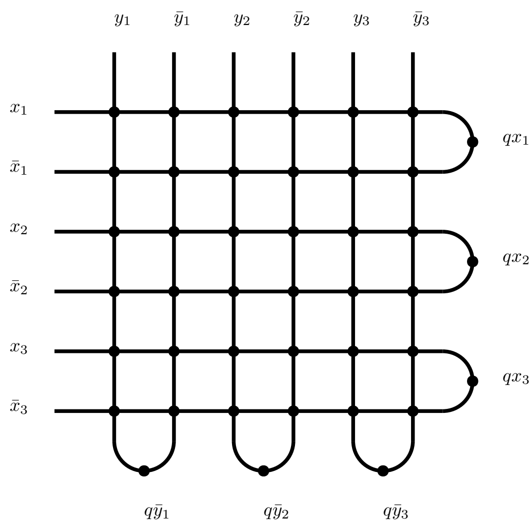


Figure 3.4: The grid corresponding to VHSASMs.

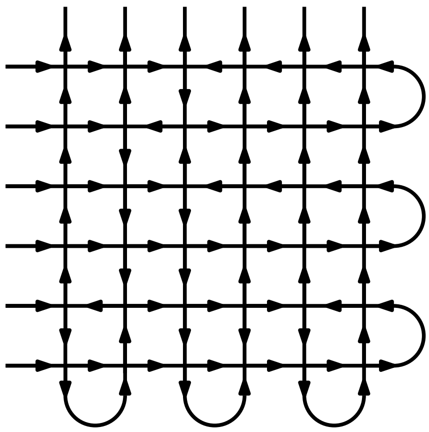


Figure 3.5: The six-vertex configuration of our example.

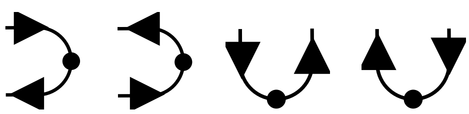


Figure 3.6: Weights of the U-turns.

Theorem 3.3. *The UU-turn partition function of order* n *is*

$$\begin{aligned} &Z_{UU}(n;x_1,\dots,x_n;y_1,\dots,y_n) \\ &\quad= \frac{\sigma(q^2)^{n-4n^2}\prod_{i=1}^{n}\sigma(q^2\bar{y}_i^2)\sigma(q^2x_i^2)\prod_{i,j=1}^{n}\alpha(x_iy_j)^2\alpha(x_i\bar{y}_j)^2}{\prod_{1\leqslant i<j\leqslant n}\sigma(\bar{x}_ix_j)^2\sigma(y_i\bar{y}_j)^2\prod_{1\leqslant i\leqslant j\leqslant n}\sigma(\bar{x}_i\bar{x}_j)^2\sigma(y_iy_j)^2} \\ &\qquad\qquad\times \det_{1\leqslant i,j\leqslant n}(M_U)\det_{1\leqslant i,j\leqslant n}(M_{UU}), \end{aligned}$$

where $\alpha(x) = \sigma(qx)\sigma(q\bar{x})$,

$$(M_U)_{i,j} = \left(\frac{1}{\alpha(x_i\bar{y}_j)} - \frac{1}{\alpha(x_iy_j)}\right),$$

and

$$(M_{UU})_{i,j} = \left(\frac{\sigma(b\bar{y}_j)\sigma(cx_i)}{\sigma(qx_i\bar{y}_j)} - \frac{\sigma(b\bar{y}_j)\sigma(c\bar{x}_i)}{\sigma(q\bar{x}_i\bar{y}_j)} - \frac{\sigma(by_j)\sigma(cx_i)}{\sigma(qx_iy_j)} + \frac{\sigma(by_j)\sigma(c\bar{x}_i)}{\sigma(q\bar{x}_iy_j)}\right),$$

and all determinants are of order n.

In the following, we will specialize

$$(x_1,\dots,x_n) = (x,1,\dots,1) \quad \text{and} \quad (y_1,\dots,y_n) = (1,\dots,1),$$

as well as

$$b = \bar{q}, \quad c = \bar{q} \quad \text{and} \quad q + \bar{q} = 1, \tag{3.13}$$

in the partition function. First of all, we note that $b = \bar{q}$ and $x_i = 1$ for $i > 1$ implies that the configurations that have at least one down-pointing U-turn in positions $2, 3, \dots, n$ have weight 0 and can therefore be omitted. For the remaining configurations we can distinguish between the cases where the topmost U-turn is down-pointing (Case 1) or not (Case 2). Also $c = \bar{q}$ means that configurations which have at least one left pointing U-turns in the bottom boundary have weight 0, and so they are omitted.

Case 1. If the topmost U-turn is down-pointing, then the top row is forced and all vertex configurations are of type . In the second row, there is precisely one configuration of type , say in position i counted from the left, and the configurations right of it are all of type , while the configurations left of it are of type . The top U-turn contributes $\sigma(x)$, while all other $2n-1$ U-turns contribute $\sigma(\bar{q}^2)$. In total such a configuration has the following weight

$$\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{4n-i}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{i-1}\sigma(x)\sigma(\bar{q}^2)^{2n-1}.$$

Case 2. In this case, there is a unique occurrence of  in the top row, say in position i. There is either one occurrence of  in the second row, say in position j with $1 \leqslant j < i$, or no such occurrence. In the first case, the weight is

$$\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-j-2}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-2i+j-1}\sigma(\bar{q}^2\bar{x})\sigma(\bar{q}^2)^{2n-1},$$

while in the second case the weight is

$$\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^i \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-i-1} \sigma(\bar{q}^2\bar{x})\sigma(\bar{q}^2)^{2n-1}.$$

From this, it follows that

$$\begin{aligned}
&Z_{UU}(n;x,\underbrace{1,\dots,1}_{n-1};\underbrace{1,\dots,1}_{n}) \\
&= \sum_{i=1}^{2n} B_{VH}(4n+3,i)\left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^i \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-1} \sigma(\bar{q}^2\bar{x})\sigma(\bar{q}^2)^{2n-1} \\
&\quad + \sum_{i=1}^{2n} B_{VH}(4n+3,i)\left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^i \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{4n} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-1} \sigma(x)\sigma(\bar{q}^2)^{2n-1} \\
&\quad + \sum_{j=1}^{2n} B_{VH}(4n+3,j)\left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^j \sum_{i=j+1}^{2n} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-2i-2} \\
&\qquad\qquad \times \sigma(\bar{q}^2\bar{x})\sigma(\bar{q}^2)^{2n-1}. \quad (3.14)
\end{aligned}$$

From here, substituting $z = \dfrac{\sigma(q\bar{x})}{\sigma(qx)}$, we shall arrive at

$$\begin{aligned}
&-\sigma(q^2)^{2n}\sigma(q\bar{x})^{-4n}\frac{1-z^2}{1-2z} Z_{UU}(n;x,\underbrace{1,\dots,1}_{n-1};\underbrace{1,\dots,1}_{n}) \\
&\qquad = \sum_{i=1}^{2n} B_{VH}(4n+3,i)\left(z^{i-4n-1} - z^{-i+1}\right). \quad (3.15)
\end{aligned}$$

Okada [31, Theorem 2.4] showed that

$$\begin{aligned}
&\prod_{i=1}^{n} \sigma(q^2\bar{y}_i^2)^{-1}\sigma(q^2x_i^2)^{-1} Z_{UU}(n;x_1,\dots,x_n;y_1,\dots,y_n) \\
&= 3^{-2n^2+n} Sp_{4n}(n-1,n-1,n-2,n-2,\dots,0,0;x_1^2,\dots,x_n^2,y_1^2,\dots,y_n^2) \\
&\qquad \times Sp_{4n+2}(n,n-1,n-1,n-2,n-2,\dots,0,0;x_1^2,\dots,x_n^2,y_1^2,\dots,y_n^2,1),
\end{aligned}$$

provided (3.13) is satisfied. From this, we get the following for our special case

$$\begin{aligned}
&Z_{UU}(n;x,1,\dots,1;1,\dots,1) = \sigma(q^2)^{2n-1}\sigma(q^2x^2)3^{-2n^2+n} \\
&\times Sp_{4n}(n-1,n-1,n-2,n-2,\dots,0,0;x^2,1,\dots,1) \\
&\qquad \times Sp_{4n+2}(n,n-1,n-1,n-2,n-2,\dots,0,0;x^2,1,\dots,1). \quad (3.16)
\end{aligned}$$

Combining equations (3.15) and (3.16), we get

$$\begin{aligned}
&-3^{-2n^2+n}\sigma(q^2x^2)\sigma(q^2)^{4n-1}\sigma(q\bar{x})^{-4n}\frac{1-z^2}{1-2z} \\
&\times Sp_{4n}(n-1,n-1,n-2,n-2,\dots,0,0;x^2,1,\dots,1) \\
&\times Sp_{4n+2}(n,n-1,n-1,n-2,n-2,\dots,0,0;x^2,1,\dots,1) \\
&\qquad = \sum_{i=1}^{2n} B_{VH}(4n+3,i)\left(z^{i-4n-1} - z^{-i+1}\right). \quad (3.17)
\end{aligned}$$

From equation (A.5) in Appendix A, we have

$$\mathrm{Sp}_{4n+2}(n, n-1, n-1, n-2, n-2, \dots, 0, 0; x^2, 1, \dots, 1) = \sum_{1 \leqslant j \leqslant i \leqslant n+1} Q_{n,i} x^{2i-4j+2}, \tag{3.18}$$

where

$$Q_{n,i} = \frac{3^{n(n-1)}}{2^{n-1}(4n-1)!} \prod_{j=0}^{n-1} \frac{(4j+3)(6j+6)!}{(2n+2j+1)!} \sum_{j=0}^{n} \left[\frac{27^j (3j-2n-i+2)_{4n-3}(3n-3j+1)}{(3j)!(n-j)!(3j+1)_{3n}} \right. \\ \left. \times \left(\frac{\left(n-j+\frac{4}{3}\right)_{2j}(2n+3j-i-1)_2}{(3n+3j+1)_2} - \frac{\left(n-j+\frac{2}{3}\right)_{2j}(-2n+3j-i)_2}{(3n-3j+1)_2} \right) \right] \tag{3.19}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$. The purpose of Appendix A is also to provide a combinatorial interpretation of $Q_{n,i}$ in terms of rhombus tilings.

Using equations (3.9), (3.17), (3.18) as well as $q + \bar{q} = 1$ and

$$B_{VH}(4n+3, i) = A_{VH}(4n+3, i+1) - A_{VH}(4n+3, i)$$

with some simplifications, we now arrive at the following equation

$$3^{-n^2}(z^2 - z + 1)^n (1 - z^2) \left(\sum_{i=2}^{2n} A_O(2n, i) z^{-i} \right) \left(\sum_{1 \leqslant j \leqslant i \leqslant n+1} Q_{n,i} x^{2i-4j+2} \right) \\ = \sum_{i=1}^{2n+1} \left(A_{VH}(4n+3, i+1) - A_{VH}(4n+3, i) \right) \left(z^{i-2n-1} - z^{-i+2n+1} \right). \tag{3.20}$$

Replacing $x^2 = \dfrac{zq-1}{q-z}$, using $q + \bar{q} = 1$ we get

$$3^{-n^2}(1 - z^2) \left(\sum_{i=2}^{2n} A_O(2n, i) z^{-i} \right) \\ \times \left(\sum_{1 \leqslant j \leqslant i \leqslant n+1} Q_{n,i} (zq-1)^{n+i-2j+1} (q-z)^{n-i+2j-1} (-q)^{-n} \right) \\ = \sum_{i=1}^{2n+1} \left(A_{VH}(4n+3, i+1) - A_{VH}(4n+3, i) \right) \left(z^{i-2n-1} - z^{-i+2n+1} \right).$$

Thus, we have proved the following result.

Theorem 3.4. *Let $A_{VH}(4n+3, i)$ denote the number of VHSASMs of order $4n+3$, with the first occurrence of a 1 in the second row be in the i-th column. Then, for all $n \geqslant 1$ the following is satisfied*

$$3^{-n^2}(1-z^2)\left(\sum_{i=2}^{2n} A_O(2n,i)z^{-i}\right)$$
$$\times\left(\sum_{1\leqslant j\leqslant i\leqslant n+1} Q_{n,i}(zq-1)^{n+i-2j+1}(q-z)^{n-i+2j-1}(-q)^{-n}\right)$$
$$= \sum_{i=1}^{2n+1} \left(A_{VH}(4n+3,i+1) - A_{VH}(4n+3,i)\right)\left(z^{i-2n-1} - z^{-i+2n+1}\right),$$

where every quantity appearing on the left-hand side is explicitly known, and $A_{VH}(4n+3, 1) = 0$ for all n.

Remark 2. *We can write equation (3.17) differently by using results of Ayyer and Behrend [2] as*

$$-3^{-2n^2-3n-2}(z-1-z^2)^{2n+1}(1-z^2)$$
$$\times s(2n+1, 2n, 2n, \ldots, 1, 1, 0, 0; x^2, \bar{x}^2, 1, \ldots, 1)$$
$$= \sum_{i=1}^{2n} B_{VH}(4n+3,i)\left(z^{i-1} - z^{4n-i+1}\right).$$

where

$$s(\lambda_1, \ldots, \lambda_n; x_1, \ldots, x_n) = \frac{V(\lambda_1+n-1, \lambda_2+n-2, \ldots, \lambda_n; x_1, \ldots, x_n)}{V(n-1, n-2 \ldots, 1, 0; x_1, \ldots, x_n)}$$

is the character of the irreducible representation of the general linear group $GL_n(\mathbb{C})$ corresponding to the partition $(\lambda_1, \ldots, \lambda_n)$ (i.e., a Schur function) and

$$V(\alpha_1, \ldots, \alpha_n; x_1, \ldots, x_n) = \det_{1\leqslant i,j\leqslant n}\left(x_i^{\alpha_j}\right).$$

To see that this is true, we first use equation (7), then Proposition 5, followed by equation (8) and finally Corollary 11, equation (55) from the work of Ayyer and Behrend [2].

From this and using the relation (3.12), we shall arrive at

$$-3^{-2n^2-3n-1}(z-1-z^2)^{2n+1}(1-z^2)$$
$$\times s(2n+1, 2n, 2n, \ldots, 1, 1, 0, 0; x^2, \bar{x}^2, 1, \ldots, 1)$$
$$= \sum_{i=1}^{2n}\left(A_{VH}(4n+3,i+1) - A_{VH}(4n+3,i)\right)\left(z^{i+1} - z^{4n-i+3}\right). \quad (3.21)$$

3.2.2 *VHSASMs of order* $4n+1$

We now focus on the VHSASMs of order $4n+1$. An example of such a matrix of order 9 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The observations on the two top rows of order $4n+3$ VHSASMs follow in this case as well. The aim of this subsection is to give a generating function result for the refined enumeration of order $4n+1$ VHSASMs with respect to the first 1 in the second row. However, we need to modify our arguments for this case, as the six vertex configurations of VHSASMs of order $4n+1$ are slightly different than the one for order $4n+3$.

It is clear that any order $4n+1$ VHSASM corresponds to a $(2n+1)\times(2n+1)$ matrix with the following properties.

1. The non-zero entries alternate in each row and column.
2. The topmost non-zero entry of each column is 1; the last column is equal to $(1,-1,1,\ldots,-1)^T$.
3. The first non-zero entry of each row is 1; the last row is equal to $(1,-1,1,\ldots,-1)$.

The 5×5 matrix with these properties that corresponds to the VHSASM from above is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}.$$

(We deleted the last $2n$ columns and rows.)

Let us denote the number of order $4n+1$ VHSASMs with the first occurrence of a 1 in its second row being placed in the i-th column to be $A_{VH}(4n+1,i)$. That is, $A_{VH}(4n+1,i)$ counts the number of $(2n+1)\times(2n+1)$ matrices described above where the unique 1 in its second row is situated in the i-th column. Throughout the remainder of this section, we consider $n>1$. It is easy to see that for $n=1$, $A_{VH}(5,1)=0$ and $A_{VH}(5,2)=1$.

We shall use the correspondence between ASMs and the six-vertex model as in the previous section. The grid for order $4n+1$ VHSASMs is the same as in Figure

3.4. However, for order $4n+1$ VHSASMs the U-turns in the right boundary are now down-pointing, as opposed to the up-pointing ones for order $4n+3$ VHSASMs and the U-turns on the bottom are now left-pointing, as opposed to the right-pointing ones for order $4n+3$ VHSASMs (see Figure 3.5). This is because we do not delete the first row and the first column here. Again, for the partition function $Z_{UU}(n; x_1, \ldots, x_n; y_1, \ldots, y_n)$, we allow both up-pointing and down-pointing U-turns for the right boundary as well as both right-pointing and left-pointing U-turns for the bottom boundary, and the weights are as indicated in Figure 3.6. The partition function is still the one given in Theorem 3.3.

In the following, we will specialize

$$(x_1, \ldots, x_n) = (x, 1, \ldots, 1) \quad \text{and} \quad (y_1, \ldots, y_n) = (1, \ldots, 1),$$

as well as

$$b = q, \quad c = q \quad \text{and} \quad q + \bar{q} = 1, \tag{3.22}$$

in the partition function. This will give us two cases similar to the cases described in the previous section on VSASMs. Analogous to equation (3.2) we shall get the following

$$\begin{aligned}
&Z_{UU}(n; x, \underbrace{1, \ldots, 1}_{n-1}; \underbrace{1, \ldots, 1}_{n}) \\
&= \sum_{i=1}^{2n} A_{VH}(4n+1, i) \left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^{i} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-1} \sigma(\bar{x})\sigma(q^2)^{2n-1} \\
&+ \sum_{i=1}^{2n} A_{VH}(4n+1, i) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{i} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{4n} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-1} \sigma(q^2x)\sigma(q^2)^{2n-1} \\
&+ \sum_{j=1}^{2n} A_{VH}(4n+1, j) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{j} \sum_{i=j+1}^{2n} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-2i-2} \\
&\qquad \times \sigma(\bar{x})\sigma(q^2)^{2n-1},
\end{aligned} \tag{3.23}$$

provided (3.22) is satisfied.

Like earlier, we want to perform the following transformation of variable

$$z = \frac{\sigma(q\bar{x})}{\sigma(qx)}$$

and eliminate x. After a straightforward calculation, analogous to how we obtained equation (3.3), we shall get the following

$$\begin{aligned}
&-\sigma(q^2)^{2n}\sigma(q\bar{x})^{-4n}\frac{1+z}{1-2z} Z_{UU}(n; x, \underbrace{1, \ldots, 1}_{n-1}; \underbrace{1, \ldots, 1}_{n}) \\
&\qquad = \sum_{i=1}^{2n} A_{VH}(4n+1, i)\left(z^{i-4n-1} + z^{-i}\right).
\end{aligned} \tag{3.24}$$

We define

$$W^+(\alpha_1,\ldots,\alpha_n;x_1,\ldots,x_n)=\det_{1\leqslant i,j\leqslant n}\left(x_i^{\alpha_j}+x_i^{-\alpha_j}\right)$$

and

$$O_{2n}(\lambda_1,\ldots,\lambda_n;x_1,\ldots,x_n)=\frac{2W^+(\lambda_1+n-1,\lambda_2+n-2,\ldots,\lambda_n;x_1,\ldots,x_n)}{W^+(n-1,n-2\ldots,0;x_1,\ldots,x_n)}.$$

Then $O_{2n}(\lambda_1,\ldots,\lambda_n;x_1,\ldots,x_n)$ is the character of the irreducible representation of the double cover of the even orthogonal group O_{2n} corresponding to the partition $(\lambda_1,\ldots,\lambda_n)$, if $\lambda_n\neq 0$. It suffices for our purposes to have only the case for $\lambda_n\neq 0$. Okada [31, Theorem 2.4] showed that

$$\begin{aligned}&\prod_{i=1}^{n}(x_i+\bar{x}_i)(y_i+\bar{y}_i)\prod_{i=1}^{n}\sigma(q^2\bar{y}_i^2)^{-1}\sigma(q^2x_i^2)^{-1}Z_{UU}(n;x_1,\ldots,x_n;y_1,\ldots,y_n)\\&=3^{-2n^2+n}Sp_{4n}(n-1,n-1,n-2,n-2,\ldots,0,0;x_1^2,\ldots,x_n^2,y_1^2,\ldots,y_n^2)\\&\times O_{4n}\left(n+\frac{1}{2},n-\frac{1}{2},n-\frac{1}{2},n-\frac{3}{2},n-\frac{3}{2},\ldots,\frac{3}{2},\frac{3}{2},\frac{1}{2};x_1^2,\ldots,x_n^2,y_1^2,\ldots,y_n^2\right),\end{aligned}$$

provided (3.22) is satisfied.

The above for our special values gives us

$$\begin{aligned}&2^{2n-1}(x+\bar{x})\sigma(q^2)^{-2n+1}\sigma(q^2x^2)^{-1}Z_{UU}(n;x,\underbrace{1,\ldots,1}_{n-1};\underbrace{1,\ldots,1}_{n})\\&=3^{-2n^2+n}Sp_{4n}(n-1,n-1,n-2,n-2,\ldots,0,0;x^2,1,\ldots,1)\\&\quad\times O_{4n}\left(n+\frac{1}{2},n-\frac{1}{2},n-\frac{1}{2},n-\frac{3}{2},n-\frac{3}{2},\ldots,\frac{3}{2},\frac{3}{2},\frac{1}{2};x^2,1,\ldots,1\right).\end{aligned}\tag{3.25}$$

Now, by using a formula of Ayyer and Behrend [2, Proposition 5, then use equation (7)] we can rewrite equation (3.25) as follows.

$$\begin{aligned}&Z_{UU}(n;x,\underbrace{1,\ldots,1}_{n-1};\underbrace{1,\ldots,1}_{n})=3^{-2n^2+3n-2}\sigma(q^2)^{2n-1}\sigma(q^2x^2)\\&\times(x^2+1+\bar{x}^2)Sp_{4n}(n-1,n-1,n-2,n-2,\ldots,0,0;x^2,1,\ldots,1)\\&\qquad\times Sp_{4n-2}(n-1,n-2,n-2,\ldots,1,0;x^2,1,\ldots,1)\end{aligned}\tag{3.26}$$

From equations (3.9), (3.18), (3.24) and (3.26), after some simplifications we shall get the following equation:

$$\begin{aligned}&(-1)^{n+1}3^{-n^2+2n-1}(1+z)(z-1-z^2)^{n-1}\\&\times\left(\sum_{i=2}^{2n}A_O(2n,i)z^{-i}\right)\left(\sum_{1\leqslant i\leqslant j\leqslant n}Q_{n-1,i}x^{2i-4j+2}\right)\\&\qquad=\sum_{i=1}^{2n}A_{VH}(4n+1,i)(z^{i-2n-2}+z^{2n-1-i}),\end{aligned}$$

where $Q_{n,i}$ is given by (3.19). From the above, using $x^2=\dfrac{zq-1}{q-z}$ we shall arrive at the following result.

Theorem 3.5. *Let* $A_{VH}(4n+1,i)$ *denote the number of VHSASMs of order* $4n+1$*, with the first occurrence of a* 1 *in the second row be in the* i*-th column. Then, for all* $n>1$ *the following is satisfied*

$$\begin{aligned} &3^{-n^2+2n-1}(1+z)\left(\sum_{i=2}^{2n} A_O(2n,i)z^{-i}\right)\\ &\times\left(\sum_{1\leqslant i\leqslant j\leqslant n} Q_{n-1,i}(zq-1)^{n+i-2j}(q-z)^{n-i+2j-2}(-q)^{-n+1}\right)\\ &\qquad\qquad = \sum_{i=1}^{2n} A_{VH}(4n+1,i)(z^{i-2n-2}+z^{2n-1-i}),\end{aligned}$$

where every quantity appearing in the left-hand side is explicitly known.

Remark 3. *We can write equation* (3.25) *differently by using a result of Ayyer and Behrend [2, Equation (8) and then use Corollary 11, equation (54)] as*

$$\begin{aligned} Z_{UU}(n;x,\underbrace{1,\dots,1}_{n-1};\underbrace{1,\dots,1}_{n}) &= (-1)^{n^2}3^{-2n^2-n}\sigma(q^2)^{2n-1}\sigma(q^2x^2)(z^2+1-z)z^{-1}\\ &\times s(2n,2n-1,2n-1,\dots,1,1,0,0;x^2,\bar{x}^2,1,\dots,1).\end{aligned}$$

From this and using equation (3.24)*, we shall arrive that*

$$\begin{aligned} &(-1)^{n^2}3^{-2n^2-n}(z-1-z^2)^{2n}(1+z)\\ &\times s(2n,2n-1,2n-1,\dots,1,1,0,0;x^2,\bar{x}^2,\dots,1)\\ &\qquad\qquad = \sum_{i=1}^{2n} A_{VH}(4n+1,i)\left(z^i+z^{4n+1-i}\right). \qquad (3.27)\end{aligned}$$

3.3 VERTICALLY AND HORIZONTALLY PERVERSE ASMS

A $(4n+1)\times(4n+3)$ matrix is called a vertically and horizontally perverse ASM (VHPASM) if it satisfies the alternating sign conditions and has the same symmetries as a VHSASM, except the central entry $(\star)$ which has opposite signs when read horizontally and vertically. An example of such a VHPASM of dimension 9×11 is

$$\begin{pmatrix} 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&-1&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&1&-1&1&0&-1&0&1&-1&1&0\\ 1&-1&1&-1&1&\star&1&-1&1&-1&1\\ 0&1&-1&1&0&-1&0&1&-1&1&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&-1&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0 \end{pmatrix}.$$

This class was first considered by Kuperberg [28], and then enumerated by Okada [31].

A VHPASM with the dimensions stated above is said to be of order $4n+2$. It is clear that we can ask for the refined enumerations of VHPASMs with respect to the position of the first occurrence of a 1 in the second column as well as in the second row. These numbers are different due to the different lengths of the rows and columns. Let us denote by $A^{C}_{\mathrm{VHP}}(4n+2,i)$ (resp. $A^{R}_{\mathrm{VHP}}(4n+2,i)$) the number of order $4n+2$ VHPASMs with the first occurrence of a 1 in the second column (resp. row) at the i-th row (resp. i-th column). In this section, we give enumeration results for these numbers. Since, the technique is similar to the one used in Sections 3.1 and 3.2, for the sake of brevity we omit certain easily verifiable details. We also assume $n \geqslant 1$ unless otherwise mentioned. For $n=0$, there are no VHPASMs because of the restriction imposed by the special entry $\star$.

It is clear that VHPASMs of order $4n+2$ correspond to $(2n+1)\times(2n+1)$ matrices with entries in $\{\pm 1, 0, \star\}$ that have the following properties.

1. The non-zero entries alternate in each row and column.
2. The topmost non-zero entry of each column is 1; the last column is equal to $(1,-1,\dots,-1,\star)^T$.
3. The first non-zero entry of each row is 1; the last row is equal to $(-1,1,\dots,1,\star)$.

The 5×5 matrix with these properties that corresponds to the VHPASM from above is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 1 & \star \end{pmatrix}.$$

(We deleted the first column, the last $2n+1$ columns as well as the bottom $2n$ rows.)

We again use the correspondence between ASMs and the six-vertex model. The grid for order $4n+2$ VHPASMs is the same as in Figure 3.4 (we ignore the special entry $\star$ to get the grid). However, for order $4n+2$ VHPASMs the U-turns in the right boundary are now down-pointing and the U-turns on the bottom boundary are right-pointing. Again, for the partition function $Z_{\mathrm{UU}}(n;x_1,\dots,x_n;y_1,\dots,y_n)$, we allow both up-pointing and down-pointing U-turns for the right boundary as well as both right-pointing and left-pointing U-turns for the bottom boundary, and the weights are as indicated in Figures 2.2 and 3.6. The partition function is still the one given in Theorem 3.3.

In the following, we will specialize

$$(x_1,\dots,x_n)=(x,1,\dots,1) \quad \text{and} \quad (y_1,\dots,y_n)=(1,\dots,1),$$

as well as

$$b=q, \quad c=\bar{q} \quad \text{and} \quad q+\bar{q}=1, \tag{3.28}$$

for refined enumeration with respect to rows and

$$b = \bar{q}, \quad c = q \quad \text{and} \quad q + \bar{q} = 1, \tag{3.29}$$

for refined enumeration with respect to columns, in the partition function. This will give us two cases similar to the cases described in the previous sections. Analogous to equations (3.2) and (3.14), we shall get the following sets of equations.

$$\begin{aligned}
&(-1)^n Z_{UU}(n; x, \underbrace{1, \dots, 1}_{n-1}; \underbrace{1, \dots, 1}_{n}) \\
&= \sum_{i=1}^{2n} A^R_{VHP}(4n+2, i+1) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^i \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{4n} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-1} \sigma(q^2 x)\sigma(q^2)^{2n-1} \\
&+ \sum_{i=1}^{2n} A^R_{VHP}(4n+2, i+1) \left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^i \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-1} \sigma(\bar{x})\sigma(q^2)^{2n-1} \\
&+ \sum_{j=1}^{2n} A^R_{VHP}(4n+2, j+1) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^j \sum_{i=j+1}^{2n} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-2i-2} \\
&\qquad \times \sigma(\bar{x})\sigma(q^2)^{2n-1}.
\end{aligned} \tag{3.30}$$

provided (3.28) holds, and

$$\begin{aligned}
&(-1)^n Z_{UU}(n; x, \underbrace{1, \dots, 1}_{n-1}; \underbrace{1, \dots, 1}_{n}) \\
&= \sum_{i=1}^{2n} B_{VHP}(4n+2, i) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^i \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{4n} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-1} \sigma(x)\sigma(\bar{q}^2)^{2n-1} \\
&+ \sum_{i=1}^{2n} B_{VHP}(4n+2, i) \left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^i \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-1} \sigma(\bar{q}^2\bar{x})\sigma(\bar{q}^2)^{2n-1} \\
&+ \sum_{j=1}^{2n} B_{VHP}(4n+2, j) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^j \sum_{i=j+1}^{2n} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{4n-2i-2} \\
&\qquad \times \sigma(\bar{q}^2\bar{x})\sigma(\bar{q}^2)^{2n-1},
\end{aligned} \tag{3.31}$$

provided (3.29) holds and where

$$B_{VHP}(n, i) = A^C_{VHP}(n, i) - A^C_{VHP}(n, i-1).$$

From here, substituting $z = \dfrac{\sigma(q\bar{x})}{\sigma(qx)}$, equation (3.30) gives us

$$\begin{aligned}
&(-1)^n \sigma(q^2)^{2n} \sigma(q\bar{x})^{-4n} \frac{1+z}{2z-1} Z_{UU}(n; x, \underbrace{1, \dots, 1}_{n-1}; \underbrace{1, \dots, 1}_{n}) \\
&\qquad = \sum_{i=1}^{2n} A^R_{VHP}(4n+2, i+1) \left(z^{i-4n-1} + z^{-i}\right).
\end{aligned} \tag{3.32}$$

provided (3.28) holds, and equation (3.31) gives us

$$(-1)^n \sigma(q^2)^{2n} \sigma(q\bar{x})^{-4n} \frac{1-z^2}{2z-1} Z_{UU}(n; x, \underbrace{1, \ldots, 1}_{n-1}; \underbrace{1, \ldots, 1}_{n}) \\ = \sum_{i=1}^{2n} B_{VHP}(4n+2, i) \left(z^{i-4n-1} - z^{-i+1}\right), \quad (3.33)$$

provided (3.29) holds.

Okada [31, Theorem 2.4] showed that, if (3.28) holds, then

$$(-1)^n \prod_{i=1}^{n} \sigma(q^2\bar{y}_i^2)^{-1} \sigma(q^2 x_i^2)^{-1} (y_i^2 + 1 + y_i^{-2})^{-1} Z_{UU}(n; x_1, \ldots, x_n; y_1, \ldots, y_n) \\ = 3^{-2n^2+n} \left(Sp_{4n}(n-1, n-1, n-2, n-2, \ldots, 0, 0; x_1^2, \ldots, x_n^2, y_1^2, \ldots, y_n^2)\right)^2, \quad (3.34)$$

and if (3.29) holds then

$$(-1)^n \prod_{i=1}^{n} \sigma(q^2\bar{y}_i^2)^{-1} \sigma(q^2 x_i^2)^{-1} (x_i^2 + 1 + x_i^{-2})^{-1} Z_{UU}(n; x_1, \ldots, x_n; y_1, \ldots, y_n) \\ = 3^{-2n^2+n} \left(Sp_{4n}(n-1, n-1, n-2, n-2, \ldots, 0, 0; x_1^2, \ldots, x_n^2, y_1^2, \ldots, y_n^2)\right)^2, \quad (3.35)$$

Using equations (3.9), (3.32), (3.33), (3.34) and (3.35), after some simplification we shall arrive at

$$(z^3+1) \left(\sum_{i=2}^{2n} A_O(2n, i) z^{-i} \right)^2 = \sum_{i=1}^{2n} A_{VHP}^{R}(4n+2, i+1) \left(z^{i-4n-1} + z^{-i}\right), \quad (3.36)$$

and

$$(1-z^2) \left(\sum_{i=2}^{2n} A_O(2n, i) z^{-i} \right)^2 = \sum_{i=1}^{2n} B_{VHP}(4n+2, i) \left(z^{i-4n-1} - z^{-i+1}\right). \quad (3.37)$$

Now, by comparing coefficients in (3.36) and (3.37) we obtain the following results.

Theorem 3.6. *The number of order $4n+2$ VHPASMs with the leftmost occurrence of 1 in the second row in i-th column is*

$$\sum_{k=0}^{i-2} A_O(2n, k+2) \left(A_O(2n, i-k) + A_O(2n, i-3-k)\right),$$

where $A_O(2n, j)$ is given by (3.8) and we take $A_O(2n, -1) = 0$.

Theorem 3.7. *The number of order $4n+2$ VHPASMs with the topmost occurrence of 1 in the second column in the i-th row is*

$$\sum_{k=0}^{i-2} A_O(2n, k+2) \left(A_O(2n, i-k) + A_O(2n, i-1-k)\right),$$

where $A_O(2n, j)$ is given by (3.8).

Remark 4. *From above it follows that*

$$A^C_{\mathrm{VHP}}(4n+2,i) = A^R_{\mathrm{VHP}}(4n+2,i) + A^C_{\mathrm{VHP}}(4n+2,i-1) - A^C_{\mathrm{VHP}}(4n+2,i-2). \quad (3.38)$$

4 ASMS WITH OFF-DIAGONAL SYMMETRY

This chapter deals with ASMs with off-diagonal symmetry; in particular we look at ASMs with off-diagonal and off-antidiagonal symmetry (OOSASMs) and ASMs with off-diagonal and vertical symmetry (VOSASMs).

4.1 OFF-DIAGONALLY AND OFF-ANTIDIAGONALLY SYMMETRIC ASMS

An ASM which is diagonally and anti-diagonally symmetric with each entry in the diagonal and antidiagonal equal to 0 are called off-diagonally and off-antidiagonally symmetric ASMs (OOSASMs). These matrices occur for order $4n$ and no product formula for their enumeration is currently known or conjectured. However, Ayyer, Behrend and Fischer [3] introduced the concept of an odd order OOSASM while studying extreme behaviour of odd order diagonally and anti-diagonally symmetric ASMs (DASASMs). We explain this briefly below.

Consider the DASASM of order 9 in Subsection 3.2.2, which is given below again

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We notice that this DASASM is determined completely by the entries in red. This portion of the matrix is called a *fundamental triangle*. In general a DASASM of order $2n+1$ with entries $a_{i,j}$ $(1 \leqslant i,j \leqslant 2n+1)$ is determined by the fundamental triangle $\{(i,j) | 1 \leqslant i \leqslant n+1, i \leqslant j \leqslant 2n+2-i\}$. Any DASASM of order $2n+1$ with $2n$ entries equal to 0 along the portions of the diagonals that lie in this fundamental triangle is called an OOSASM of order $2n+1$. The central entry of such a matrix is always $(-1)^n$. The matrix above is an example of an odd order OOSASM. Ayyer, Behrend and Fischer [3] proved that the number of order $4n-1$ OOSASMs is the same as the number of order $4n+1$ VHSASMs and that the number of order $4n+1$ OOSASMs is the same as the number of order $4n+3$ VHSASMs. The aim of this section is to give generating functions for the refined enumeration of OOSASMs with respect to the position of the unique 1 in the first row of such matrices.

The correspondence between ASMs and the six-vertex model extends to this case as well via the grid in Figure 4.1. The correspondence between the degree 4 vertices

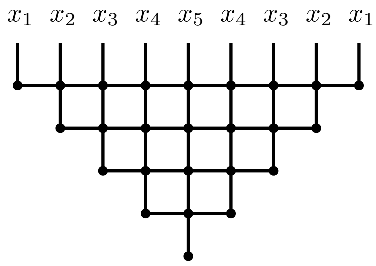


Figure 4.1: The grid corresponding to DASASMs.

of the grid and the entries in the fundamental triangle is same as in Figure 2.2. The vertices of degree 1, namely and corresponds to entries 1 and -1 respectively. Both of them carry a weight of 1. The remaining vertices of degree 2 are also fixed in our case as the boundary entries are always a 0. So we take their weights to be 1 as well, as they do not make any difference in enumeration results. We also assume $n \geqslant 1$ unless otherwise mentioned. For $n = 0$, there is no OOSASM of order $2n + 1 = 1$.

Ayyer, Behrend and Fischer proved the following theorem for the partition function of OOSASMs $Z_{OO}(n; x_1, x_2, \dots, x_{n+1})$ [3, Theorem 7.1] (up to some normalization factor).

Theorem 4.1. *The OOSASM partition function of order* n *is given by*

$$Z_{OO}(n; x_1, x_2, \dots, x_{n+1}) = Z_O\left(\left\lceil \frac{n}{2} \right\rceil; x_1, x_2, \dots, x_{2\lceil \frac{n}{2} \rceil}\right) \times Q\left(\left\lceil \frac{n+1}{2} \right\rceil; x_1, x_2, \dots, x_{2\lceil \frac{n+1}{2} \rceil - 1}\right),$$

where $Z_O(m; x_1, x_2, \dots, x_{2m})$ *is given by equation* (3.5) *and*

$$Q(m; x_1, x_2, \dots, x_{2m-1}) = \sigma(q^2)^{-(m-1)(2m-1)} \prod_{1 \leqslant i < j \leqslant 2m} \frac{\sigma(q x_i x_j)\sigma(q \bar{x}_i \bar{x}_j)}{\sigma(x_i \bar{x}_j)} \times \underset{1 \leqslant i < j \leqslant 2m}{\mathrm{Pf}} \left(\begin{cases} \frac{\sigma(x_i \bar{x}_j)}{\sigma(q x_i x_j)} + \frac{\sigma(x_i \bar{x}_j)}{\sigma(q \bar{x}_i \bar{x}_j)}, & j < 2m \\ 1, & j = 2m \end{cases} \right).$$

In the following, we will specialize

$$(x_1, x_2, \dots, x_{n+1}) = (x, 1, 1, \dots, 1) \quad \text{as well as} \quad q + \bar{q} = 1$$

in the partition function. We will now explore how this specialization can be expressed in terms of $A_{OO}(2n + 1, i)$, the number of OOSASMs of order $2n + 1$ where the unique 1 in the first row is at the i-th column.

We notice that there is a unique occurrence of a in the first row, say at position i. This forces the other degree 4 vertices to its left to be of type , and to its right to be of type . The left boundary vertex is forced to be and the right boundary vertex is forced to be . This gives us

$$Z_{OO}(n; x, \underbrace{1, \dots, 1}_{n}) = \sum_{i=1}^{2n+1} A_{OO}(2n+1, i) \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{i-2} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-i}. \tag{4.1}$$

We now perform the change of variable

$$z = \frac{\sigma(q\bar{x})}{\sigma(qx)}$$

and use $q + \bar{q} = 1$ to eliminate x from equation (4.1) to get

$$Z_{OO}(n; x, \underbrace{1, \dots, 1}_{n}) = \frac{(-1)^{n-1} z^{2n}}{(z - 1 - z^2)^{n-1}} \sum_{i=1}^{2n+1} A_{OO}(2n+1, i) z^{-i}. \tag{4.2}$$

Ayyer, Behrend and Fischer [3, Theorem 7.2] also showed that

$$\begin{aligned} &Z_{OO}(2n-1; x_1, x_2, \dots, x_{2n}) \\ &= 3^{-(n-1)(2n-1)} Sp_{4n}(n-1, n-1, n-2, n-2, \dots, 0, 0; x_1^2, x_2^2, \dots, x_{2n}^2) \\ &\quad \times Sp_{4n-2}(n-1, n-2, n-2, n-3, n-3, \dots, 1, 1; x_1^2, x_2^2, \dots, x_{2n-1}^2) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &Z_{OO}(2n; x_1, x_2, \dots, x_{2n+1}) \\ &= 3^{-n(2n-1)} Sp_{4n}(n-1, n-1, n-2, n-2, \dots, 0, 0; x_1^2, x_2^2, \dots, x_{2n}^2) \\ &\quad \times Sp_{4n+2}(n, n-1, n-1, n-2, n-2, \dots, 0, 0; x_1^2, x_2^2, \dots, x_{2n+1}^2). \end{aligned} \tag{4.4}$$

By comparing equations (3.9), (3.18), (4.2), (4.3) and (4.4) we get the following pairs of equations

$$\begin{aligned} &(-1)^{n-1} 3^{-n^2+2n-1} (z - 1 - z^2)^{n-1} \left(\sum_{i=2}^{2n} A_O(2n, i) z^{-i} \right) \\ &\quad \times \left(\sum_{1 \leqslant i \leqslant j \leqslant n} Q_{n-1,i} x^{2i-4j+2} \right) = \sum_{i=1}^{4n-1} A_{OO}(4n-1, i) z^{2n-2-i}, \end{aligned} \tag{4.5}$$

where $n > 1$; and

$$\begin{aligned} &(-1)^n 3^{-n^2} (z - 1 - z^2)^n \left(\sum_{i=2}^{2n} A_O(2n, i) z^{-i} \right) \\ &\quad \times \left(\sum_{1 \leqslant i \leqslant j \leqslant n+1} Q_{n,i} x^{2i-4j+2} \right) = \sum_{i=1}^{4n+1} A_{OO}(4n+1, i) z^{2n-i}. \end{aligned} \tag{4.6}$$

From equation (4.5) we get the following result.

Theorem 4.2. *Let* $A_{OO}(4n-1,i)$ *denote the number of diagonally and off-diagonally symmetric ASMs of order* $4n-1$ *with the unique* 1 *of the first row in the* i*-th column. Then for all* $n>1$ *the following is satisfied*

$$3^{-n^2+2n-1}\left(\sum_{i=2}^{2n} A_O(2n,i)z^{-i}\right)$$
$$\times\left(\sum_{1\leqslant i\leqslant j\leqslant n} Q_{n-1,i}(zq-1)^{n+i-2j}(q-z)^{n-i+2j-2}(-q)^{-n+1}\right)$$
$$=\sum_{i=1}^{4n-1} A_{OO}(4n-1,i)z^{2n-2-i},$$

where every quantity appearing in the left-hand side is explicitly known.

Further, from equation (4.6), we get the following result.

Theorem 4.3. *Let* $A_{OO}(4n+1,i)$ *denote the number of diagonally and off-diagonally symmetric ASMs of order* $4n+1$ *with the unique* 1 *of the first row in the* i*-th column. Then, for all* $n\geqslant 1$ *the following is satisfied*

$$3^{-n^2}\left(\sum_{i=2}^{2n} A_O(2n,i)z^{-i}\right)$$
$$\times\left(\sum_{1\leqslant i\leqslant j\leqslant n+1} Q_{n,i}(zq-1)^{n+i-2j+1}(q-z)^{n-i+2j-1}(-q)^{-n}\right)$$
$$=\sum_{i=1}^{4n+1} A_{OO}(4n+1,i)z^{2n-i},$$

where every quantity appearing in the left-hand side is explicitly known.

Remark 5. *From Theorems 3.5 and 4.2 we get the following*

$$A_{VH}(4n+1,i)=A_{OO}(4n-1,i)+A_{OO}(4n-1,i-1), \tag{4.7}$$

and, from Theorems 3.4 and 4.3 we get the following

$$A_{VH}(4n+3,i)=A_{OO}(4n+1,i)+A_{OO}(4n+1,i-1). \tag{4.8}$$

In the above we assume $A_{OO}(2n+1,-1)=0$*. These are similar to the relationship between refined enumeration of VSASMs and OSASMs (cf. equation* (3.11)*).*

Remark 6. *Analogous to equations* (3.21) *and* (3.27) *we shall get the following pairs of equations*

$$(-1)^{n^2}3^{-2n^2-n}(z-1-z^2)^{2n}s(2n,2n-1,2n-1,\dots,1,1,0,0;x^2,\bar{x}^2,\dots,1)$$
$$=\sum_{i=1}^{4n-1} A_{OO}(4n-1,i)z^{4n-i}$$

and

$$-3^{-2n^2-3n-1}(z-1-z^2)^{2n+1}s(2n+1,2n,2n,\dots,1,1,0,0;x^2,\bar{x}^2,1,\dots,1)$$
$$=\sum_{i=1}^{4n+1}A_{OO}(4n+1,i)z^{4n+2-i}.$$

4.2 VERTICALLY AND OFF-DIAGONALLY SYMMETRIC ASMS

An ASM which is vertically symmetric as well as off-diagonally symmetric with a null diagonal except for the central entry is called a vertically and off-diagonally symmetric ASM (VOSASM). These matrices occur for odd orders $8n+1$ and $8n+3$. This class was first considered by Okada [31], who proved enumeration formulas for them. VOSASMs are also OOSASMs of odd order as described in the previous section. The off-antidiagonal symmetry follows from the vertical symmetry, which in turn makes them vertically and horizontally symmetric. In fact VOSASMs are special cases of totally symmetric ASMs (TSASMs). No product formula for TSASMs is currently known or conjectured.

An example of such a VOSASM is the matrix from the previous section

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly the array of numbers in red are sufficient to construct the whole matrix in our example. Since VOSASMs have vertical symmetry so we can ask for their refined enumeration with respect to the position of the first occurrence of a 1 in the second row. Let these numbers be denoted by $A_{VOS}(n,i)$ for order n VOSASMs. The aim of this section is to give generating functions for these numbers.

The correspondence between VOSASMs and the six-vertex model is via the grid shown in Figure 4.2, which is now a combination of U-turns and the triangular grid from last section. This grid was first considered by Kuperberg [28]. The weights of the degree 4 vertices remain the same as in Figure 2.2, the U-turns have the weights described in Figure 3.3 and the degree 2 vertices will have weight 1 as they do not make any difference in our results.

We shall use the following formula for the partition function that was derived by Kuperberg [28, Theorem 10] (up to some normalization factor).

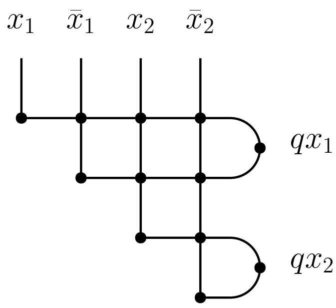


Figure 4.2: The grid corresponding to VOSASMs.

Theorem 4.4.

$$Z_{UO}(n; x_1, x_2, \dots, x_{2n}) = \frac{\sigma(q^2)^{3n-8n^2}\sigma(q)^{2n}\prod_{i\leqslant 2n}\sigma(q^2x_i^2)}{\prod_{i<j\leqslant 2n}\sigma(\bar{x}_i x_j)^2\prod_{i\leqslant j\leqslant 2n}\sigma(x_i x_j)^2}$$
$$\times \prod_{i<j\leqslant 2n}\sigma(q\bar{x}_i x_j)^2\sigma(qx_i\bar{x}_j)^2\sigma(qx_ix_j)^2\sigma(q\bar{x}_i\bar{x}_j)^2$$
$$\times \operatorname{Pf}_{1\leqslant i<j\leqslant 2n}\left(\sigma(\bar{x}_i x_j)\sigma(x_ix_j)\left(\frac{1}{\sigma(qx_ix_j)\sigma(q\bar{x}_i\bar{x}_j)} - \frac{1}{\sigma(q\bar{x}_ix_j)\sigma(qx_i\bar{x}_j)}\right)\right)$$
$$\times \operatorname{Pf}_{1\leqslant i,j\leqslant 2n}\left(\sigma(\bar{x}_i x_j)\sigma(x_ix_j)\left(\frac{\sigma(bx_i)\sigma(bx_j)}{\sigma(qx_ix_j)} - \frac{\sigma(bx_i)\sigma(b\bar{x}_j)}{\sigma(qx_i\bar{x}_j)} - \frac{\sigma(b\bar{x}_i)\sigma(bx_j)}{\sigma(q\bar{x}_ix_j)} + \frac{\sigma(b\bar{x}_i)\sigma(b\bar{x}_j)}{\sigma(q\bar{x}_i\bar{x}_j)}\right)\right)$$

We shall again specialize

$$(x_1, x_2, \dots, x_{2n}) = (x, 1, \dots, 1) \quad \text{as well as} \quad q + \bar{q} = 1,$$

in the following. But first, we notice that due to the imposed symmetries we have

$$A_{VOS}(n, 1) = A_{VOS}(n, 2) = 0 \quad \text{for all } n.$$

There will be two cases depending on whether the VOSASM is of order $8n+1$ or $8n+3$. For order $8n+1$ we further specialize

$$q + \bar{q} = 1 \quad \text{as well as} \quad b = q, \tag{4.9}$$

and also assume $n > 1$ for this case. When $n = 1$, the only such VOSASM of order 9 is the one above, and for $n = 0$ no VOSASM exist. For order $8n+3$ we specialize

$$q + \bar{q} = 1 \quad \text{as well as} \quad b = \bar{q}, \tag{4.10}$$

and assume $n \geqslant 1$. When $n = 0$ in this case, then the only such VOSASM is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The calculations for these cases are similar to the cases for VSASMs and order $4n+3$ VHSASMs. Analogous to equations (3.2) and (3.14) we shall get the following pairs of equations.

$$
\begin{aligned}
&Z_{UO}(n;x,\underbrace{1,\dots,1}_{2n-1}) \\
&= \sum_{i=3}^{4n} A_{VOS}(8n+1,i)\left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{i}\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{8n-2}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-3}\sigma(q^2x)\sigma(q^2)^{2n-1} \\
&+ \sum_{i=3}^{4n} A_{VOS}(8n+1,i)\left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^{i}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{8n-3}\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{-2}\sigma(\bar{x})\sigma(q^2)^{2n-1} \\
&+ \sum_{j=3}^{4n} A_{VOS}(8n+1,j)\left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{j+2}\sum_{i=j+1}^{4n}\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{8n-2i-6} \\
&\qquad\times \sigma(\bar{x})\sigma(q^2)^{2n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
&Z_{UO}(n;x,\underbrace{1,\dots,1}_{2n-1}) \\
&= \sum_{i=2}^{4n} B_{VOS}(8n+3,i)\left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{i}\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{8n-2}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-3}\sigma(x)\sigma(\bar{q}^2)^{2n-1} \\
&+ \sum_{i=2}^{4n} B_{VOS}(8n+3,i)\left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^{i}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{8n-3}\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{-2}\sigma(\bar{q}^2\bar{x})\sigma(\bar{q}^2)^{2n-1} \\
&+ \sum_{j=2}^{4n} B_{VOS}(8n+3,j)\left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{j+2}\sum_{i=j+1}^{4n}\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{8n-2i-6} \\
&\qquad\times \sigma(\bar{q}^2\bar{x})\sigma(\bar{q}^2)^{2n-1},
\end{aligned}
$$

where

$$B_{VOS}(8n+3,i) = A_{VOS}(8n+3,i+1) - A_{VOS}(8n+3,i).$$

We substitute $z = \dfrac{\sigma(q\bar{x})}{\sigma(qx)}$ and eliminate x from the above equations to get the following pairs of equations

$$
\begin{aligned}
&-\sigma(q^2)^{6n-4}\sigma(q\bar{x})^{-8n+4}\frac{1+z}{1-2z}Z_{UO}(n;x,\underbrace{1,\dots,1}_{2n-1}) \\
&\qquad\qquad = \sum_{i=3}^{4n} A_{VOS}(8n+1,i)(z^{i-8n+1}+z^{-i+2}), \quad (4.11)
\end{aligned}
$$

provided (4.9) holds; and

$$-\sigma(q^2)^{6n-4}\sigma(q\bar{x})^{-8n+4}\frac{1-z^2}{1-2z}Z_{UO}(n;x,\underbrace{1,\ldots,1}_{2n-1}) = \sum_{i=2}^{4n} B_{VOS}(8n+3,i)(z^{i-8n+1}-z^{-i+3}), \quad (4.12)$$

provided (4.10) holds.

Okada [31, Theorem 2.5] proved that

$$\prod_{i=1}^{2n}(x_i+\bar{x}_i)\prod_{i=1}^{2n}\sigma(q^2x_i^2)^{-1}Z_{UO}(n;x_1,x_2,\ldots,x_{2n})$$
$$=3^{-4n^2+3n}(Sp_{4n}(n-1,n-1,n-2,n-2,\ldots,0,0;x_1^2,\ldots,x_{2n}^2))^3$$
$$\times O_{4n}\left(n+\frac{1}{2},n-\frac{1}{2},n-\frac{1}{2},n-\frac{3}{2},n-\frac{3}{2},\ldots,\frac{3}{2},\frac{3}{2},\frac{1}{2};x_1^2,\ldots,x_{2n}^2\right) \quad (4.13)$$

provided (4.9) holds; and

$$\prod_{i=1}^{2n}\sigma(q^2x_i^2)^{-1}Z_{UO}(n;x_1,x_2,\ldots,x_{2n})$$
$$=3^{-4n^2+3n}(Sp_{4n}(n-1,n-1,n-2,n-2,\ldots,0,0;x_1^2,\ldots,x_{2n}^2))^3$$
$$\times Sp_{4n+2}(n,n-1,n-1,n-2,n-2,\ldots,0,0;x_1^2,\ldots,x_{2n}^2,1) \quad (4.14)$$

provided (4.10) holds. Transforming the even orthogonal group character in equation (4.13) for our special case, into a symplectic group character using a result of Ayyer and Behrend [2, Proposition 5, and then use equation (7)], we get the following for our special values

$$Z_{UO}(n;x,\underbrace{1,\ldots,1}_{2n-1}) = 3^{-4n^2+5n-2}\sigma(q^2)^{2n-1}\sigma(q^2x^2)(x^2+1+\bar{x}^2)$$
$$\times(Sp_{4n}(n-1,n-1,n-2,n-2,\ldots,0,0;x^2,1,\ldots,1))^3$$
$$\times Sp_{4n-2}(n-1,n-2,n-2,\ldots,0,0;x^2,1,\ldots,1), \quad (4.15)$$

provided (4.9) holds.

From equations (3.9), (3.18), (4.11) and (4.15), after some simplifications we get

$$(-1)^{n-1}3^{-n^2+2n-1}(1+z)(z-1-z^2)^{n-1}\left(\sum_{i=2}^{2n}A_O(2n,i)z^{-i}\right)^3$$
$$\times\left(\sum_{1\leqslant j\leqslant i\leqslant n}Q_{n-1,i}x^{2i-4j+2}\right)=\sum_{i=3}^{4n}A_{VOS}(8n+1,i)(z^{i-6n-4}+z^{2n-i-3}), \quad (4.16)$$

provided (4.9) holds and where $Q_{n-1,i}$ is given by (3.19). On the other hand, from equations (3.9), (3.18), (4.12) and (4.14) we get

$$(-1)^n 3^{-n^2}(1-z^2)z^{-2n+4}(z-1-z^2)^n \left(\sum_{i=2}^{2n} A_O(2n,i)z^{-i}\right)^3 \times \left(\sum_{1\leqslant i\leqslant i\leqslant n+1} Q_{n,i}x^{2i-4j+2}\right) = \sum_{i=2}^{4n} B_{VOS}(8n+3,i)(z^{i-8n+1}-z^{-i+3}), \tag{4.17}$$

provided (4.10) holds and where $Q_{n,i}$ is given by (3.19).

After some simplifications the above pairs of equations give the following theorems.

Theorem 4.5. *Let $A_{VOS}(8n+1,i)$ denote the number of order $8n+1$ VOSASMs with the first 1 in the second row in the i-th column. Then, for all $n>1$ the following is satisfied*

$$3^{-n^2+2n-1}(1+z)\left(\sum_{i=2}^{2n} A_O(2n,i)z^{-i}\right)^3 \times \left(\sum_{1\leqslant j\leqslant i\leqslant n} Q_{n-1,i}(zq-1)^{n+i-2j}(q-z)^{n-i+2j-2}(-q)^{-n+1}\right) = \sum_{i=3}^{4n} A_{VOS}(8n+1,i)(z^{i-6n-4}+z^{2n-i-3}),$$

where every quantities appearing on the left hand side is explicitly known.

Theorem 4.6. *Let $A_{VOS}(8n+3,i)$ denote the number of order $8n+3$ VOSASMs with the first 1 in the second row in the i-th column. Then, for all $n\geqslant 1$ the following is satisfied*

$$3^{-n^2}(1-z^2)\left(\sum_{i=2}^{2n} A_O(2n,i)z^{-i}\right)^3 \times \left(\sum_{1\leqslant i\leqslant i\leqslant n+1} Q_{n,i}(zq-1)^{n+i-2j+1}(q-z)^{n-i+2j-1}(-q)^{-n}\right) = \sum_{i=2}^{4n}(A_{VOS}(8n+3,i+1)-A_{VOS}(8n+3,i))(z^{i-6n-3}-z^{2n-i-1}),$$

where every quantities appearing on the left hand side is explicitly known, and $A_{VOS}(8n+3,1)=A_{VOS}(8n+3,2)=0$ for all n.

Remark 7. *From Theorems 3.5 and 4.5 we get*

$$\left(\sum_{i=2}^{2n} A_O(2n,i)z^{-i}\right)^2 \left(\sum_{i=2}^{2n} A_{VH}(4n+1,i)(z^{i-2n-2}+z^{2n-1-i})\right) = \sum_{i=3}^{4n} A_{VOS}(8n+1,i)(z^{i-6n-4}+z^{2n-i-3}) \tag{4.18}$$

and from Theorems 3.4 and 4.6 we get

$$\left(\sum_{i=2}^{2n} A_O(2n,i) z^{-i}\right)^2 \times \left(\sum_{i=1}^{2n+1} \left(A_{VH}(4n+3,i+1) - A_{VH}(4n+3,i)\right) \left(z^{i-2n-1} - z^{-i+2n+1}\right)\right) = \sum_{i=2}^{4n} (A_{VOS}(8n+3,i+1) - A_{VOS}(8n+3,i))(z^{i-6n-3} - z^{2n-i-1}). \quad (4.19)$$

5 ASMS WITH QUARTER-TURN SYMMETRY

This chapter deals with ASMs with quarter-turn symmetry; that is ASMs invariant under a 90° rotation. In particular we look at even and odd order quarter-turn symmetric ASMs (QTSASMs) and a related class of matrices called quasi quarter-turn symmetric ASMs (qQTSASMs).

5.1 quarter-turn symmetric asms

As a first observation about QTSASMs, we see that these ASMs cannot occur for order $4n+2$ [1, Lemma 4], consider the QTSASM of order $2n$ where the entries are given by $a_{i,j}$ $(1 \leqslant i, j \leqslant 2n)$. Then we have

$$2n = \sum_{1 \leqslant i,j \leqslant 2n} a_{i,j} = 4 \sum_{1 \leqslant i,j \leqslant n} a_{i,j},$$

and this implies that $2|n$. So for the even case they occur only for order $4n$. Order $4n$ QTSASMs were enumerated by Kuperbeg [28], while Razumov and Stroganov [36] enumerated them for odd order. In this section, we will give refined enumeration formulas for this class of ASMs with respect to the position of the unique 1 in the first row. These results were conjectured by Robbins [38]. We deal with the even case in Subsection 5.1.1 and with the odd case in Subsection 5.1.2.

Since the proofs are similar for all cases (in this section and also in Section 5.2), we will only derive the results in full for the even QTSASM case and indicate the steps for the other cases. Throughout this section, we will consider the case $\vec{x} = (x, 1, 1, \ldots, 1)$ and $q + \bar{q} = 1$, unless otherwise mentioned.

5.1.1 *Even Order QTSASMs*

Kuperberg [28] enumerated this class of QTSASMs using the six vertex model; since a quarter of such an ASM is enough to determine the whole ASM, we can see that the six vertex model corresponding to even order QTSASMs will be the one shown in Figure 5.1, where there are $2n$ many spectral parameters associated with it if the QTSASM is of order $4n$, which are denoted by x_i's. The vertex weights and bijection with the normal ASMs which was shown in Section 2 carry forward to this class as well, with one difference: the arrows of the configuration will change sign when they move through the circular turns. For instance, consider the simplest possible QTSASM of even order

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

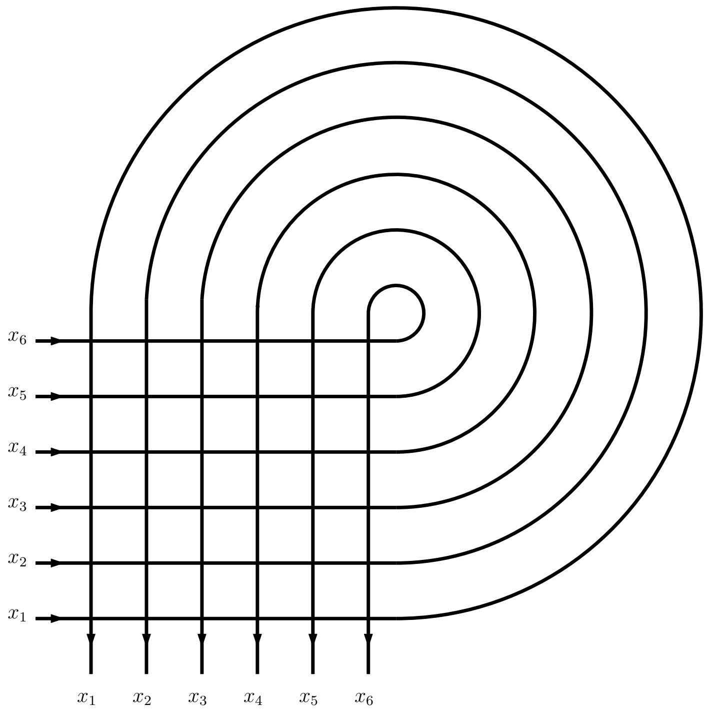


Figure 5.1: Six vertex configuration of QTSASM of order $4n$.

The six-vertex configuration of this matrix is now given by Figure 5.2.

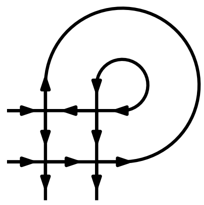


Figure 5.2: Six vertex configuration of an even order QTSASM.

Let $A_{QT}(n, i)$ be the total number of QTSASMs of order n such that the unique 1 in the last row is at the i-th position. By symmetry of the ASMs, this is also the number of QTSASMs of order n such that the unique 1 in the first row is at the i-th position. In the sequel we will use this description instead of the last row, without further commentary. The weight of the last row of such a QTSASM of order $4n$ would be

$$\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{i-2}\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{4n-i-1}.$$

(The first entry is always a 0 for QTSASMs, and the weight of that entry is $\frac{\sigma(qx\bar{x})}{\sigma(q^2)} = 1$ when $q + \bar{q} = 1$.) If $Z_Q(4n; x_1, x_2, \ldots, x_{2n})$ denotes the partition function of QTSASMs of order $4n$, then we have

$$\sum_{i=2}^{4n-1} A_{QT}(4n, i) z^{i-2} = \frac{Z_Q(4n; x, 1, 1, \ldots, 1)}{\sigma(q^2)^{3-4n}\sigma(qx)^{4n-3}}, \tag{5.1}$$

where $z = \dfrac{\sigma(q\bar{x})}{\sigma(qx)}$ and from Kuperberg [28, Theorem 10] we have (up to some normalization factor)

$$Z_Q(4n; x_1, x_2, \ldots, x_{2n}) = \sigma(q^2)^{4n-4n^2} \frac{\prod_{i,j=1}^{2n} \alpha(\bar{x}_i x_j)^2}{\prod_{i,j=1}^{2n} \sigma(\bar{x}_i x_j)^2} \times \mathrm{Pf}_{1\leqslant i<j\leqslant 2n} \left(\frac{\sigma(\bar{x}_i x_j)}{\alpha(\bar{x}_i x_j)} \right) \mathrm{Pf}_{1\leqslant i<j\leqslant 2n} \left(\frac{\sigma(\bar{x}_i^2 x_j^2)}{\alpha(\bar{x}_i x_j)} \right).$$

Further we have the following analogous equations from Razumov and Stroganov [35, Equation (38)],

$$\sum_{i=1}^{n} A(n,i) z^{i-1} = \frac{Z(n; x, 1, 1, \ldots, 1)}{\sigma(q^2)^{1-n} \sigma(qx)^{n-1}} \tag{5.2}$$

and

$$\sum_{i=1}^{2n} A_{HT}(2n,i) z^{i-1} = \frac{Z_H(2n; x, 1, 1, \ldots, 1)}{\sigma(q^2)^{1-2n} \sigma(qx)^{2n-1}} \tag{5.3}$$

where $A(n,i)$ denotes the number of ASMs of order n with the unique 1 in the first row in the i-th column, given by Zeilberger [44]

$$A(n,i) = \binom{n+i-2}{n-1} \frac{(2n-i-1)!}{(n-i)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}$$

and $A_{HT}(n,i)$ denotes the number of order n half-turn symmetric ASMs (HTSASMs; ASMs which are invariant under a 180° rotation are called half-turn symmetric ASMs) with the unique 1 in the first row in the i-th column, given by Stroganov [41]

$$A_{HT}(2n,i) = \frac{(2n-1)!^2}{(n-1)!^2 (3n-3)! (3n-1)!} \prod_{j=0}^{n-1} \frac{(3j+2)(3j+1)!^2}{(3j+1)(n+j)!^2} \times \sum_{j=1}^{i} \left(\frac{(n^2 - nj + (j-1)^2 (n+j-3)!)(2n-j-1)!(n+i-j-1)!(2n-i+j-2)!}{(j-1)!(n-j+1)!(i-j)!(n-i+j-1)!} \right).$$

Also

$$Z(n; x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$

is the partition function of order n ASMs and

$$Z_H(2n; x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$

is the partition function of order $2n$ HTSASMs, which are given by Kuperberg [28, Theorem 10] as follows (up to some normalization factor)

$$Z(n; x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{\sigma(q^2)^{n-n^2} \prod_{i,j} \alpha(x_i \bar{x}_j)}{\prod_{i<j} \sigma(\bar{x}_i x_j) \sigma(y_i \bar{y}_j)} \det_{1\leqslant i,j\leqslant n} \left(\frac{1}{\alpha(x_i \bar{x}_j)} \right),$$

and

$$Z_H(2n; x_1, \dots, x_n, y_1, \dots, y_n) = \frac{\sigma(q^2)^{n-2n^2} \prod_{i,j} \alpha(x_i\bar{x}_j)^2}{\prod_{i<j} \sigma(\bar{x}_i x_j)^2 \sigma(y_i \bar{y}_j)^2} \times \det_{1 \leqslant i,j \leqslant n} \left(\frac{1}{\alpha(x_i \bar{x}_j)} \right) \det_{1 \leqslant i,j \leqslant n} \left(\frac{1}{\sigma(q\bar{x}_i y_j)} + \frac{1}{\sigma(q x_i \bar{y}_j)} \right).$$

By results of Okada [31, Theorems 2.4 and 2.5], when $q + \bar{q} = 1$, we have

$$Z_Q(4n; x, 1, \dots, 1) = (Z(n; x, 1, \dots, 1))^2 Z_H(2n; x, 1, \dots, 1). \tag{5.4}$$

Combining equations (5.1) to (5.4), we get the following result.

Theorem 5.1.

$$\sum_{i=2}^{4n-1} A_{QT}(4n, i) z^{i-2} = \left(\sum_{i=1}^{n} A(n, i) z^{i-1} \right)^2 \left(\sum_{i=1}^{2n} A_{HT}(2n, i) z^{i-1} \right).$$

Theorem 5.1 allows us to give a formula for the refined enumeration of even order QTSASMs by comparing the coefficients of z from both sides of the equation.

5.1.2 *Odd Order QTSASMs*

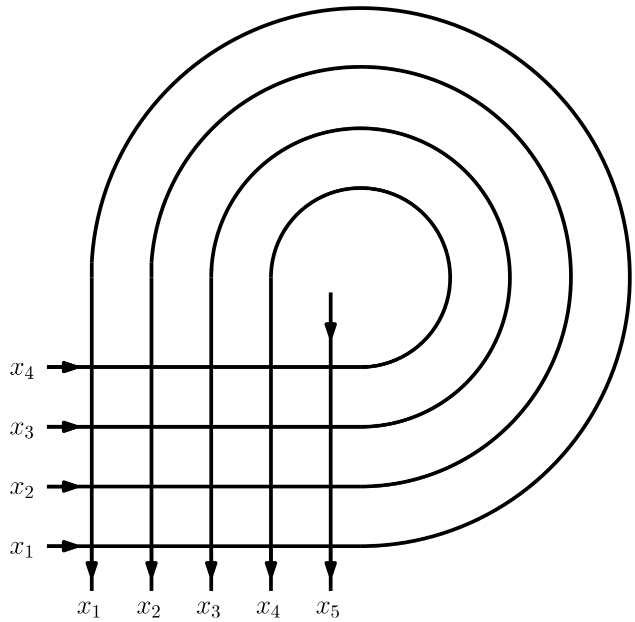


Figure 5.3: Six vertex configuration of QTSASM of order $4n + 1$.

For odd order QTSASMs (say of order $2m + 1$), we can notice that the central entry of the matrix will be $(-1)^m$. Similar to the grid described in Subsection 5.1.1, we will have the configuration in Figure 5.3 if $m = 2n$ and in Figure 5.4 if $m = 2n + 1$. These were used by Razumov and Stroganov to enumerate odd order QTSASMs

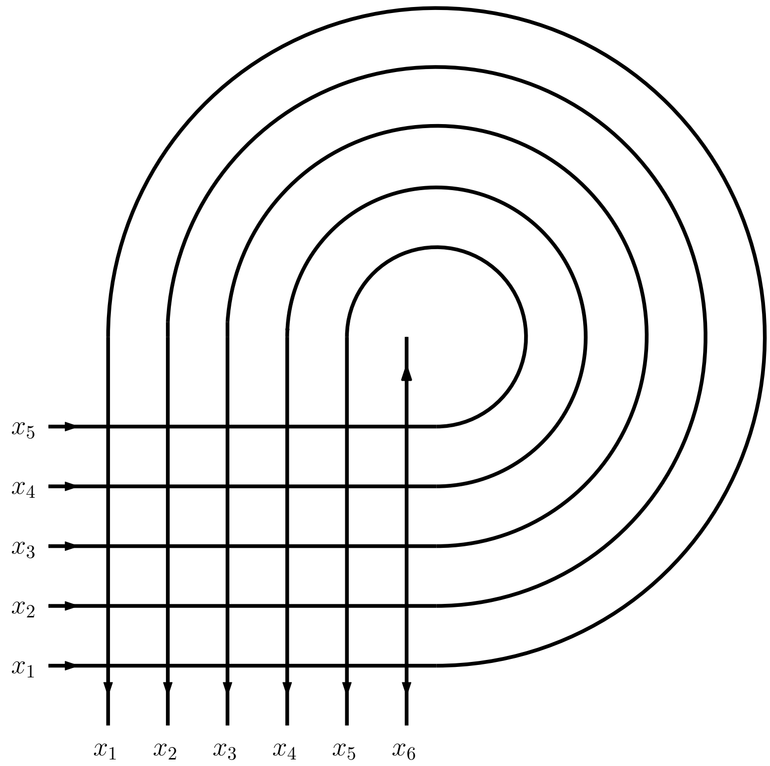


Figure 5.4: Six vertex configuration of QTSASM of order $4n+3$.

[36], and they found a formula similar to the one found by Kuperberg for even order QTSASMs. We also notice that for an order $2m+1$ QTSASM, the grid has $m+1$ spectral parameters.

We proceed in a similar way, as in the case for even QTSASMs. Analogous to equation (5.1), we will have the following equations.

$$\sum_{i=2}^{4n} A_{QT}(4n+1, i)z^{i-2} = \frac{Z_Q(4n+1; x, 1, 1, \dots, 1)}{\sigma(q^2)^{2-4n}\sigma(qx)^{4n-2}}, \tag{5.5}$$

and

$$\sum_{i=2}^{4n+2} A_{QT}(4n+3, i)z^{i-2} = \frac{Z_Q(4n+3; x, 1, 1, \dots, 1)}{\sigma(q^2)^{-4n}\sigma(qx)^{4n}}, \tag{5.6}$$

where $z = \dfrac{\sigma(q\bar{x})}{\sigma(qx)}$ and the partition functions are given by Razumov and Stroganov [36, Equations (10) and (11)] (up to some normalization factor). Further we have the following analogous equation from Razumov and Stroganov [35],

$$\sum_{i=1}^{2n+1} A_{HT}(2n+1, i; w)t^{i-1} = \frac{Z_H(2n+1; x, 1, 1, \dots, 1)}{\sigma(q^2)^{-2n}\sigma(qx)^{2n}}, \tag{5.7}$$

where $Z_H(2n+1; x_1, x_2, \dots, x_{n+1}, y_1, y_2, \dots, y_{n+1})$ is the partition function of odd order HTSASMs, which was found by Razumov and Stroganov [35, Theorem 1] (up to some normalization factor).

Again, analogous to equation (5.4), we have the following equations from Razumov and Stroganov [36], when $q + \bar{q} = 1$

$$Z_Q(4n+1; x, 1, \ldots, 1) = (Z(n; x, 1, \ldots, 1))^2 Z_H(2n+1; x, 1, \ldots, 1), \tag{5.8}$$

and

$$Z_Q(4n+3; x, 1, \ldots, 1) = (Z(n+1; x, 1, \ldots, 1))^2 Z_H(2n+1; x, 1, \ldots, 1). \tag{5.9}$$

Now, combining equations (5.2) and (5.5) to (5.9), we get the following result.

Theorem 5.2.

$$\sum_{i=2}^{4n} A_{QT}(4n+1, i) z^{i-2} = \left(\sum_{i=1}^{n} A(n, i) z^{i-1} \right)^2 \left(\sum_{i=1}^{2n+1} A_{HT}(2n+1, i) z^{i-1} \right),$$

and

$$\sum_{i=2}^{4n+2} A_{QT}(4n+3, i) z^{i-2} = \left(\sum_{i=1}^{n+1} A(n+1, i) z^{i-1} \right)^2 \left(\sum_{i=1}^{2n+1} A_{HT}(2n+1, i) z^{i-1} \right).$$

5.2 QUASI QUARTER-TURN SYMMETRIC ASMS

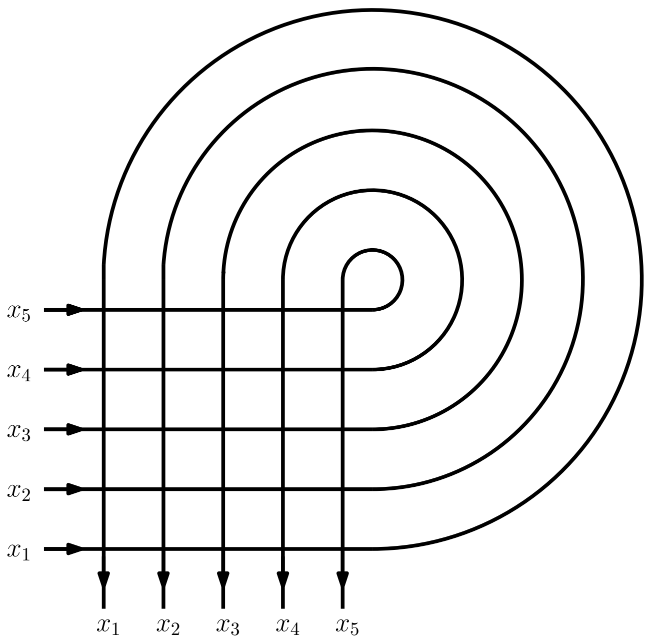


Figure 5.5: Six vertex configuration of qQTSASM of order $4n+2$.

As pointed out in Section 5.1, there are no even order QTSASMs of order $4n+2$. However, Duchon [13] introduced a new type of ASM, called quasi QTSASMs (qQTSASMs) which follows all the conditions of an ASM and has quarter-turn symmetry for all entries except the four entries in the middle, which can be either $\{1, 0, 0, 1\}$ or $\{0, -1, -1, 0\}$. Below we give examples of both these types of matrices.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Philippe Duchon [13] stated two conjectures related to the enumeration of qQTSASMs: the first of these is on the unrestricted enumeration of this class of ASMs, which was subsequently proved by Jean-Christophe Aval and Duchon [1] by studying the six vertex configuration associated with the even order qQTSASMs (shown in Figure 5.5); the second conjecture deals with the refined enumeration of qQTSASMs with respect to the position of the unique 1 in the first row, which we prove below.

Theorem 5.3. *Let* $A_{qQT}(n, i)$ *denote the number of order* n *qQTSASMs with the unique* 1 *in the first row in the* i*-th column, then we have*

$$\sum_{i=2}^{4n+1} A_{qQT}(4n+2, i) z^{i-2} = \left(\sum_{i=1}^{n} A(n, i) z^{i-1} \right) \left(\sum_{i=1}^{n+1} A(n+1, i) z^{i-1} \right) \times \left(\sum_{i=1}^{2n+1} A_{HT}(2n+1, i) z^{i-1} \right).$$

Proof. The proof is similar to the proof of Theorem 5.1, hence we will just sketch it. Analogous to equation (5.1) we have the following equation in this case, provided $q + \bar{q} = 1$

$$\sum_{i=2}^{4n+1} A_{qQT}(4n+2, i) z^{i-2} = \frac{Z_{qQ}(4n+2; x, 1, 1, \ldots, 1)}{\sigma(q^2)^{1-4n} \sigma(qx)^{4n-1}}, \tag{5.10}$$

where $Z_{qQ}(4n+2; x_1, x_2, \ldots, x_{2m+1})$ is the partition function of order $4n+2$ qQTSASMs with spectral parameters $\vec{x}$, which was found by Aval and Duchon [1] (up to some normalization factor).

We now use the following result of Aval and Duchon [1, Theorem 6], if $q + \bar{q} = 1$ then we have

$$Z_{qQ}(4n+2; x, 1, \ldots, 1) = Z(n; x, 1, \ldots, 1) Z(n+1; x, 1, \ldots, 1) Z_H(2n+1; x, 1, \ldots, 1). \tag{5.11}$$

Combining equations (5.10),(5.2),(5.3) and (5.11) we obtain the result. □

Part II

DOMINO TILINGS OF AZTEC RECTANGLES

In this part we enumerate domino tilings of an Aztec rectangle with arbitrary defects of size one on all boundary sides. This result extends previous work by different authors: Mills-Robbins-Rumsey and Elkies-Kuperberg-Larsen-Propp. We use the method of graphical condensation developed by Kuo and generalized by Ciucu, to prove our results; a common generalization of both Kuo's and Ciucu's result is also presented here. This part corresponds to work already published [39].

6 GRAPHICAL CONDENSATION AND DOMINO TILINGS

In this chapter we exploit the connection between domino tilings and perfect matchings of planar graphs (discussed in Section 1.3), to use a powerful technique to count such perfect matchings, called *Kuo condensation*. We also present a generalization of this technique. We shall use this method extensively in the next two chapters.

Let G be a weighted graph, where the weights are associated with each edge of G, and let $M(G)$ denote the sum of the weights of the perfect matchings of G, where the weight of a perfect matching is taken to be the product of the weights of its constituent edges. We are interested in graphs with edge weights all equaling 1, which corresponds to tilings of the region in our special case. The relevant results of Eric Kuo [25] that are needed in the next chapter are the following.

Theorem 6.1. *[25, Theorem 2.3] Let $G = (V_1, V_2, E)$ be a plane bipartite graph in which $|V_1| = |V_2|$. Let w, x, y and z be vertices of G that appear in cyclic order on a face of G. If $w, x \in V_1$ and $y, z \in V_2$ then*

$$M(G - \{w,z\})\, M(G - \{x,y\}) = M(G)\, M(G - \{w,x,y,z\}) + M(G - \{w,y\})\, M(G - \{x,z\}).$$

Theorem 6.2. *[25, Theorem 2.5] Let $G = (V_1, V_2, E)$ be a plane bipartite graph in which $|V_1| = |V_2| + 2$. Let the vertices w, x, y and z appear in that cyclic order on a face of G. Let $w, x, y, z \in V_1$, then*

$$M(G - \{w,y\})\, M(G - \{x,z\}) = M(G - \{w,x\})\, M(G - \{y,z\}) + M(G - \{w,z\})\, M(G - \{x,y\}).$$

Theorem 6.3. *[25, Theorem 2.1] Let $G = (V_1, V_2, E)$ be a plane bipartite graph with $|V_1| = |V_2|$ and w, x, y, z be vertices of G that appear in cyclic order on a face of G. If $w, y \in V_1$ and $x, z \in V_2$ then*

$$M(G)\, M(G - \{w,x,y,z\}) = M(G - \{w,x\})\, M(G - \{y,z\}) + M(G - \{w,z\})\, M(G - \{x,y\}).$$

In fact, Theorems 6.1, 6.2 and 6.3 follow from the following non-bipartite version of Kuo condensation.

Theorem 6.4. *[26, Proposition 1.1] Let G be a planar graph and w, x, y, z be vertices of G that appear in cyclic order on a face of G. Then*

$$\begin{aligned} M(G)\, M(G - \{w,x,y,z\}) &+ M(G - \{w,y\})\, M(G - \{x,z\}) \\ &= M(G - \{w,x\})\, M(G - \{y,z\}) + M(G - \{w,z\})\, M(G - \{x,y\}). \end{aligned}$$

To prove our main result in Chapter 8 we shall use the following result of Ciucu [10].

Theorem 6.5 (Ciucu, [10])**.** *Let G be a planar graph with the vertices $a_1, a_2, \ldots, a_{2k}$ appearing in that cyclic order on a face of G. Consider the skew-symmetric matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant 2k}$ with entries given by*

$$a_{ij} := M(G \setminus \{a_i, a_j\}), \qquad \text{if } i < j. \tag{6.1}$$

Then we have that

$$M(G \setminus \{a_1, a_2, \dots, a_{2k}\}) = \frac{\mathrm{Pf}(A)}{[M(G)]^{k-1}}. \tag{6.2}$$

For $A = (a_{i,j})$, a $2n \times 2n$ anti-symmetric matrix, the Pfaffian of A (denoted $\mathrm{Pf}(A)$) is defined as

$$\mathrm{Pf}(A) = \sum_{\pi=\{(i_1,j_1),\dots,(i_n,j_n)\}\in\Gamma_n} \operatorname{sgn}\pi \prod_{k=1}^{n} a_{i_k,j_k}$$

where $\operatorname{sgn}\pi = \operatorname{sgn} i_1 j_1 i_2 j_2 \dots i_n j_n$ and Γ_n is the set of all perfect matchings of K_{2n}. There are many ways to write π, so to have $\mathrm{Pf}(A)$ well-defined we assume that $i_k < j_k$ and $i_1 < i_2 < \cdots < i_n$.

Although Theorem 6.5 is enough for our purposes, we state and prove a slightly more general version of the theorem below. It turns out that our result is a common generalization for the condensation results of Kuo [25] as well as Theorem 6.5 which follows immediately from Theorem 6.6 below if we consider $a_1, \dots, a_{2k} \in V(G)$. We also mention that Corollary 6.7 of Theorem 6.6, does not follow from Theorem 6.5.

To state and prove our result, we will need to make some notations and concepts clear. We consider the symmetric difference on the vertices and edges of a graph. Let H be a planar graph and G be an induced subgraph of H and let $W \subseteq V(H)$. Then we define $G + W$ to be the induced subgraph of H with vertex set $V(G + W) = V(G)\Delta V(W)$, where Δ denotes the symmetric difference of sets. Now we are in a position to state our result below.

Theorem 6.6. *Let H be a planar graph and let G be an induced subgraph of H with the vertices $a_1, a_2, \dots, a_{2k}$ appearing in that cyclic order on a face of H. Consider the skew-symmetric matrix $A = (a_{ij})_{1\leqslant i,j\leqslant 2k}$ with entries given by*

$$a_{ij} := M(G + \{a_i, a_j\}), \qquad \textit{if } i < j. \tag{6.3}$$

Then we have that

$$M(G + \{a_1, a_2, \dots, a_{2k}\}) = \frac{\mathrm{Pf}(A)}{[M(G)]^{k-1}}. \tag{6.4}$$

Corollary 6.7. *[25, Theorem 2.4] Let $G = (V_1, V_2, E)$ be a bipartite planar graph with $|V_1| = |V_2| + 1$; and let w, x, y and z be vertices of G that appear in cyclic order on a face of G. If $w, x, y \in V_1$ and $z \in V_2$ then*

$$M(G-\{w\})\,M(G-\{x,y,z\}) + M(G-\{y\})\,M(G-\{w,x,z\}) = M(G-\{x\})\,M(G-\{w,y,z\}).$$

Proof. Take $n = 2$, $a_1 = w, a_2 = x, a_3 = y, a_4 = z$ and $G = H \setminus \{a_1\}$ in Theorem 6.6. □

The proof of Theorem 6.5 follows from the use of some auxiliary results. Similar to those results, we need the following proposition to complete our proof of Theorem 6.6.

Proposition 6.8. *Let* H *be a planar graph and* G *be an induced subgraph of* H *with the vertices* $a_1, \dots, a_{2k}$ *appearing in that cyclic order among the vertices of some face of* H*. Then*

$$M(G)\,M(G+\{a_1,\dots,a_{2k}\}) + \sum_{l=2}^{k} M(G+\{a_1,a_{2l-1}\})\,M(G+\overline{\{a_1,a_{2l-1}\}}) = \sum_{l=1}^{k} M(G+\{a_1,a_{2l}\})\,M(G+\overline{\{a_1,a_{2l}\}}), \quad (6.5)$$

where $\overline{\{a_i,a_j\}}$ *stands for the complement of* $\{a_i,a_j\}$ *in the set* $\{a_1,\dots,a_{2k}\}$*.*

Our proof follows closely that of the proof of an analogous proposition given by Ciucu [10, Proposition 1] with very little difference and hence we refer to it for the sake of brevity.

Proof. We recast equation (6.5) in terms of disjoint unions of cartesian products as follows

$$\mathcal{M}(G)\times\mathcal{M}(G+\{a_1,\dots,a_{2k}\}) \cup \mathcal{M}(G+\{a_1,a_3\})\times\mathcal{M}(G+\overline{\{a_1,a_3\}}) \cup \dots \cup \mathcal{M}(G+\{a_1,a_{2k-1}\})\times\mathcal{M}(G+\overline{\{a_1,a_{2k-1}\}}) \quad (6.6)$$

and

$$\mathcal{M}(G+\{a_1,a_2\})\times\mathcal{M}(G+\overline{\{a_1,a_2\}}) \cup \mathcal{M}(G+\{a_1,a_4\})\times\mathcal{M}(G+\overline{\{a_1,a_4\}}) \cup \dots \cup \mathcal{M}(G+\{a_1,a_{2k}\})\times\mathcal{M}(G+\overline{\{a_1,a_{2k}\}}) \cup \quad (6.7)$$

where $\mathcal{M}(F)$ denotes the set of perfect matchings of the graph F. For each element (μ,ν) of (6.6) or (6.7), we think of the edges of μ as being marked by solid lines and that of ν as being marked by dotted lines, on the same copy of the graph H. If there are any edges common to both then we mark them with both solid and dotted lines.

We now define the weight of (μ,ν) to be the product of the weight of μ and the weight of ν. Thus, the total weight of the elements in the set (6.6) is same as the left hand side of equation (6.5) and the total weight of the elements in the set (6.7) equals the right hand side of equation (6.5). To prove our result, we have to construct a weight-preserving bijection between the sets (6.6) and (6.7). The construction is similar to the one given by Ciucu [10, Proposition 1], so we mention only the essential details below and refer the reader to Ciucu's proof.

Let (μ,ν) be an element in (6.6). Then we have two possibilities as discussed in the following. If $(\mu,\nu)\in\mathcal{M}(G)\times\mathcal{M}(G+\{a_1,\dots,a_{2k}\})$, we consider the path containing a_1 and change a solid edge to a dotted edge and a dotted edge to a solid edge in order to obtain a new pair of matchings. Let this pair of matchings be (μ',ν'). For a clearer view of this *shifting along the path* process, we refer the reader to Ciucu's proof. The path we have obtained must connect a_1 to one of the even-indexed vertices. So (μ',ν') is an element of (6.7).

If $(\mu,\nu)\in\mathcal{M}(G+\{a_1,a_3\})\times\mathcal{M}(G+\overline{\{a_1,a_3\}})$, then we map it to a pair of matchings (μ',ν') obtained by reversing the solid and dotted edges along the path containing

a_3. With a similar reasoning like above, this path must connect a_3 to one of the even-indexed vertices and a similar argument will show that indeed (μ', ν') is an element of (6.7). If $(\mu, \nu) \in \mathcal{M}(G + \{a_1, a_{2i+1}\}) \times \mathcal{M}(G + \overline{\{a_1, a_{2i+1}\}})$ with $i > 1$, we have the same construction with a_3 replaced by a_{2i+1}.

The map $(\mu, \nu) \mapsto (\mu', \nu')$ is invertible because given an element in (μ', ν') of (6.7), the pair (μ, ν) that is mapped to it is obtained by shifting along the path that contains the vertex a_{2i}, such that $(\mu', \nu') \in \mathcal{M}(G + \{a_1, a_{2i}\}) \times \mathcal{M}(G + \overline{\{a_1, a_{2i}\}})$. The map we have defined is weight-preserving and this proves the proposition. □

Now we can prove Theorem 6.6, which is essentially the same proof as that of Theorem 6.5, but now uses our more general Proposition 6.8.

Proof of Theorem 6.6. We prove the statement by induction on k. For $k = 1$ it follows from the fact that

$$\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a.$$

For the induction step, we assume that the statement holds for $k-1$ with $k \geqslant 2$. Let A be the matrix

$$\begin{pmatrix} 0 & M(G+\{a_1,a_2\}) & M(G+\{a_1,a_3\}) & \cdots & M(G+\{a_1,a_{2k}\}) \\ -M(G+\{a_1,a_2\}) & 0 & M(G+\{a_2,a_3\}) & \cdots & M(G+\{a_2,a_{2k}\}) \\ -M(G+\{a_1,a_3\}) & -M(G+\{a_2,a_3\}) & 0 & \cdots & M(G+\{a_3,a_{2k}\}) \\ \vdots & \vdots & \vdots & & \vdots \\ -M(G+\{a_1,a_{2k}\}) & -M(G+\{a_2,a_{2k}\}) & -M(G+\{a_3,a_{2k}\}) & \cdots & 0 \end{pmatrix}.$$

By a well-known property of Pfaffians, we have

$$\mathrm{Pf}(A) = \sum_{i=2}^{2k} (-1)^i M(G + \{a_1, a_i\}) \, \mathrm{Pf}(A_{1i}). \tag{6.8}$$

Now, the induction hypothesis applied to the graph G and the $2k-2$ vertices in $\overline{\{a_i, a_j\}}$ gives us

$$[M(G)]^{k-2} M(G + \overline{\{a_1, a_i\}}) = \mathrm{Pf}(A_{1i}), \tag{6.9}$$

where A_{1i} is same as in equation (6.8). So using equations (6.8) and (6.9) we get

$$\mathrm{Pf}(A) = [M(g)]^{k-2} \sum_{i=2} 2k(-1)^i M(G + \{a_1, a_i\}) \, M(G + \overline{\{a_1, a_i\}}). \tag{6.10}$$

Now using Proposition 6.8, we see that the above sum is $M(G)\, M(G + \{a_1, \ldots, a_{2k}\})$ and hence equation (6.10) implies (6.4). □

7

REGIONS WITH DEFECTS ON BOUNDARIES

This chapter deals with various Aztec rectangles with defects on one or two boundary sides. We use the methods described in the previous chapter to prove the results.

We define the binomial coefficients that appear in this chapter as follows

$$\binom{c}{d} := \begin{cases} \dfrac{c(c-1)\cdots(c-d+1)}{d!}, & \text{if } d \geqslant 0 \\ 0, & \text{otherwise} \end{cases}.$$

Our formulas also involve hypergeometric series. We recall that the hypergeometric series of parameters $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ is defined as

$${}_rF_s\left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!},$$

where the Pochhammer symbol is defined as $(n)_m := n(n+1)(n+2)\cdots(n+m-1)$. Most of the proofs are quite similar and hence we prove only Propositions 7.5 and 7.7 in full and omit certain details from some of the proofs of the other propositions in this chapter.

We also fix a notation for the remainder of this thesis as follows: if we remove the squares labelled $2, 4, 7$ from the southeastern boundary of $\mathcal{AR}_{4,7}$ (marked from bottom to top), we denote it by $\mathcal{AR}_{4,7}(2,4,7)$. In the derivation of the results in this chapter, the following corollaries of Theorem 1.2 will be used.

Corollary 7.1. *The number of tilings of* $\mathcal{AR}_{a,a+1}(i)$ *is given by*

$$2^{a(a+1)/2}\binom{a}{i-1}.$$

Corollary 7.2. *The number of tilings of* $\mathcal{AR}_{a,b}(2, \ldots, b-a+1)$ *is given by*

$$2^{a(a+1)/2}\binom{b-1}{a-1}.$$

7.1 AUXILIARY TILING RESULTS

Our main results in Chapter 8 are given in terms of tilings of several intermediate regions which we study in this chapter. In this section, some auxiliary results used in enumerating domino tilings of these intermediate regions are proved.

The following proposition does not appear explicitly anywhere, but it is used implicitly in deriving Proposition 7.4.

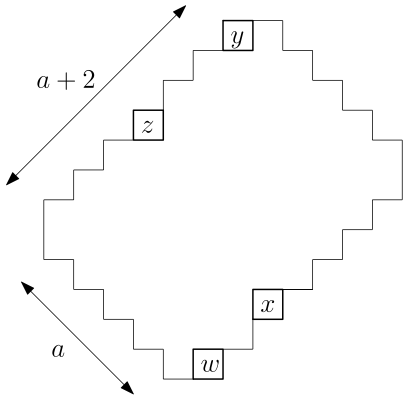


Figure 7.1: An $a \times (a+2)$ Aztec rectangle with some labelled squares; here $a = 5$.

Proposition 7.3. *Let $1 \leqslant a$ be a positive integer, then the number of tilings of $\mathcal{AR}_{a,a+2}$ with a defect at the i-th position on the southeastern side counted from the south corner and a defect on the j-th position on the northwestern side counted from the west corner is given by*

$$2^{a(a+1)/2}\left[\binom{a}{i-2}\binom{a}{j-1}+\binom{a}{i-1}\binom{a}{j-2}\right]. \tag{7.1}$$

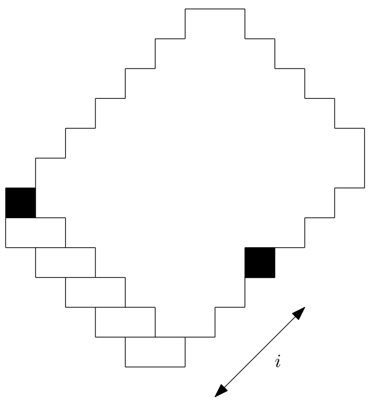


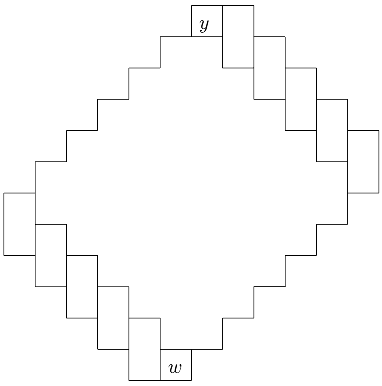
Figure 7.2: An $a \times (a+2)$ Aztec rectangle with defects marked; here $a = 5$, $i = 1$.

Proof. If $j = 1$ or $j = a+2$, then the region we want to tile reduces to the type in Theorem 1.2 due to forced dominoes in any tiling, as can be seen from Figure 7.2 for the case when $j = 1$. The case when $j = a+2$ is similar. We do not worry about the case when both i and j equals 1 or $a+2$ because then the expression in (7.1) is 0. It is easy to see that the expression (7.1) is satisfied in all of the other cases. By symmetry, this also takes care of the cases $i = 1$ and $i = a+2$.

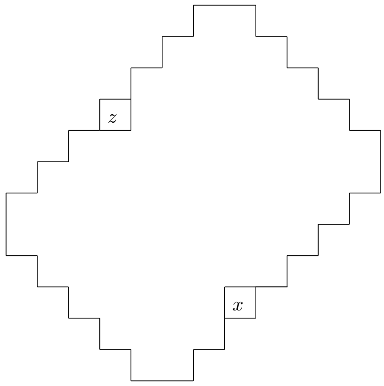
In the rest of the proof, we now assume that $1 < i, j < a+2$ and let us denote the region we are interested in by $O(a)_{i,j}$. We now use Theorem 6.2 with the vertices as indicated in Figure 7.1 to obtain the following identity (Figure 7.3).

$$\begin{aligned} M(AD(a))\,M(O(a)_{i,j}) = {} & M(\mathcal{AR}_{a,a+1}(i-1))\,M(\mathcal{AR}_{a,a+1}(j)) \\ & + M(\mathcal{AR}_{a,a+1}(j-1))\,M(\mathcal{AR}_{a,a+1}(i)). \end{aligned} \tag{7.2}$$

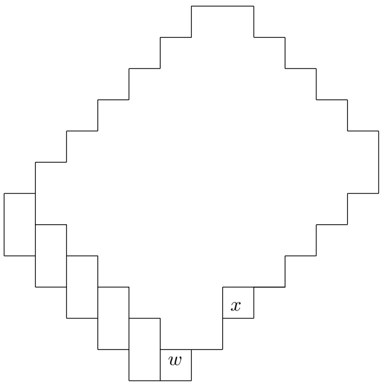
Now, using Theorem 1.1 and Corollary 7.1 in equation (7.2) we get (7.1). □



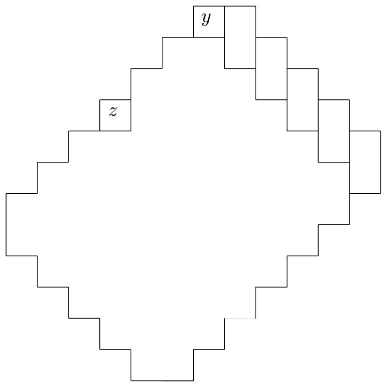
(a) $\mathrm{AD}(a)$.



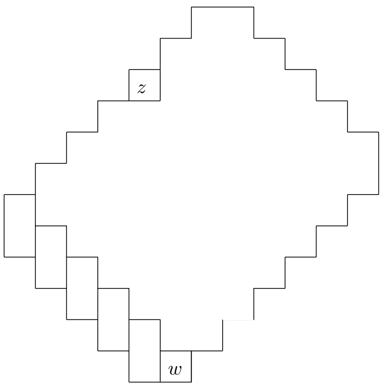
(b) $\mathrm{O}(a)_{i,j}$.



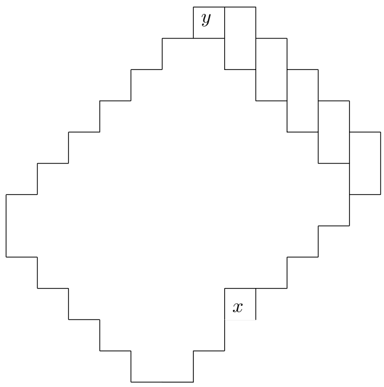
(c) $\mathcal{AR}_{a,a+1}(i-1)$.



(d) $\mathcal{AR}_{a,a+1}(j)$.



(e) $\mathcal{AR}_{a,a+1}(j-1)$.



(f) $\mathcal{AR}_{a,a+1}(i)$.

Figure 7.3: Some forced dominoes in the proof of Proposition 7.3 where the vertices we remove are labelled.

Remark 8. *Ciucu and Fischer [11] have a similar result for the number of lozenge tilings of a hexagon with defects on opposite sides (Proposition 4 in their paper). They also make use of Kuo's condensation result, Theorem 6.1 and obtain the following identity*

$$\begin{aligned}\mathrm{OPP}(a,b,c)_{i,j}\,\mathrm{OPP}(a-2,b,c)_{i-1,j-1}\\ =&\,\mathrm{OPP}(a-1,b,c)_{i-1,j-1}\,\mathrm{OPP}(a-1,b,c)_{i,j}\\ &-\mathrm{OPP}(a-1,b-1,c+1)_{i,j-1}\,\mathrm{OPP}(a-1,b+1,c-1)_{i-1,j}\end{aligned}$$

where $\operatorname{OPP}(a,b,c)_{i,j}$ *denotes the number of lozenge tilings of a hexagon* $H_{a,b,c}$ *with opposite side lengths* a,b,c *and with two defects in positions* i *and* j *on opposite sides of length* a*, where* a,b,c,i,j *are positive integers with* $1 \leqslant i,j \leqslant a$.

In their use of Kuo's result, they take the graph G *to be the planar dual graph of* $H_{a,b,c}$ *with two defects in positions* i *and* j *on opposite sides of length* a *(the resulting number of lozenge tilings of this region is then* $\operatorname{OPP}(a,b,c)_{i,j}$*), but if we take the graph* G *to be the planar dual graph of* $H_{a,b,c}$ *and use Theorem 6.1 with an appropriate choice of labels, we get the following identity*

$$\begin{aligned}&\operatorname{OPP}(a,b,c)_{i,j}\,H(a-1,b,c)\\&=H(a,b,c)\operatorname{OPP}(a-1,b,c)_{i,j}\\&\qquad\qquad+H(a,c-1,b+1,a-1,c,b)_i\,H(a,c,b,a-1,c+1,b-1)_{a-j+1}\end{aligned}$$

where $H(a,b,c)$ *denotes the number of lozenge tilings of the hexagon with opposite sides of length* a,b,c *and* $H(m,n,o,p,q,r)_k$ *denotes the number of lozenge tilings of a hexagon with side lengths* m,n,o,p,q,r *with a defect at position* k *on the side of length* m*. Then, Proposition 4 of Ciucu and Fischer [11] follows more easily without the need for contiguous relations of hypergeometric series that they use in their paper.*

The following result does not appear explicitly in the statements of our main theorems, but this result is essential in deriving Proposition 7.6 later.

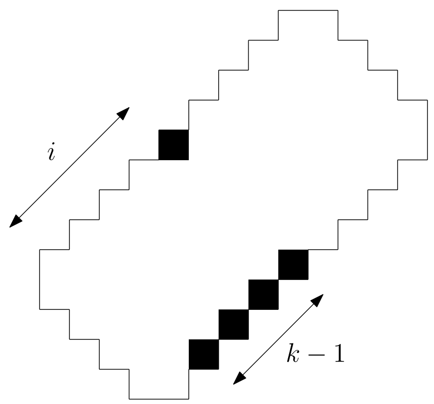


Figure 7.4: An $a \times b$ Aztec rectangle with defects marked in black; here $a=4, b=9, k=5, i=5$.

Proposition 7.4. *Let* $1 \leqslant a,i \leqslant b$ *be positive integers with* $k=b-a>0$ *and* $y=\min\{i,k\}$*, then the number of domino tilings of* $\mathcal{AR}_{a,b}(2,3,\dots,k)$ *with a defect on the northwestern side in the* i*-th position counted from the west corner as shown in the Figure 7.4 is given by*

$$2^{a(a+1)/2}\binom{a+y-2}{a-1}\binom{a}{i-y}\,{}_3F_2\left[\begin{matrix}1,1-y,i-a-y\\ i-y+1,2-a-y\end{matrix};-1\right].$$

Proof. Our proof will be by induction on $b=a+k$. The base case of induction will follow if we verify the result for $a=2,k=1$ in which case $b=3$. To check our base case it is now enough to verify the formula for $a=2,k=1,i=1,2,3$, in which case

$y = 1$. This is easily verified as when $i = 1$, we have many forced dominoes and we get the region shown in Figure 7.5, which is AD(2). When $i = 2$, we see that the region we obtain is of the type as described in Corollary 7.1 and finally when $i = 3$, then we get a region of the type in Theorem 1.2. It is easily verified that in all these cases, the regions satisfy the statement of the result.

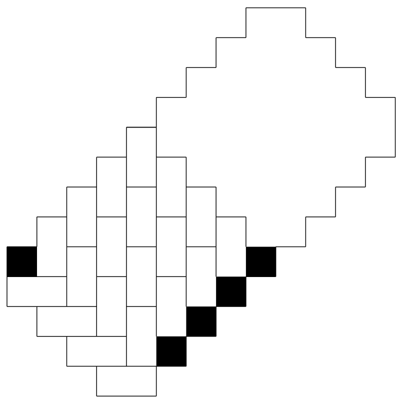


Figure 7.5: Forced tilings for $i = 1$ in Proposition 7.6.

We now deal with the case when $b > 3$, and $i = 1$ or $i = b$, before dealing with the other cases for values of i. If $i = 1$ we have many forced dominoes and we get the region shown in Figure 7.5, which is AD(a). Again, if $i = b$, then we get a region of the type in Theorem 1.2 due to forced dominoes. In both of these cases the number of domino tilings of these regions satisfy the formula mentioned in the statement. From now on, we assume $b > 3$ and $1 < i < b$. We denote the region of the type shown in Figure 7.4 by $\mathcal{AR}^{i}_{a,b,k-1}$. We use Theorem 6.2 here, with the vertices w, x, y and z marked as shown in Figure 7.6, where we add a series of unit squares to the northeastern side to make it into an $a \times (b+1)$ Aztec rectangle. Note that the square in the i-th position to be removed is included in this region and is labelled by z. The identity we now obtain is the following (see Figure 7.7 for forcings)

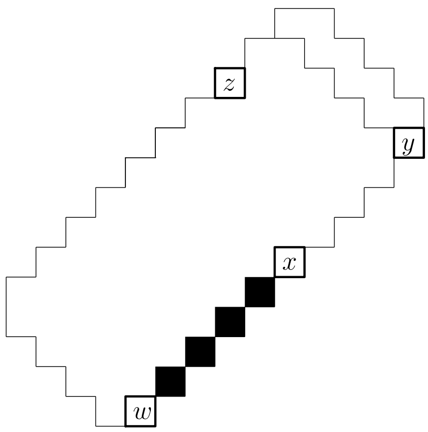


Figure 7.6: Labelled $a \times (b+1)$ Aztec rectangle; here $a = 4, b = 9$.

$$\begin{aligned} &M(AD(a))\,M(\mathcal{AR}^{i}_{a,b+1,k}) \\ &\qquad = M(AD(a))\,M(\mathcal{AR}^{i}_{a,b,k-1}) + Y \cdot M(\mathcal{AR}_{a,b}(2,3,\ldots,k,k+1)) \end{aligned} \tag{7.3}$$

where

$$Y := \begin{cases} 0, & \text{if } i \leqslant k \\ M(\mathcal{AR}_{a,a+1}(a+k+2-i)), & \text{if } i \geqslant k+1 \end{cases}. \tag{7.4}$$

Using equation (7.4) in equation (7.3), we can simplify the relation further to the following

$$M(\mathcal{AR}^{i}_{a,b+1,k}) = M(\mathcal{AR}^{i}_{a,b,k-1}) + Z \cdot M(\mathcal{AR}_{a,b}(2,3,\ldots,k+1)) \tag{7.5}$$

where

$$Z := \begin{cases} 0, & \text{if } i \leqslant k \\ \dfrac{M(\mathcal{AR}_{a,a+1}(a+k+2-i))}{M(AD(a))}, & \text{if } i \geqslant k+1 \end{cases}. \tag{7.6}$$

It now remains to show that the expression in the statement satisfies equation (7.5). This is now a straightforward application of the induction hypothesis and some algebraic manipulation (see the proof of Proposition 7.7 below for a more detailed proof of a similar type). □

7.2 REGIONS WITH DEFECTS ON ONE OR TWO BOUNDARY SIDES

The statement of our main results in the next chapter involves regions with defects on one or two boundary sides. These regions are studied in this section.

Proposition 7.5. *Let $1 \leqslant a \leqslant b$ be positive integers with $k = b - a > 0$, then the number of domino tilings of $\mathcal{AR}_{a,b}(j)$ with $k-1$ squares added to the southeastern side starting at the second position (and not at the bottom) as shown in the Figure 7.8 for $j \geqslant k$ is given by*

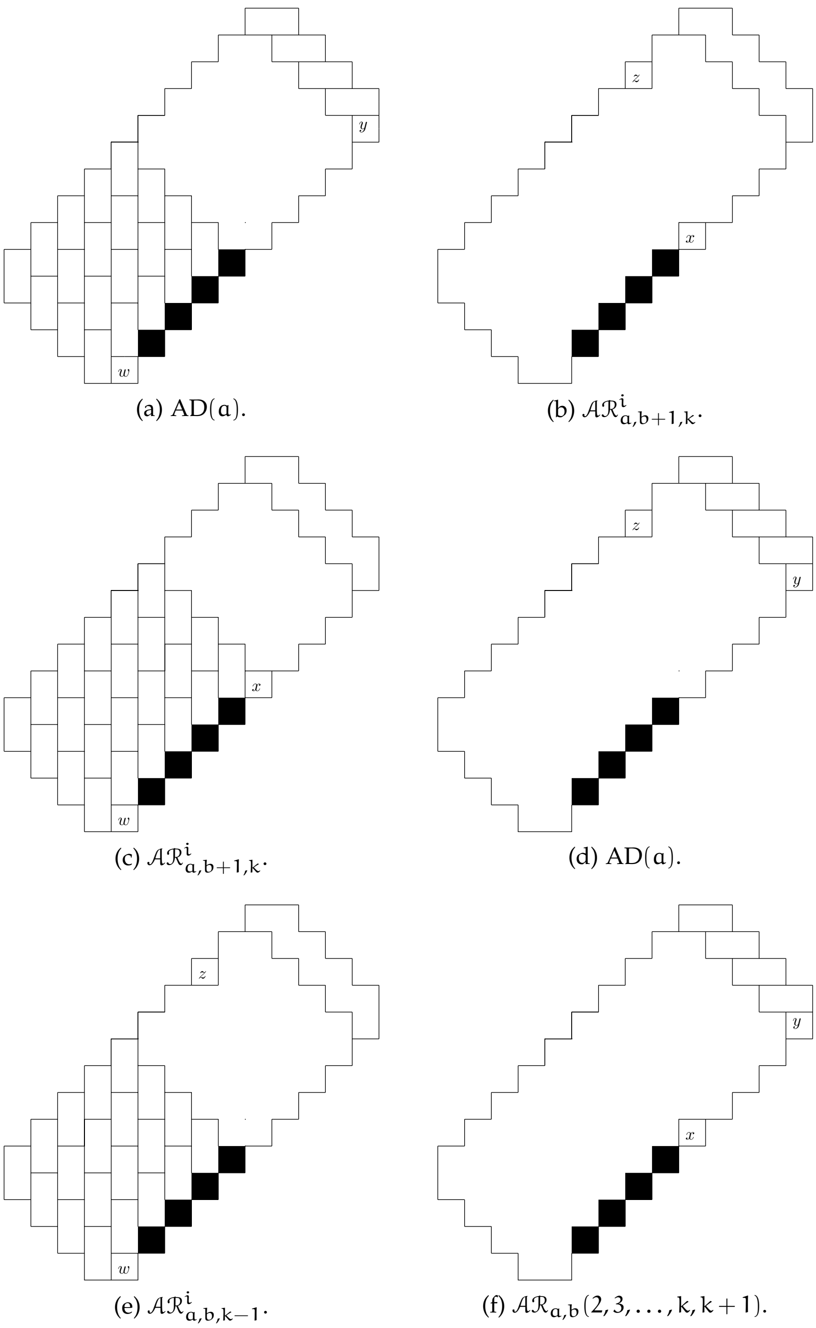
$$2^{a(a+1)/2}\binom{a+k-1}{j-1}\binom{j-2}{k-1}\,{}_3F_2\left[\begin{matrix}1, 1-j, 1-k\\ 2-j, 1-a-k\end{matrix}; 1\right]. \tag{7.7}$$

If $j \leqslant k-1$, then the number of tilings of $\mathcal{AR}_{a,b}(j)$ with $k-1$ squares added to the southeastern side starting at the second position (and not at the bottom) as shown in the Figure 7.9 is given by

$$2^{a(a+1)/2}.$$

Proof. Let us denote the region in Figure 7.8 by $\mathcal{AR}^{k-1,j}_{a,b}$ and we work with the planar dual graph of the region $\mathcal{AR}^{k-1,j}_{a,b}$ and count the number of matchings of that graph.

We first deal with the case when $j \geqslant k$. We notice that the first added square in any domino tiling of the region in Figure 7.8 by dominoes has two possibilities to be



(a) AD(a). (b) $\mathcal{AR}^{i}_{a,b+1,k}$.

(c) $\mathcal{AR}^{i}_{a,b+1,k}$. (d) AD($a$).

(e) $\mathcal{AR}^{i}_{a,b,k-1}$. (f) $\mathcal{AR}_{a,b}(2,3,\dots,k,k+1)$.

Figure 7.7: Forced dominoes in the proof of Proposition 7.4 where the vertices we remove are labelled

matched up with squares marked in grey in Figure 7.10. This observation allows us to write the number of tilings of $\mathcal{AR}^{k-1,j}_{a,b}$ in terms of the following recursion

$$M(\mathcal{AR}^{k-1,j}_{a,b}) = M(\mathcal{AR}^{k-2,j-1}_{a,b-1}) + M(\mathcal{AR}_{a,b}(2,3,\dots,k,j)). \tag{7.8}$$

which can be verified from Figure 7.11.

We can now continue to descend further in a similar way until we have no added squares on the southeastern side remaining. Thus, we can repeat this process for

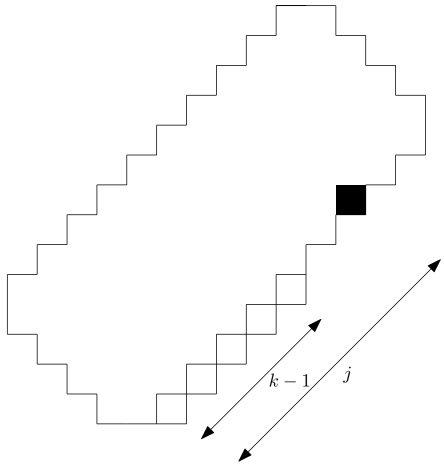


Figure 7.8: An Aztec rectangle with $k-1$ squares added on the southeastern side and a defect on the j-th position shaded in black; here $a=4, b=10, k=6, j=8$.

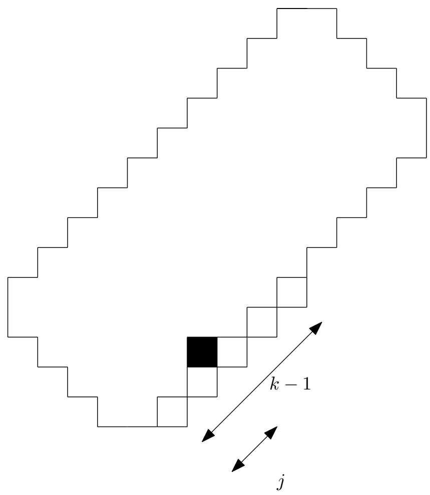


Figure 7.9: An Aztec rectangle with $k-1$ squares added on the southeastern side and a defect on the j-th position shaded in black; here $a=4, b=10, k=6, j=3$.

$\mathcal{AR}^{k-2,j-1}_{a,b-1}$, then for $\mathcal{AR}^{k-3,j-2}_{a,b-2}$ and so on. Repeatedly using equation (7.8) as a template for this descend process $k-1$ times successively, we shall finally obtain

$$M(\mathcal{AR}^{k-1,j}_{a,b}) = \sum_{l=0}^{k-2} M(\mathcal{AR}_{a,b-l}(2,3,\dots,k-l,j-l)) + M(\mathcal{AR}_{a,a+1}(j-k+1)). \quad (7.9)$$

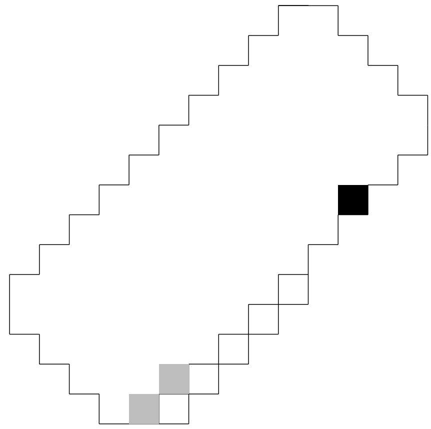


Figure 7.10: $\mathcal{AR}_{a,b}^{k-1,j}$ with the possible choices for the first added square in a tiling; here $a = 4, b = 10, k = 6, j = 8$.

Now, plugging in the values of the quantities in the right hand side of equation (7.9) from Theorem 1.2 and Corollary 7.1 we shall obtain the following.

$$\begin{aligned} M(\mathcal{AR}_{a,b}^{k-1,j}) &= \sum_{l=0}^{k-2} 2^{a(a+1)/2} \binom{a+k-l-1}{a+k-j} \binom{j-l-2}{k-l-1} \\ &\quad + 2^{a(a+1)/2} \binom{a}{a+k-j} \\ &= 2^{a(a+1)/2} \sum_{l=0}^{k-1} \binom{a+k-l-1}{a+k-j} \binom{j-l-2}{k-l-1}. \end{aligned} \tag{7.10}$$

Using standard techniques, if we transform the above binomial sum into hypergeometric notation, then we shall obtain equation (7.7).

For the case when $j \leqslant k-1$, we see that there are many forced dominoes in any tiling (see Figure 7.12) and the region we want to tile is reduced to an Aztec diamond of order a, and this completes the proof. □

Proposition 7.6. *Let $1 \leqslant a, i \leqslant b$ be positive integers with $k = b - a > 0$ and $y = \min\{i, k\}$, then the number of domino tilings of $\mathcal{AR}_{a,b}$ with $k-1$ squares added to the southeastern side starting at the second position (and not at the bottom) as shown in the Figure 7.13 and a defect on the northwestern side at the i-th position counted from the western corner is given by*

$$2^{a(a+1)/2} \binom{a}{i-y} \sum_{l=0}^{y-1} \binom{a+y-l-2}{y-l-1} {}_3F_2 \left[\begin{matrix} 1, 1-y+l, i-a-y \\ i-y+1, 2-a-y+l \end{matrix}; -1 \right]. \tag{7.11}$$

Proof. We follow a similar approach for this proof, like we did in the proof of Proposition 7.5. Let us denote the region in Figure 7.13 by $\mathcal{AR}_{a,b}(k-1;i)$ and we

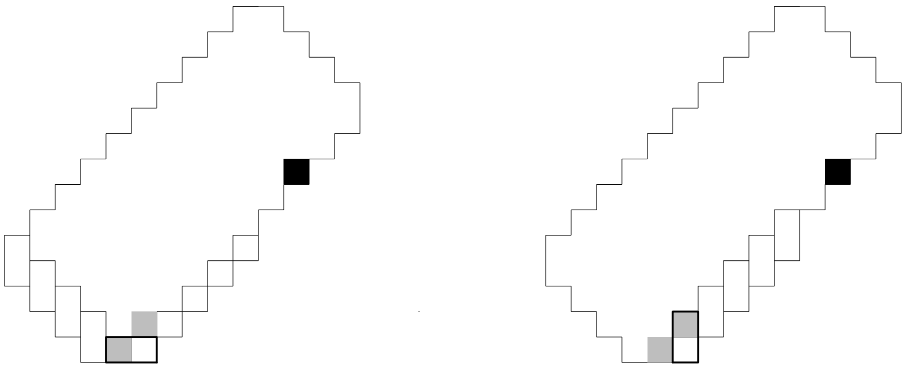


Figure 7.11: Choices for the tilings of $\mathcal{AR}_{a,b}^{k-1,j}$ with forced dominoes; here $a = 4, b = 10, k = 6, j = 8$.

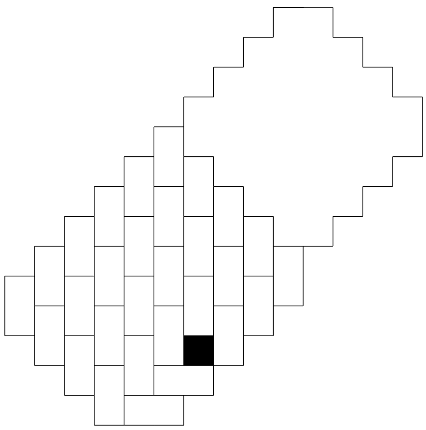


Figure 7.12: $\mathcal{AR}_{a,b}^{k-1,j}$, when $j \leqslant k-1$; here $a = 4, b = 10, k = 6, j = 3$.

work with the planar dual graph of this region and count the number of matchings of that graph. We first notice that the first added square in any tiling of the region in Figure 7.13 by dominoes has two possibilities to be matched up with squares marked in grey in Figure 7.14. This observation allows us to write the number of tilings of $\mathcal{AR}_{a,b}(k-1;i)$ in terms of the following recursion (see Figure 7.15)

$$M(\mathcal{AR}_{a,b}(k-1;i)) = M(\mathcal{AR}_{a,b-1}(k-2;i-1)) + M(\mathcal{AR}_{a,b,k-1}^{i}). \tag{7.12}$$

As in the proof of Proposition 7.5, repeatedly using equation (7.12) $y-1$ times on successive iterations, we shall finally obtain

$$M(\mathcal{AR}_{a,b}(k-1;i)) = \sum_{l=0}^{y-2} M(\mathcal{AR}_{a,b-l,k-l}^{i-l}) + M(\mathcal{AR}_{a,a+1}(i-y+1)) \tag{7.13}$$

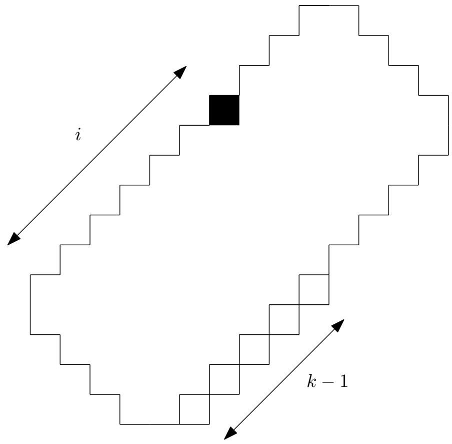


Figure 7.13: An Aztec rectangle with $k-1$ squares added on the southeastern side and a defect on the i-th position shaded in black; here $a=4, b=10, k=6, i=7$.

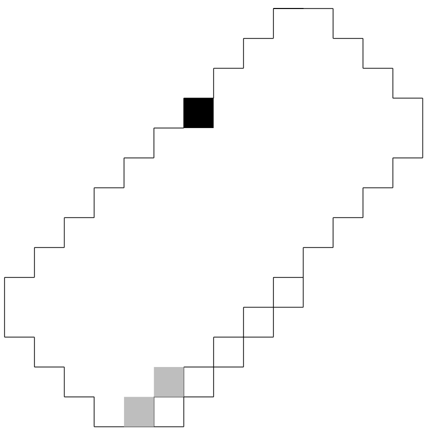


Figure 7.14: $\mathcal{AR}_{a,b}(k-1;i)$ with the possible choices for the first added square in a tiling; here $a=4, b=10, k=6, i=7$.

where $y=\min\{i,k\}$.

Now, plugging in the values of the quantities in the right hand side of equation (7.13) from Proposition 7.4 and Corollary 7.1, and then transforming the binomial sum into hypergeometric notations we shall obtain equation (7.11). □

Proposition 7.7. *Let* a, i, j *be positive integers such that* $1 \leqslant i, j \leqslant a$, *then the number of domino tilings of* AD(a) *with one defect on the southeastern side at the* i*-th position counted*

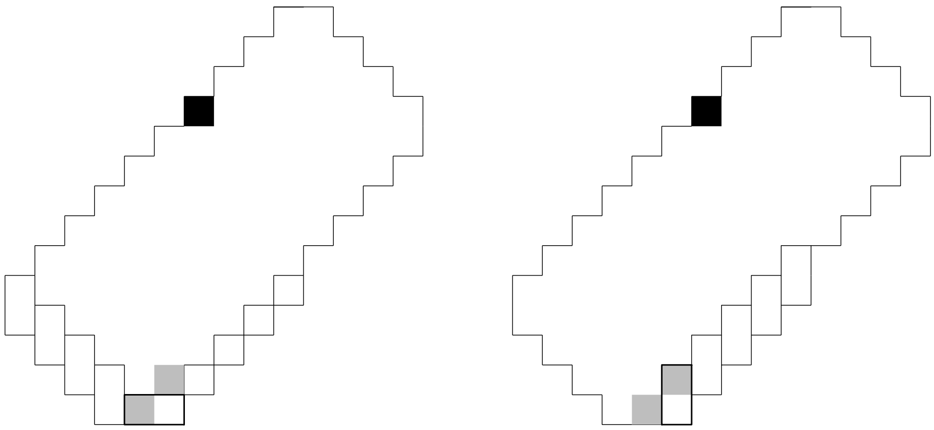


Figure 7.15: Choices for the tilings of $\mathcal{AR}_{a,b}(k-1;i)$ with forced dominoes; here $a=4, b=10, k=6, i=7$.

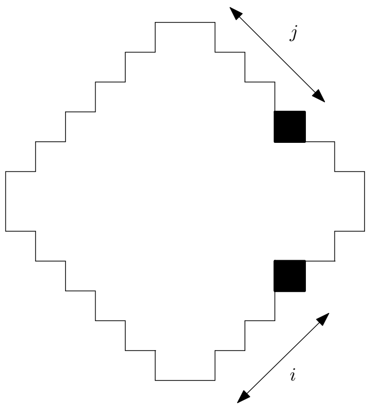


Figure 7.16: An Aztec diamond with defects on adjacent sides; here $a=6$, $i=4$, $j=4$.

from the south corner and one defect on the northeastern side at the j*-th position counted from the north corner as shown in Figure 7.16 is given by*

$$2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\,{}_3F_2\left[\begin{matrix}1,1-i,1-j\\1-a,1-a\end{matrix};2\right].$$

Proof. We use induction with respect to a. The base case of induction is $a=2$, $i=1,2$, $j=1,2$. We check for the cases when, $i=1, j=1, i=a$ and $j=a$ separately. So, for our base case, the only possibilities are $i=1$ or $i=a$ and $j=1$ or $j=a$, so we do not have to consider this case, once we consider the other mentioned cases.

We now note that when either i or j is 1 or a, some dominoes are forced in any tiling and hence we are reduced to an Aztec rectangle of size $(a-1)\times a$. It is easy to see that our formula is correct for this.

In the rest of the proof we assume $a\geqslant 3$ and $1<i,j<a$. Let us now denote the region we are interested in this proposition as $AD_a(i,j)$. Using the dual graph of this region and applying Theorem 6.1 with the vertices as labelled in Figure 7.17 we obtain the following identity (see Figure 7.18 for details),

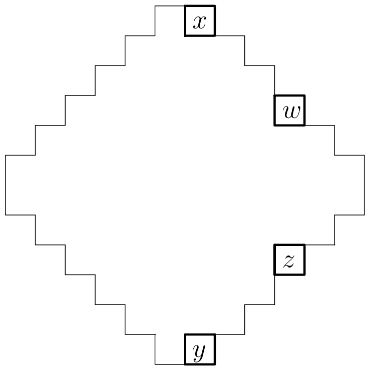
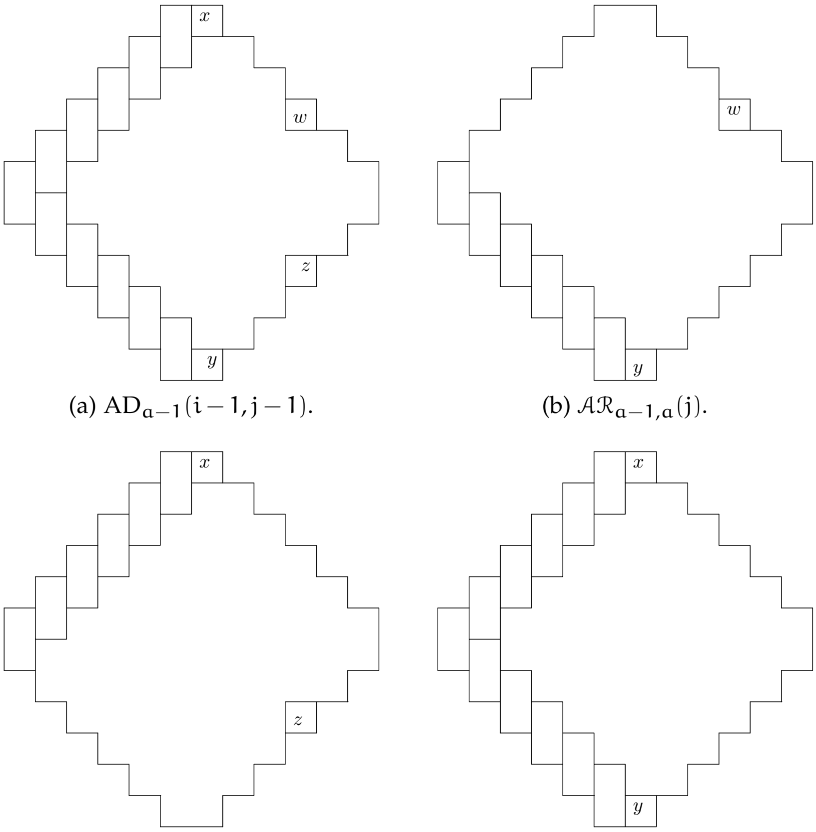


Figure 7.17: An Aztec diamond with some labelled squares; here $a = 6$.



(a) $AD_{a-1}(i-1, j-1)$.

(b) $\mathcal{AR}_{a-1,a}(j)$.

(c) $\mathcal{AR}_{a,a+1}(i)$.

(d) $AD(a-1)$.

Figure 7.18: Forced dominoes in the proof of Proposition 7.7 where the vertices we remove are labelled.

$$M(AD_a(i,j))\,M(AD(a-1)) = M(AD(a))\,M(AD_{a-1}(i-1,j-1)) + M(\mathcal{AR}_{a-1,a}(j))\,M(\mathcal{AR}_{a-1,a}(i)). \tag{7.14}$$

Simplifying equation (7.14), we get the following

$$M(AD_a(i,j)) = 2^a\, M(AD_{a-1}(i-1,j-1)) + 2^{a(a-1)/2}\binom{a-1}{j-1}\binom{a-1}{i-1} \tag{7.15}$$

where we used Theorem 1.1 and Corollary 7.1.

Now, using our inductive hypothesis on equation (7.15) we have the following

$$\begin{aligned}
&M(AD_a(i,j)) = 2^a \cdot 2^{(a-1)(a-2)/2}\binom{a-2}{i-2}\binom{a-2}{j-2}\, {}_3F_2\left[\begin{matrix}1, 2-i, 2-j\\ 2-a, 2-a\end{matrix}; 2\right]\\
&+ 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\\
&= 2\cdot 2^{a(a-1)/2}\binom{a-2}{i-2}\binom{a-2}{j-2}\sum_{k=0}^{\infty}\frac{(1)_k(2-i)_k(2-j)_k}{(2-a)_k(2-a)_k}\frac{2^k}{k!}\\
&+ 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\\
&= 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\left[2\cdot\frac{(1-i)(1-j)}{(1-a)(1-a)}\sum_{k=0}^{\infty}2^k\cdot\frac{(2-i)_k(2-j)_k}{(2-a)_k(2-a)_k}+1\right]\\
&\qquad\qquad = 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\sum_{k=0}^{\infty}2^k\cdot\frac{(1-i)_k(1-j)_k}{(1-a)_k(1-a)_k}.
\end{aligned}$$

This completes the proof of our proposition. □

Remark 9. *Ciucu and Fischer [11] have a similar result for the number of lozenge tilings of a hexagon with defects on adjacent sides (Proposition 3 in their paper). They also make use of Kuo's condensation result, Theorem 6.3 and obtain the following identity*

$$\begin{aligned}
&ADJ(a,b,c)_{j,k}\, ADJ(a-1,b,c-1)_{j,k}\\
&\qquad = ADJ(a,b,c-1)_{j,k}\, ADJ(a-1,b,c)_{j,k}\\
&\qquad\quad + ADJ(a-1,b+1,c-1)_{j,k}\, ADJ(a,b-1,c)_{j,k}
\end{aligned}$$

where $ADJ(a,b,c)_{j,k}$ *denotes the number of lozenge tilings of a hexagon* $H_{a,b,c}$ *with opposite side lengths* a, b, c *with two defects on adjacent sides of length* a *and* c *in positions* j *and* k *respectively, where* a, b, c, j, k *are non-negative integers with* $1 \leqslant j \leqslant a$ *and* $1 \leqslant k \leqslant c$.

In their use of Theorem 6.3, they take the graph G *to be the planar dual graph of* $H_{a,b,c}$ *with two defects on adjacent sides of length* a *and* c *in positions* j *and* k *(the resulting number of lozenge tilings of such a region is then* $ADJ(a,b,c)_{j,k}$*), but if we take the graph* G *to be the planar dual graph of* $H_{a,b,c}$ *and use Theorem 6.1 with an appropriate choice of labels we obtain the following identity*

$$\begin{aligned}
&H(a-1,b,c)\, ADJ(a,b,c)_{j,k}\\
&= H(a,b,c)\, ADJ(a-1,b,c)_{j,k}\\
&\qquad + H(c,a-1,b+1,c-1,a,b)_k\, H(b-1,c+1,a-1,b,c,a)_j
\end{aligned}$$

with the same notations as in Remark 8. Then, Proposition 3 of Ciucu and Fischer [11] follows more easily without the need for contiguous relations of hypergeometric series that they use in their paper.

8

AZTEC RECTANGLES WITH BOUNDARY DEFECTS

This chapter deals with the problem of enumerating domino tilings of an Aztec rectangle with the most general case of boundary defects possible.

8.1 MAIN RESULTS

As already mentioned in Chapter 1, in order to create a region that can be tiled by dominoes we have to remove $b-a$ (henceforth denoted by k) more white squares than black squares along the boundary of $\mathcal{AR}_{a,b}$. There are $2b$ white squares and $2a$ black squares on the boundary of $\mathcal{AR}_{a,b}$. We choose $n+k$ of the white squares that share an edge with the boundary and denote them by $\beta_1, \beta_2, \dots, \beta_{n+k}$ (we will refer to them as defects of type β). We choose any n squares from the black squares which share an edge with the boundary and denote them by $\alpha_1, \alpha_2, \dots, \alpha_n$ (we refer to them as defects of type α). We consider regions of the type $\mathcal{AR}_{a,b} \setminus \{\beta_1, \dots, \beta_{n+k}, \alpha_1, \dots, \alpha_n\}$, which are more general than the type considered in previous work [14, 37].

We now state the main results of this part below. The first result is concerned with the case when the defects are confined to three of the four sides of the Aztec rectangle (defects do not occur on one of the sides with shorter length), and provides a Pfaffian expression for the number of tilings of such a region, with each entry in the Pfaffian being given by a simple product or by a sum or product of quotients of factorials and powers of 2. The second result gives a nested Pfaffian expression for the general case when we do not restrict the occurrence of defects on any boundary side. The third result deals with the case of an Aztec diamond with arbitrary defects on the boundary and gives a Pfaffian expression for the number of tilings of such a region, with each entry in the Pfaffian being given by a simple sum of quotients of factorials and powers of 2.

We define the region $\mathcal{AR}^k_{a,b}$ to be the region obtained from $\mathcal{AR}_{a.b}$ by adding a string of k unit squares along the boundary of the southeastern side as shown in Figure 8.1. We denote this string of k unit squares by $\gamma_1, \gamma_2, \dots, \gamma_k$ and refer to them as defects of type γ.

Theorem 8.1. *Assume that one of the two sides on which defects of type α can occur does not actually have any defects on it. Without loss of generality, we assume this to be the southwestern side. Let $\delta_1, \dots, \delta_{2n+2k}$ be the elements of the set $\{\beta_1, \dots, \beta_{n+k}\} \cup \{\alpha_1, \dots, \alpha_n\} \cup \{\gamma_1, \dots, \gamma_k\}$ listed in a cyclic order.*

Then we have

$$M(\mathcal{AR}_{a,b} \setminus \{\beta_1, \dots, \beta_{n+k}, \alpha_1, \dots, \alpha_n\}) = \frac{1}{[M(\mathcal{AR}^k_{a,b})]^{n+k-1}} \operatorname{Pf}[(M(\mathcal{AR}^k_{a,b} \setminus \{\delta_i, \delta_j\}))_{1 \leqslant i < j \leqslant 2n+2k}], \quad (8.1)$$

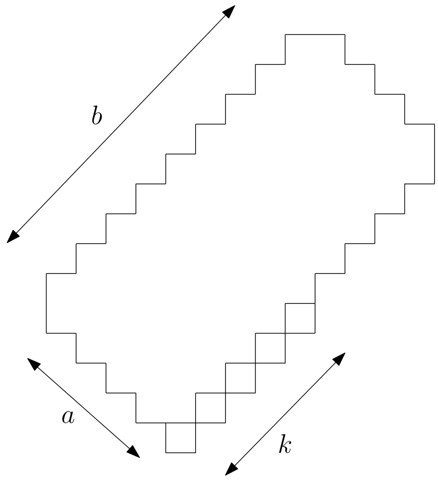


Figure 8.1: $\mathcal{AR}_{a,b}^{k}$ with $a = 4, b = 9, k = 5$.

where all the terms on the right hand side are given by explicit formulas:

1. $M(\mathcal{AR}_{a,b}^{k})$ *is given by Theorem 1.1,*
2. $M(\mathcal{AR}_{a,b}^{k} \setminus \{\beta_i, \alpha_j\})$ *is given by Proposition 7.7 if* β_i *is on the southeastern side and not above a* γ *defect; otherwise it is* 0*,*
3. $M(\mathcal{AR}_{a,b}^{k} \setminus \{\beta_i, \gamma_j\})$ *is given by Theorem 1.1 if* β_i *is above a* γ *defect; it is given by Proposition 7.6 if the* β *defect is in the northwestern side and its distance from the western corner is more than the distance of the* γ *defect from the southern corner; it is given by Propositions 7.5 if the* β *defect is on the southeastern side; otherwise it is* 0*,*
4. $M(\mathcal{AR}_{a,b}^{k} \setminus \{\beta_i, \beta_j\}) = M(\mathcal{AR}_{a,b}^{k} \setminus \{\alpha_i, \alpha_j\}) = M(\mathcal{AR}_{a,b}^{k} \setminus \{\alpha_i, \gamma_j\}) = M(\mathcal{AR}_{a,b}^{k} \setminus \{\gamma_i, \gamma_j\}) = 0$.

Theorem 8.2. *Let* $\beta_1, \dots, \beta_{n+k}$ *be arbitrary defects of type* β *and* $\alpha_1, \dots, \alpha_n$ *be arbitrary defects of type* α *along the boundary of* $\mathcal{AR}_{a,b}$*. Then* $M(\mathcal{AR}_{a,b} \setminus \{\beta_1, \dots, \beta_{n+k}, \alpha_1, \dots, \alpha_n\})$ *is equal to the Pfaffian of a* $2n \times 2n$ *matrix whose entries are Pfaffians of* $(2k+2) \times (2k+2)$ *matrices of the type in the statement of Theorem 8.1.*

In the special case when the number of defects of both types are the same; that is, when $k = 0$ we get an Aztec diamond with arbitrary defects on the boundary and the number of tilings can be given by a Pfaffian where the entries of the Pfaffian are explicit, as stated in the theorem below.

Theorem 8.3. *Let* $\beta_1, \dots, \beta_n$ *be arbitrary defects of type* β *and* $\alpha_1, \dots, \alpha_n$ *be arbitrary defects of type* α *along the boundary of* $AD(a)$*, and let* $\delta_1, \dots, \delta_{2n}$ *be a cyclic listing of the elements of the set* $\{\beta_1, \dots, \beta_n\} \cup \{\alpha_1, \dots, \alpha_n\}$*. Then*

$$M(AD(a) \setminus \{\beta_1, \dots, \beta_n, \alpha_1, \dots, \alpha_n\}) = \frac{1}{[M(AD(a))]^{n-1}} \operatorname{Pf}[(M(AD(a) \setminus \{\delta_i, \delta_j\}))_{1 \leqslant i < j \leqslant 2n}], \quad (8.2)$$

where the values of $M(AD(a) \setminus \{\delta_i, \delta_j\}))$ *are given explicitly as follows:*

1. $M(AD(a) \setminus \{\beta_i, \alpha_j\})$ *is given by Proposition 7.7,*

2. $M(AD(a) \setminus \{\beta_i, \beta_j\}) = M(AD(a) \setminus \{\alpha_i, \alpha_j\}) = 0.$

8.2 PROOFS OF THE MAIN RESULTS

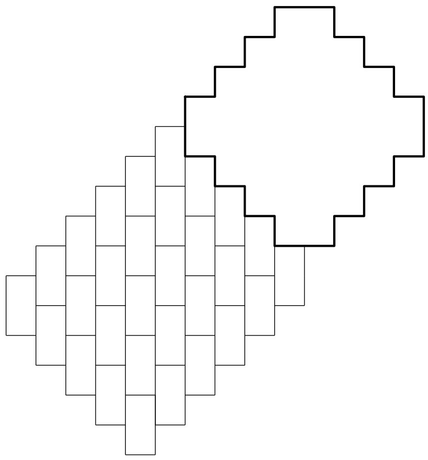


Figure 8.2: Removing the forced dominoes from $\mathcal{AR}^k_{a,b}$; here $a = 5, b = 10, k = 5$.

Proof of Theorem 8.1. We shall apply the formula in Theorem 6.5 to the planar dual graph of our region $\mathcal{AR}^k_{a,b}$, and the vertices $\delta_1, \dots, \delta_{2n+2k}$. Then the left hand side of equation (6.2) becomes the left hand side of equation (8.1), and the right hand side of equation (6.2) becomes the right hand side of (8.1). We just need to verify that the quantities expressed in equation (8.1) are indeed given by the formulas described in the statement of Theorem 8.1.

The first statement follows immediately by noting that the added squares on the southeastern side of $\mathcal{AR}^k_{a,b}$ forces some dominoes. After removing this forced dominoes we are left with an Aztec diamond of order a as shown in Figure 8.2, whose number of tilings is given by Theorem 1.1.

The possibilities in the second statement are as follows. If a β square shares an edge with some γ_l, then the region cannot be covered by any domino as illustrated in the right image of Figure 8.3. Again, if β_i is on the northwestern side at a distance of at most k from the western corner, then the strips of forced dominoes along the southwestern side interfere with the β_i and hence there cannot be any tiling in this case as illustrated in the left image of Figure 8.3. If neither of these situations is the case, then due to the squares $\gamma_1, \dots, \gamma_k$ on the southeastern side, there are forced dominoes as shown in Figure 8.4 and then β_i and α_j are defects on an Aztec diamond on adjacent sides and then the second statement follows from Proposition 7.7.

To prove the validity of the third statement, we notice that if an β and γ defect share an edge then, there are two possibilities, either the β defect is above the γ defect in which case we have some forced dominoes as shown in the left of Figure 8.6 and we are reduced to finding the number of domino tilings of an Aztec diamond;

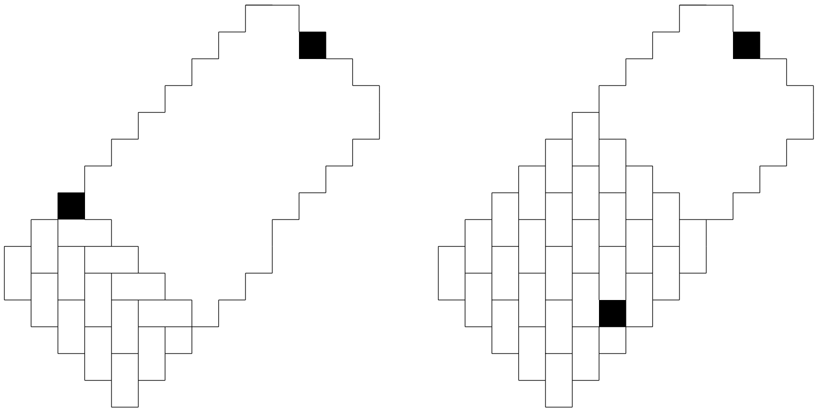


Figure 8.3: Choices of β defects that lead to no tiling of $\mathcal{AR}_{a,b}^k$.

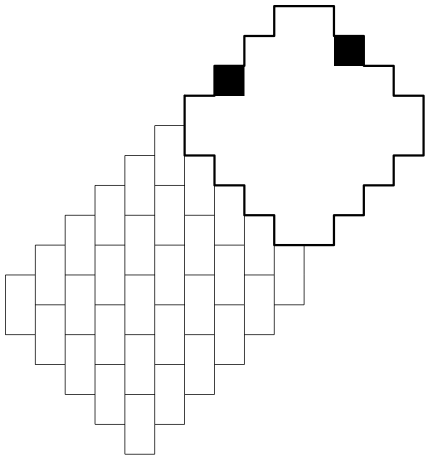


Figure 8.4: Choice of β defect, not sharing an edge with some γ_l.

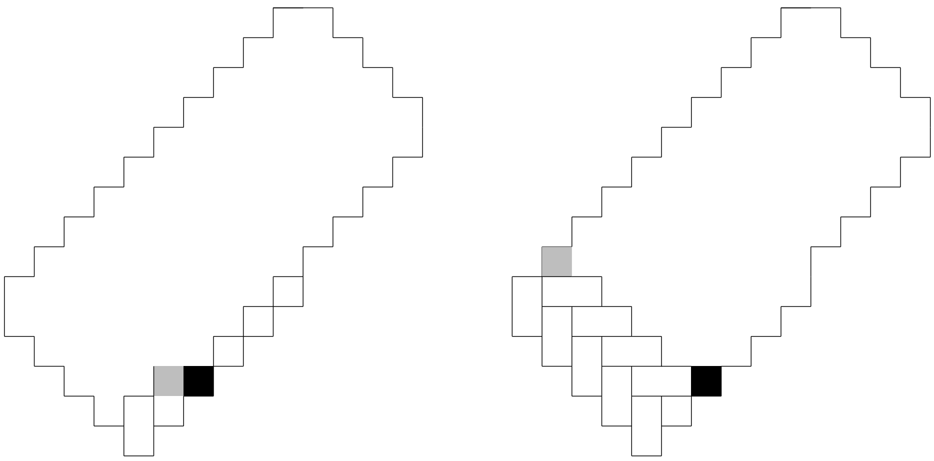


Figure 8.5: Choices of β and γ defects that lead to no tiling of $\mathcal{AR}_{a,b}^k$.

or the β defect is to the left of a γ defect, in which case, we get no tilings as shown in the left of Figure 8.5 as the forced dominoes interfere in this case.

If β_i and γ_j share no edge in common, then we get no tiling if the β defect is on the northwestern side at a distance of at most $k-1$ from the western corner as illustrated in the right of Figure 8.5. If the β defect is in the northwestern side at a distance more than $k-1$ from the western corner then the situation is as shown in the right of Figure 8.6 and is described in Proposition 7.6. If the β defect is in the southeastern side then the situation is as shown in the middle of Figure 8.6 and is described in Proposition 7.5.

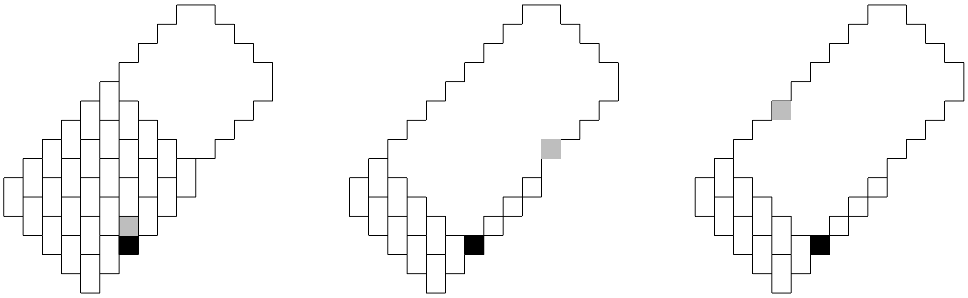


Figure 8.6: Choices of β and γ defects that lead to tiling of $\mathcal{AR}^k_{a,b}$.

The fourth statement follows immediately from the checkerboard drawing (see Figure 1.2) of an Aztec rectangle and the condition that a tiling by dominoes exists for such a board if and only if the number of white and black squares are the same. In all other cases, the numbers of tilings is 0. □

Proof of Theorem 8.2. Let $\mathcal{AR}$ be the region obtained from $\mathcal{AR}^k_{a,b}$ by removing k of the squares $\beta_1, \ldots, \beta_{n+k}$. We now apply Theorem 6.5 to the planar dual graph of $\mathcal{AR}$, with the removed squares chosen to be the vertices corresponding to the n β_i's inside $\mathcal{AR}$ and to $\alpha_1, \ldots, \alpha_n$. The left hand side of equation (6.2) is now the required number of tilings and the right hand side of equation (6.2) is the Pfaffian of a $2n \times 2n$ matrix with entries of the form $M(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$, where β_i is not one of the unit squares that we removed from $\mathcal{AR}^k_{a,b}$ to get $\mathcal{AR}$.

We now notice that $M(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$ is the number of domino tilings of an Aztec rectangle with all its defects confined to three of the sides. So, we can apply Theorem 8.1 and it gives us an expression for $M(\mathcal{AR} \setminus \{\beta_i, \alpha_j\})$ as the Pfaffian of a $(2k+2) \times (2k+2)$ matrix of the type described in the statement of Theorem 8.1. □

Proof of Theorem 8.3. We shall now apply Theorem 6.5 to the planar dual graph of $AD(a)$ with removed squares chosen to correspond to $\beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_n$. The right hand side of equation (6.2) is precisely the right hand side of equation (8.2). If δ_i and δ_j are of the same type then $AD(a) \setminus \{\delta_i, \delta_j\}$ does not have any tiling as the numbers of black and white squares in the checkerboard setting of an Aztec diamond will not be the same (see Figure 1.2). Finally, the proof is complete once we note that $AD(a) \setminus \{\beta_i, \alpha_j\}$ is an Aztec diamond with two defects removed from adjacent sides for any choice of β_i and α_j and is given by Proposition 7.7. □

We now illustrate Theorem 8.1 with the help of an example. Throughout this example, we will use the notations from previous sections without commentary.

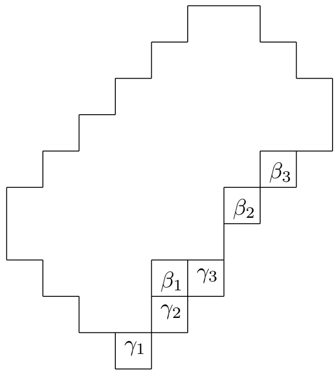


Figure 8.7: $\mathcal{AR}^3_{3,6}$ with the α, β, γ defects marked.

We consider the example in Figure 8.7 with $a = 3, b = 6, k = 3, n = 0$ and the set $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\} = \{\gamma_1, \gamma_2, \beta_1, \gamma_3, \beta_2, \beta_3\}$. Then by equation (8.1) we have

$$M(\mathcal{AR}_{3,6} \setminus \{\beta_1, \beta_2, \beta_3\}) = \frac{1}{[M(AD(3))]^2} \, \text{Pf}(A)$$

where the matrix A is

$$\begin{pmatrix} 0 & 0 & M(AD(3)) & 0 & M(\mathcal{AR}^{2,4}_{3,6}) & M(\mathcal{AR}^{2,5}_{3,6}) \\ 0 & 0 & M(AD(3)) & 0 & M(\mathcal{AR}^{1,3}_{3,5}) & M(\mathcal{AR}^{1,4}_{3,5}) \\ -M(AD(3)) & -M(AD(3)) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M(\mathcal{AR}_{3,4}(2)) & M(\mathcal{AR}_{3,4}(3)) \\ -M(\mathcal{AR}^{2,4}_{3,6}) & -M(\mathcal{AR}^{1,3}_{3,5}) & 0 & -M(\mathcal{AR}_{3,4}(2)) & 0 & 0 \\ -M(\mathcal{AR}^{2,5}_{3,6}) & -M(\mathcal{AR}^{2,4}_{3,5}) & 0 & -M(\mathcal{AR}_{2,3}(3)) & 0 & 0 \end{pmatrix}.$$

If we now calculate all the quantities that appear in the matrix A using the results mentioned in this paper, we shall get $\text{Pf}(A) = 3932160$ and hence

$$M(\mathcal{AR}_{3,6} \setminus \{\beta_1, \beta_2, \beta_3\}) = 960.$$

It is easy to see that this also agrees if we use Theorem 1.2 to calculate $M(\mathcal{AR}_{3,6} \setminus \{\beta_1, \beta_2, \beta_3\})$.

APPENDIX

A

EVALUATION OF SYMPLECTIC CHARACTERS

Symplectic characters can be interpreted as a generating function of weighted rhombus tilings of certain regions in the triangular lattice, which follows from the work of Robert A. Proctor [32] and Henry Cohn, Larsen and Propp [12] (see also the recent work of Ayyer and Fischer [4]). We shall use this description now to study the symplectic character that appears in (3.17) (as well as in other chapters of Part i).

We start with a quartered hexagon in the triangular lattice, which is obtained after cutting it along its axes, see Figure A.1. Let the top boundary of this quartered hexagon be of length ℓ and the right boundary of length $2n$. The left boundary consists of a zig-zag line containing n up-pointing and n down-pointing triangles, and the bottom boundary has protruded down-pointing triangles in positions $p_1 < p_2 < \cdots < p_n$, numbered from left to right, starting with a 1. We denote such a region by $QH_{2n,\ell}^{p_1,p_2,\ldots,p_n}$. We note that the bottom row triangles force some rhombus in any tiling, we shall remove such rhombi in the figures that follows.

We are interested in the rhombus tilings of these quartered hexagons. We assign a weight of x_i to each "left-oriented" rhombus (⏢) that appears in the i-th row of a tiling of $QH_{2n,\ell}^{p_1,p_2,\ldots,p_n}$, and let all other rhombi that appear in a tiling to have weight 1. This weighted region is denoted by

$$QH_{2n,\ell}^{p_1,p_2,\ldots,p_n}(x_1,x_2,\ldots,x_{2n})$$

and the generating function of its rhombus tilings by

$$M(QH_{2n,\ell}^{p_1,p_2,\ldots,p_n}(x_1,x_2,\ldots,x_{2n})).$$

The result which we will use is the following.

Proposition A.1 (Theorem 2.8, [4]). *For a partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ allowing also zero parts, we have*

$$Sp_{2n}(\lambda; x_1,x_2,\ldots,x_n) = M(QH_{2n,\lambda_1}^{\lambda_n+1,\lambda_{n-1}+2,\ldots,\lambda_1+n}(x_1,\bar{x}_1,x_2,\bar{x}_2,\ldots,x_n,\bar{x}_n)). \tag{A.1}$$

The quantity that we are interested in equation (3.17) is

$$\begin{aligned} &Sp_{4n+2}(n,n-1,n-1,\ldots,1,1,0,0;x^2,1,\ldots,1) \\ &\qquad = M(QH_{4n+2,n}^{1,2,4,5,\ldots,3n-2,3n-1,3n+1}(x^2,\bar{x}^2,1,\ldots,1)) \end{aligned} \tag{A.2}$$

For ease in the sequel, we use the following notation

$$QH_n(x) = QH_{4n+2,n}^{1,2,4,5,\ldots,3n-2,3n-1,3n+1}(x^2,\bar{x}^2,1,\ldots,1) \text{ and } QH_n = QH_n(1).$$

In general, we call something *unweighted*, if it has weight 1.

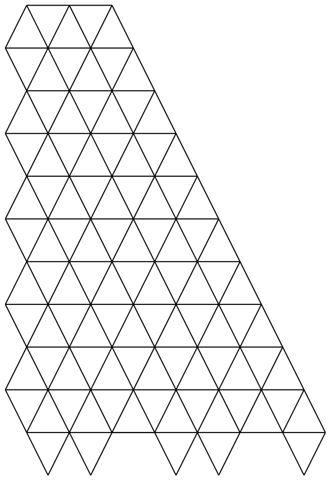


Figure A.1: $QH^{1,2,4,5,7}_{10,2}(x_1, x_2, \dots, x_{10})$.

Note that the weights in our case are manifest only in the first two rows of the quartered hexagon (i.e., one zig-zag portion of the hexagon). There is precisely one vertical rhombus (◊) in the first row (see Figure A.2), say in position j if counted from left and starting with 1, and the rhombi that appear to its right in the first row are all forced to be of the type ⏢, while no such rhombus appears left of the vertical rhombus in the first row. Thus, the first row contributes $x^{2(n+1-j)}$ to the weight. In the second row, all rhombi left of the bottom triangle of that fixed vertical rhombus are of type ⏢ and they contribute $x^{2(-j+1)}$ to the weight. Now, right of this vertical rhombus there is precisely one vertical rhombus that has its upper triangle in the second row, say in position i, and all rhombi right of this are of type ⏢, and they contribute $x^{2(-n-1+i)}$ to the weight. The total weight is then $x^{2(i-2j+1)}$.

We let $Q_{n,i}$ denote the number of tilings of QH_n with parameter i as described in the previous paragraph and letting j vary in $\{1, 2, \dots, i\}$. Equivalently, letting $QH_{n,i}$ denote the region obtained from QH_n by deleting the top two rows as well as the i-th down-pointing triangle in the (new) top row (see Figure A.3 for $QH_{n,i}$, the dotted lines are to be ignored at the moment), then $Q_{n,i}$ is the number of lozenge tilings of $QH_{n,i}$. Then, from the above discussion, we have

$$M(QH_n(x)) = \sum_{1 \leqslant j \leqslant i \leqslant n+1} Q_{n,i} x^{2i-4j+2}. \tag{A.3}$$

We shall now evaluate $Q_{n,i}$ using the Lindström-Gessel-Viennot [22, 29] technique, where the first step is to transform a rhombus tiling of the region into a family of non-intersecting lattice paths, and then evaluate the total number of such paths by means of a determinant.

We mark the left sides of the up-pointing triangles of the left boundary in our region by $s_1, s_2, \dots, s_{2n}$ from bottom to top, mark the right side of the deleted (black) down-pointing triangle in the top row by s_{2n+1} and mark the left sides of the deleted up-pointing triangles (by forcing) on the bottom boundary of our region by $e_1, e_2, \dots, e_{2n+1}$ from left to right. Any rhombus tiling of the region can be transformed into a family of non-intersecting lattice paths, where each path starts

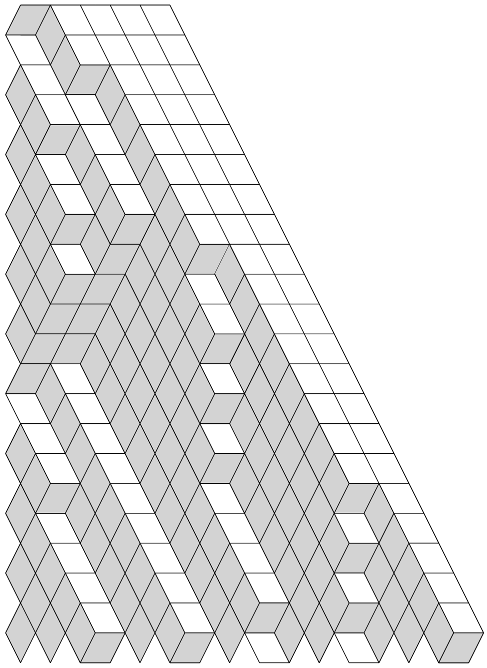


Figure A.2: A tiling of QH_5, where $k = i = 2$.

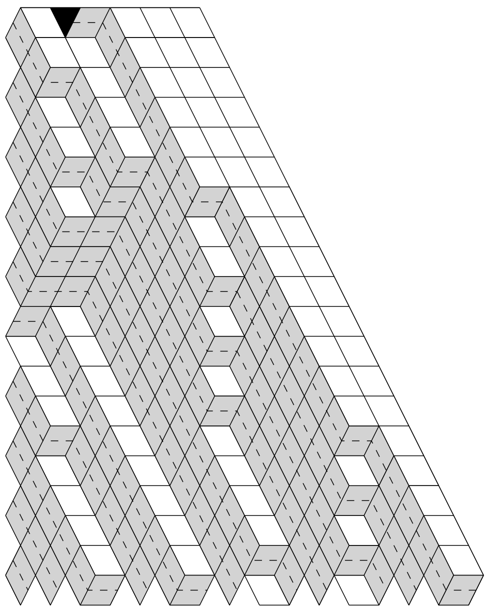


Figure A.3: A tiling of $QH_{5,2}$.

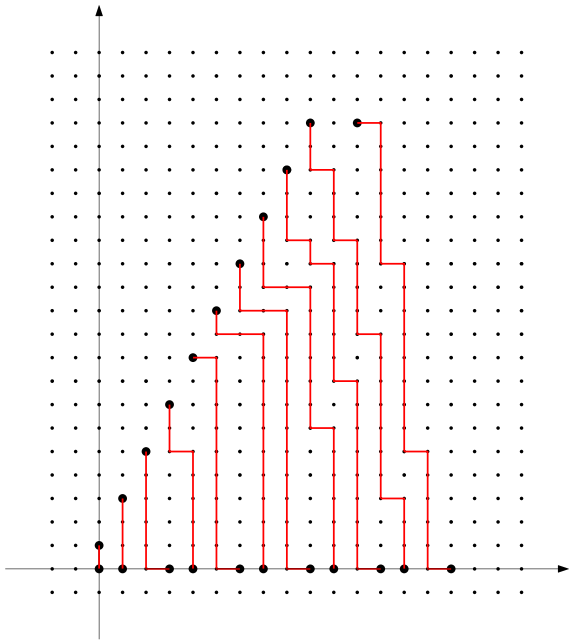


Figure A.4: The collection of non-intersecting lattice paths arising from the tiling of $QH_{5,2}$.

at s_j and ends in e_j. An example of this in shown in Figure A.3, where the lattice paths are given by the dotted lines.

We normalize the obtuse coordinate system in Figure A.3 and rotate it into the position shown in Figure A.4, where the points s_j have coordinates $(j-1, 2j-1)$ for $1 \leqslant j \leqslant 2n$, s_{2n+1} has coordinates $(2n-1+i, 4n-1)$ and e_j have coordinates $(\lfloor(3j-1)/2\rfloor - 1, 0)$, for $1 \leqslant j \leqslant 2n+1$. For ease of notation, we denote $\lfloor(3j-1)/2\rfloor$ by a_j. Now, we want to find the number of non-intersecting lattice paths beginning at s_j and ending at e_j, $1 \leqslant j \leqslant 2n+1$ with only down and east steps.

By the Lindström-Gessel-Viennot Lemma [22, 29], the number of such *unweighted* non-intersecting lattice paths is given by the determinant of the $(2n+1) \times (2n+1)$ matrix, whose (u, v)-th entry is the number $q_{u,v}$ of lattice paths from e_u to s_v. Thus, we have

$$q_{u,v} = \begin{cases} \dbinom{a_u + v - 1}{2v - 1} & 1 \leqslant u \leqslant 2n+1, 1 \leqslant v \leqslant 2n \\ \dbinom{a_u + 2n - 1 - i}{4n - 1} & 1 \leqslant u \leqslant 2n+1, v = 2n+1 \end{cases},$$

and

$$Q_{n,i} = \det(q_{u,v})_{1 \leqslant u,v \leqslant 2n+1}.$$

Expanding the determinant with respect to the $(2n+1)$-th column, we get

$$Q_{n,i} = \sum_{j=1}^{2n+1} (-1)^{j+1} D_{n,j} \binom{a_j + 2n - 1 - i}{4n - 1},$$

where

$$D_{n,j} = \det_{\substack{1 \leqslant u \leqslant 2n+1, u \neq j \\ 1 \leqslant v \leqslant 2n}} \left(\binom{a_u + v - 1}{2v - 1} \right).$$

Taking out common factors and then swapping the ν-th column with the $(2n-\nu+1)$-th column for every ν, we get that $D_{n,j}$ equals

$$(-1)^{n(2n-1)} \frac{\prod_{i=1}^{2n+1} a_i}{a_j \prod_{i=1}^{2n} (2i-1)!} \times \det_{\substack{1\leqslant i\leqslant 2n+1, i\neq j \\ 1\leqslant k\leqslant 2n}} ((a_i-2n+k)(a_i-2n+k+1)\dots(a_i-1)(a_i+1)\dots(a_i+2n-j)). \tag{A.4}$$

In order to evaluate this determinant, we need the following lemma, due to Krattenthaler.

Lemma A.2 (Lemma 4, [24]). *Let $x_1, x_2, \dots, x_r$ and $y_2, y_3, \dots, y_r$ be indeterminates, and c be a constant. Then we have*

$$\det\left(\prod_{k=j+1}^{r} (x_i - y_k - c)(x_i + y_k)\right)_{1\leqslant i,j\leqslant r} = \prod_{1\leqslant i<j\leqslant r} (x_j - x_i)(c - x_i - x_j).$$

We take $c = 0$, $x_i = a_i$ and $y_j = 2n - j + 1$ in Lemma A.2 and evaluate the expression in (A.4) to get

$$D_{n,j} = \frac{2\prod_{i=1}^{2n+1} a_i}{\prod_{i=1}^{2n}(2i-1)!} \frac{\prod_{1\leqslant p<q\leqslant 2n+1} (a_q - a_p)(a_p + a_q)}{\prod_{q=1}^{2n+1}(a_j + a_q) \prod_{q=j+1}^{2n+1}(a_q - a_j) \prod_{q=1}^{j-1}(a_j - a_q)}.$$

Therefore, we now have

$$\begin{aligned} M(QH_n(x)) &= \sum_{i=1}^{n+1} \left[\left(\frac{x^{4i}-1}{x^{2i-2}(x^4-1)} \right) \sum_{j=1}^{2n+1} (-1)^{j+1} D_{n,j} \binom{a_j + 2n - 1 - i}{4n-1} \right] \\ &= \sum_{1\leqslant j\leqslant i\leqslant n+1} Q_{n,i} x^{2i-4j+2}, \end{aligned} \tag{A.5}$$

where

$$Q_{n,i} = \frac{3^{n(n-1)}}{2^{n-1}(4n-1)!} \prod_{j=0}^{n-1} \frac{(4j+3)(6j+6)!}{(2n+2j+1)!} \sum_{j=0}^{n} \left[\frac{27^j (3j-2n-i+2)_{4n-3} (3n-3j+1)}{(3j)!(n-j)!(3j+1)_{3n}} \times \left(\frac{\left(n-j+\frac{4}{3}\right)_{2j} (2n+3j-i-1)_2}{(3n+3j+1)_2} - \frac{\left(n-j+\frac{2}{3}\right)_{2j} (-2n+3j-i)_2}{(3n-3j+1)_2} \right) \right]$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.

Remark 10. *If we do not delete the first two rows of* QH_n*, then taking a different family of non-intersecting lattice paths with a similar weighing scheme, rather than the ones used here, we would have arrived at the following generating function:*

$$\begin{aligned} &M(\mathrm{QH}_n(x)) \\ &= \det\Bigg(\left(x^2+\frac{1}{x^2}\right)\left\{\binom{4n}{2n-3j+i}-\binom{4n}{2n-3j-i}\right\}+\binom{4n}{2n-3j+i+1} \\ &\qquad -\binom{4n}{2n-3j-i-1}+\binom{4n}{2n-3j+i-1}-\binom{4n}{2n-3j-i+1}\Bigg)_{1\leqslant i,j\leqslant n}. \end{aligned}$$

BIBLIOGRAPHY

[1] Jean-Christophe Aval and Philippe Duchon. "Enumeration of alternating sign matrices of even size (quasi-)invariant under a quarter-turn rotation." In: *Electron. J. Combin.* 17.1 (2010), Research Paper 51, 20. ISSN: 1077-8926. URL: http://www.combinatorics.org/Volume_17/Abstracts/v17i1r51.html (cit. on pp. 49, 55).

[2] Arvind Ayyer and Roger E. Behrend. "Factorization theorems for classical group characters, with applications to alternating sign matrices and plane partitions." In: *J. Combin. Theory Ser. A* 165 (2019), pp. 78–105. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2019.01.001. URL: https://doi.org/10.1016/j.jcta.2019.01.001 (cit. on pp. 29, 32, 33, 46).

[3] Arvind Ayyer, Roger E. Behrend, and Ilse Fischer. "Extreme Diagonally and Antidiagonally Symmetric Alternating Sign Matrices of Odd Order." In: *arXiv:1611.03823v1 [math.CO]* (2016) (cit. on pp. 9, 39–41).

[4] Arvind Ayyer and Ilse Fischer. "Bijective proofs of skew Schur polynomial factorizations." In: *arXiv:1905.05226v1 [math.CO]* (2019) (cit. on p. 85).

[5] Arvind Ayyer and Dan Romik. "New enumeration formulas for alternating sign matrices and square ice partition functions." In: *Adv. Math.* 235 (2013), pp. 161–186. ISSN: 0001-8708. DOI: 10.1016/j.aim.2012.11.006. URL: https://doi.org/10.1016/j.aim.2012.11.006 (cit. on p. 4).

[6] Roger E. Behrend. "Multiply-refined enumeration of alternating sign matrices." In: *Adv. Math.* 245 (2013), pp. 439–499. ISSN: 0001-8708. URL: https://doi.org/10.1016/j.aim.2013.05.026 (cit. on p. 4).

[7] Roger E. Behrend, Ilse Fischer, and Matjaž Konvalinka. "Diagonally and antidiagonally symmetric alternating sign matrices of odd order." In: *Adv. Math.* 315 (2017), pp. 324–365. ISSN: 0001-8708. URL: https://doi.org/10.1016/j.aim.2017.05.014 (cit. on p. 4).

[8] David M. Bressoud. *Proofs and confirmations*. MAA Spectrum. The story of the alternating sign matrix conjecture. Mathematical Association of America, Washington, DC; Cambridge University Press, Cambridge, 1999, pp. xvi+274. ISBN: 0-521-66170-6; 0-521-66646-5. URL: http://links.jstor.org/sici?sici=0002-9890(200101)108:1<85:PACTSO>2.0.CO;2-6&origin=MSN (cit. on p. 3).

[9] Mihai Ciucu. "Enumeration of perfect matchings in graphs with reflective symmetry." In: *J. Combin. Theory Ser. A* 77.1 (1997), pp. 67–97. ISSN: 0097-3165. DOI: 10.1006/jcta.1996.2725. URL: https://doi.org/10.1006/jcta.1996.2725 (cit. on p. 6).

[10] Mihai Ciucu. "A generalization of Kuo condensation." In: *J. Combin. Theory Ser. A* 134 (2015), pp. 221–241. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2015.03.009. URL: http://dx.doi.org/10.1016/j.jcta.2015.03.009 (cit. on pp. 59, 61).

[11] Mihai Ciucu and Ilse Fischer. "Lozenge tilings of hexagons with arbitrary dents." In: *Adv. in Appl. Math.* 73 (2016), pp. 1–22. ISSN: 0196-8858. DOI: 10.1016/j.aam.2015.09.008. URL: http://dx.doi.org/10.1016/j.aam.2015.09.008 (cit. on pp. 65, 66, 76).

[12] Henry Cohn, Michael Larsen, and James Propp. "The shape of a typical boxed plane partition." In: *New York J. Math.* 4 (1998), pp. 137–165. ISSN: 1076-9803. URL: http://nyjm.albany.edu:8000/j/1998/4_137.html (cit. on p. 85).

[13] Philippe Duchon. "On the link pattern distribution of quarter-turn symmetric FPL configurations." In: *20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2008)*. Discrete Math. Theor. Comput. Sci. Proc., AJ. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2008, pp. 331–342 (cit. on pp. 9, 54, 55).

[14] Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp. "Alternating-sign matrices and domino tilings. I." In: *J. Algebraic Combin.* 1.2 (1992), pp. 111–132. ISSN: 0925-9899. DOI: 10.1023/A:1022420103267. URL: http://dx.doi.org/10.1023/A:1022420103267 (cit. on pp. 4, 6, 77).

[15] Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp. "Alternating-sign matrices and domino tilings. II." In: *J. Algebraic Combin.* 1.3 (1992), pp. 219–234. ISSN: 0925-9899. DOI: 10.1023/A:1022483817303. URL: http://dx.doi.org/10.1023/A:1022483817303 (cit. on pp. 4, 6).

[16] Ilse Fischer. "A new proof of the refined alternating sign matrix theorem." In: *J. Combin. Theory Ser. A* 114.2 (2007), pp. 253–264. ISSN: 0097-3165. URL: https://doi.org/10.1016/j.jcta.2006.04.004 (cit. on p. 3).

[17] Ilse Fischer. "An operator formula for the number of halved monotone triangles with prescribed bottom row." In: *J. Combin. Theory Ser. A* 116.3 (2009), pp. 515–538. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2008.05.013. URL: https://doi.org/10.1016/j.jcta.2008.05.013 (cit. on pp. 9, 16).

[18] Ilse Fischer. "Refined enumerations of alternating sign matrices: monotone (d, m)-trapezoids with prescribed top and bottom row." In: *J. Algebraic Combin.* 33.2 (2011), pp. 239–257. ISSN: 0925-9899. URL: https://doi.org/10.1007/s10801-010-0243-7 (cit. on p. 4).

[19] Ilse Fischer. "Short proof of the ASM theorem avoiding the six-vertex model." In: *J. Combin. Theory Ser. A* 144 (2016), pp. 139–156. ISSN: 0097-3165. URL: https://doi.org/10.1016/j.jcta.2016.06.007 (cit. on p. 3).

[20] Ilse Fischer and Dan Romik. "More refined enumerations of alternating sign matrices." In: *Adv. Math.* 222.6 (2009), pp. 2004–2035. ISSN: 0001-8708. URL: https://doi.org/10.1016/j.aim.2009.07.003 (cit. on p. 4).

[21] Ilse Fischer and Manjil P. Saikia. "Refined Enumeration of Symmetry classes of Alternating Sign Matrices." In: *arXiv:1906.07723 [math.CO]* (2019) (cit. on p. 11).

[22] Ira Gessel and Gérard Viennot. "Binomial determinants, paths, and hook length formulae." In: *Adv. in Math.* 58.3 (1985), pp. 300–321. ISSN: 0001-8708. DOI: 10.1016/0001-8708(85)90121-5. URL: https://doi.org/10.1016/0001-8708(85)90121-5 (cit. on pp. 86, 88).

[23] Matan Karklinsky and Dan Romik. "A formula for a doubly refined enumeration of alternating sign matrices." In: *Adv. in Appl. Math.* 45.1 (2010), pp. 28–35. ISSN: 0196-8858. URL: https://doi.org/10.1016/j.aam.2009.11.005 (cit. on p. 4).

[24] C. Krattenthaler. "Advanced determinant calculus." In: *Sém. Lothar. Combin.* 42 (1999). The Andrews Festschrift (Maratea, 1998), Art. B42q, 67. ISSN: 1286-4889 (cit. on p. 89).

[25] Eric H. Kuo. "Applications of graphical condensation for enumerating matchings and tilings." In: *Theoret. Comput. Sci.* 319.1-3 (2004), pp. 29–57. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2004.02.022. URL: http://dx.doi.org/10.1016/j.tcs.2004.02.022 (cit. on pp. 59, 60).

[26] Eric H. Kuo. "Graphical Condensation Generalizations Involving Pfaffians and Determinants." In: *arXiv:math 0605154 [math.CO]* (2006) (cit. on p. 59).

[27] Greg Kuperberg. "Another proof of the alternating-sign matrix conjecture." In: *Internat. Math. Res. Notices* 3 (1996), pp. 139–150. ISSN: 1073-7928. URL: https://doi.org/10.1155/S1073792896000128 (cit. on pp. 3, 13, 14).

[28] Greg Kuperberg. "Symmetry classes of alternating-sign matrices under one roof." In: *Ann. of Math. (2)* 156.3 (2002), pp. 835–866. ISSN: 0003-486X. DOI: 10.2307/3597283. URL: https://doi.org/10.2307/3597283 (cit. on pp. 4, 15, 17, 20, 24, 34, 43, 49, 51).

[29] Bernt Lindström. "On the vector representations of induced matroids." In: *Bull. London Math. Soc.* 5 (1973), pp. 85–90. ISSN: 0024-6093. DOI: 10.1112/blms/5.1.85. URL: https://doi.org/10.1112/blms/5.1.85 (cit. on pp. 86, 88).

[30] W. H. Mills, David P. Robbins, and Howard Rumsey Jr. "Alternating sign matrices and descending plane partitions." In: *J. Combin. Theory Ser. A* 34.3 (1983), pp. 340–359. ISSN: 0097-3165. URL: https://doi.org/10.1016/0097-3165(83)90068-7 (cit. on pp. 3–5).

[31] Soichi Okada. "Enumeration of symmetry classes of alternating sign matrices and characters of classical groups." In: *J. Algebraic Combin.* 23.1 (2006), pp. 43–69. ISSN: 0925-9899. DOI: 10.1007/s10801-006-6028-3. URL: https://doi.org/10.1007/s10801-006-6028-3 (cit. on pp. 4, 19, 20, 22, 27, 32, 34, 36, 43, 46, 52).

[32] Robert A. Proctor. "Young tableaux, Gel~fand patterns, and branching rules for classical groups." In: *J. Algebra* 164.2 (1994), pp. 299–360. ISSN: 0021-8693. DOI: 10.1006/jabr.1994.1064. URL: https://doi.org/10.1006/jabr.1994.1064 (cit. on p. 85).

[33] A. V. Razumov and Yu. G. Stroganov. "On a detailed enumeration of some symmetry classes of alternating-sign matrices." In: *Teoret. Mat. Fiz.* 141.3 (2004), pp. 323–347. ISSN: 0564-6162. DOI: 10.1023/B:TAMP.0000049757.07267.9d. URL: https://doi.org/10.1023/B:TAMP.0000049757.07267.9d (cit. on pp. 4, 20).

[34] A. V. Razumov and Yu. G. Stroganov. "Bethe roots and refined enumeration of alternating-sign matrices." In: *J. Stat. Mech. Theory Exp.* 7 (2006), P07004, 12. ISSN: 1742-5468 (cit. on pp. 4, 15).

[35] A. V. Razumov and Yu. G. Stroganov. "Enumeration of odd-order alternating-sign half-turn-symmetric matrices." In: *Teoret. Mat. Fiz.* 148.3 (2006), pp. 357–386. ISSN: 0564-6162. URL: https://doi.org/10.1007/s11232-006-0111-8 (cit. on pp. 4, 51, 53).

[36] A. V. Razumov and Yu. G. Stroganov. "Enumeration of odd-order alternating-sign quarter-turn symmetric matrices." In: *Teoret. Mat. Fiz.* 149.3 (2006), pp. 395–408. ISSN: 0564-6162. URL: https://doi.org/10.1007/s11232-006-0148-8 (cit. on pp. 4, 49, 53, 54).

[37] David P. Robbins. "The story of $1, 2, 7, 42, 429, 7436, \cdots$." In: *Math. Intelligencer* 13.2 (1991), pp. 12–19. ISSN: 0343-6993. DOI: 10.1007/BF03024081. URL: http://dx.doi.org/10.1007/BF03024081 (cit. on pp. 3, 7, 77).

[38] David P. Robbins. "Symmetry classes of alternating sign matrices." In: *arXiv:math 0008045 [math.CO]* (2000) (cit. on pp. 3, 9, 49).

[39] Manjil P. Saikia. "Enumeration of domino tilings of an Aztec rectangle with boundary defects." In: *Adv. in Appl. Math.* 89 (2017), pp. 41–66. ISSN: 0196-8858. DOI: 10.1016/j.aam.2017.04.002. URL: https://doi.org/10.1016/j.aam.2017.04.002 (cit. on p. 57).

[40] Richard P. Stanley. "A baker's dozen of conjectures concerning plane partitions." In: *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*. Vol. 1234. Lecture Notes in Math. Springer, Berlin, 1986, pp. 285–293. URL: https://doi.org/10.1007/BFb0072521 (cit. on p. 3).

[41] Yu. G. Stroganov. "The Izergin-Korepin determinant at a cube root of unity." In: *Teoret. Mat. Fiz.* 146.1 (2006), pp. 65–76. ISSN: 0564-6162. URL: https://doi.org/10.1007/s11232-006-0006-8 (cit. on pp. 4, 51).

[42] Osamu Tsuchiya. "Determinant formula for the six-vertex model with reflecting end." In: *J. Math. Phys.* 39.11 (1998), pp. 5946–5951. ISSN: 0022-2488. DOI: 10.1063/1.532606. URL: https://doi.org/10.1063/1.532606 (cit. on p. 17).

[43] Doron Zeilberger. "Proof of the alternating sign matrix conjecture." In: *Electron. J. Combin.* 3.2 (1996). The Foata Festschrift, Research Paper 13, approx. 84. ISSN: 1077-8926. URL: http://www.combinatorics.org/Volume_3/Abstracts/v3i2r13.html (cit. on p. 3).

[44] Doron Zeilberger. "Proof of the refined alternating sign matrix conjecture." In: *New York J. Math.* 2 (1996), 59–68, electronic. ISSN: 1076-9803. URL: http://nyjm.albany.edu:8000/j/1996/2_59.html (cit. on pp. 4, 51).

INDEX