Parity biases in partitions and restricted partitions

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Abstract. Let $p_o(n)$ (resp. $p_e(n)$) denote the number of partitions of n with more odd
parts (resp. even parts) than even parts (resp. odd parts). Recently, Kim, Kim, and
Lovejoy proved that $p_o(n) > p_e(n)$ for all $n > 2$ and conjectured that $d_o(n) > d_e(n)$
for all $n > 19$, where $d_o(n)$ (resp. $d_e(n)$) denotes the number of partitions into distinct
parts having more odd parts (resp. even parts) than even parts (resp. odd parts).
In this paper we provide combinatorial proofs for both the result and the conjecture
of Kim, Kim and Lovejoy. In addition, we discuss other results on partitions with
restricted parts where our methods also work.

Keywords: Combinatorial inequalities, Partitions, Parity of parts, Restricted Partitions

1 Introduction

In the theory of partitions, inequalities arising between two classes of partitions have a
long tradition of study, for instance Alder's conjecture [1] and the Ehrenpreis problem [2]
are the most famous examples in this direction. In recent years there have been a number
of results about partition inequalities. Very recently, Kim, Kim, and Lovejoy [4] have
given interesting inequalities which show biases in the parity of the partition functions.
Proofs of such results employ a wide range of techniques, ranging from q-series methods,
to combinatorial constructions and maps, to classical asymptotic analysis.

Let $p_o(n)$ (resp. $p_e(n)$) denote the number of partitions of n with more odd parts
(resp. even parts) than even parts (resp. odd parts). Kim, Kim, and Lovejoy [4] proved
that $p_o(n) > p_e(n)$ for all $n > 2$ and conjectured that $d_o(n) > d_e(n)$ for all $n > 19$ where
$d_o(n)$ (resp. $d_e(n)$) denotes the number of partitions into distinct parts having more odd
parts (resp. even parts) than even parts (resp. odd parts). The primary goal of the
present paper is to prove these two inequalities combinatorially.

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We define a partition λ of a non-negative integer n to be an integer sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. We say that λ is a partition of n, denoted by $\lambda \vdash n$ and $\sum_{i=1}^{\ell} \lambda_i = n$. The set of partitions of n is denoted by $P(n)$ and $|P(n)| = p(n)$. For $\lambda \vdash n$, we define $a(\lambda)$ to be the largest part of λ, $\ell(\lambda)$ to be the total number of parts of λ and $\mathrm{mult}_\lambda(\lambda_i) := m_i$ to be the multiplicity of the part λ_i in λ. We also use $\lambda = (\lambda_1^{m_1} \ldots \lambda_\ell^{m_\ell})$ as an alternative notation for a partition. For $\lambda \vdash n$ with $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu \vdash m$ with $\mu = (\mu_1, \ldots, \mu_{\ell'})$, define the union $\lambda \cup \mu \vdash m + n$ to be the partition with parts $\{\lambda_i, \mu_j\}$ arranged in non-increasing order. For a partition $\lambda \vdash n$, we split λ into λ_e and λ_o, respectively into even and odd parts; i.e., $\lambda = \lambda_e \cup \lambda_o$. We denote by $\ell_e(\lambda)$ (resp. $\ell_o(\lambda)$) the number of even parts (resp. odd parts) of λ and $\ell(\lambda) = \ell_e(\lambda) + \ell_o(\lambda)$.

The following sets of partitions are of interest in this paper.

Definition 1.1.

$$
\begin{aligned}
D(n) &:= \{\lambda \in P(n) : \mathrm{mult}_\lambda(\lambda_i) = 1 \text{ for all } i\}, \\
P_e(n) &:= \{\lambda \in P(n) : \ell_e(\lambda) > \ell_o(\lambda)\}, \\
P_o(n) &:= \{\lambda \in P(n) : \ell_o(\lambda) > \ell_e(\lambda)\}, \\
D_e(n) &:= P_e(n) \cap D(n), \quad \text{and} \\
D_o(n) &:= P_o(n) \cap D(n).
\end{aligned}
$$

Definition 1.2. *For all the sets defined above, their cardinalities will be denoted by lower case letters. For instance, $|P_e(n)| = p_e(n)$, $|D_e(n)| = d_e(n)$, and so on.*

Now, we state formally the main results proved in this paper.

Theorem 1.3 (Theorem 1, [4]). *For all positive integers $n \neq 2$, we have*

$$p_o(n) > p_e(n).$$

Theorem 1.4 (Conjectured, [4]). *For all positive integers $n > 19$, we have*

$$d_o(n) > d_e(n).$$

Before we move on further, let us describe the fundamental principle behind the proofs. Let X and Y be two given sets, our goal is to prove that $|Y| > |X|$. We choose a subset $X_0 \,(\subsetneq X)$ and define an injective map $f : X_0 \to Y$. Then to prove $|Y| > |X|$, it is enough to find a suitable subset $Y_0 \subsetneq Y \setminus f(X_0)$ such that $|Y_0| > |X \setminus X_0|$. Throughout this paper, we follow the notation $x \mapsto y$ instead of writing $f(x) = y$ when the map f is understood from the context.

The rest of the paper is organized as follows: in Section 2 we give a combinatorial proof of the result of Kim, Kim and Lovejoy [4], in Section 3 we give a proof of the conjecture of Kim, Kim and Lovejoy [4], Finally in Section 4 we present a very short discussion on further results that can be proved using our methods.

2 Proof of Theorem 1.3

We begin by presenting the following lemma, used later in the proof of Theorem 1.3. We skip the proof, as it is routine and can be found in the full version of the paper [3].

Lemma 2.1. *For all even positive integers n with $n \geq 14$, we have*

$$\sum_{k=1}^{\frac{n-6}{2}} \left\lfloor \frac{n-2k-2}{4} \right\rfloor > 1 + \sum_{k=1}^{\lfloor \frac{n-2}{6} \rfloor} \left\lfloor \frac{n-6k+2}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-6}{6} \rfloor} \left\lfloor \frac{n-6k-2}{4} \right\rfloor.$$

For all odd positive integers n with $n \geq 11$, we have

$$\sum_{k=1}^{\frac{n-5}{2}} \left\lfloor \frac{n-2k-1}{4} \right\rfloor > 1 + \left\lfloor \frac{n-4}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-5}{6} \rfloor} \left\lfloor \frac{n-6k-1}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-9}{6} \rfloor} \left\lfloor \frac{n-6k-5}{4} \right\rfloor.$$

Let

$$G_e^0(n) := \{\lambda \in P_e(n) : \ell_e(\lambda) - \ell_o(\lambda) = 1 \text{ and } a(\lambda) \equiv 0 \pmod 2\},$$
$$\overline{G_e^0}(n) := \{\lambda \in G_e^0(n) : \lambda_3 \geq 3\},$$
$$G_e^1(n) := \{\lambda \in P_e(n) : \ell_e(\lambda) - \ell_o(\lambda) = 1 \text{ and } a(\lambda) \equiv 1 \pmod 2\},$$
$$G_e^2(n) := \{\lambda \in P_e(n) : \ell_e(\lambda) - \ell_o(\lambda) \geq 2\},$$
$$\text{and } G_e(n) := G_e^1(n) \cup G_e^2(n).$$

We split the set $G_e(n)$ into the parity of length of partition as $G_e(n) = G_{e,0}(n) \cup G_{e,1}(n)$ with $G_{e,i}(n) = \{\lambda \in G_e(n) : \ell(\lambda) \equiv i \pmod 2\}$ $(i = 0, 1)$ and let $\overline{G_e}(n) := G_{e,0}(n) \cup G_{e,1}(n) \cup \overline{G_e^0}(n)$. Therefore,

$$P_e(n) \setminus \overline{G_e}(n) = \{\lambda \in G_e^0(n) : 0 \leq \lambda_3 \leq 2\}. \tag{2.1}$$

We construct a map $f : \overline{G_e}(n) \to P_o(n)$ by defining maps $f|_{G_{e,0}(n)} = f_1$, $f|_{G_{e,1}(n)} = f_2$ and $f|_{\overline{G_e^0}(n)} = f_3$ such that $\{f_i\}_{1 \leq i \leq 3}$ are injective with the following properties: $f_1(G_{e,0}(n)) \cap f_2(G_{e,1}(n)) = f_1(G_{e,0}(n)) \cap f_3(\overline{G_e^0}(n)) = f_2(G_{e,1}(n)) \cap f_3(\overline{G_e^0}(n)) = \varnothing$, so as to conclude that the map f is injective. Then we will choose a subset $\overline{P_o}(n) \subsetneq P_o(n) \setminus f(\overline{G_e}(n))$ with $|\overline{P_o}(n)| > |P_e(n) \setminus \overline{G_e}(n)|$.

Let $\lambda \in G_{e,0}(n)$ with $\lambda_e = (\lambda_{e_1}, \ldots, \lambda_{e_k})$ and $\lambda_o = (\lambda_{o_1}, \ldots, \lambda_{o_m})$ where $k + m = \ell(\lambda)$. Since $\lambda \in G_{e,0}(n)$, $\ell(\lambda) = 2r$ for some $r \in \mathbb{Z}_{>0}$ and $k > r$ because, $k - m \geq 1$ implies $2k \geq k + m + 1 = 2r + 1$.

We define $f_1 : G_{e,0}(n) \to P_o(n)$ by $f_1(\lambda) := \mu$ with

$$\mu_e = ((\lambda_{o_1} + 1), \ldots, (\lambda_{o_m} + 1))$$

and
$$\mu_o = ((\lambda_{e_1}+1),\ldots,(\lambda_{e_{k-r}}+1),(\lambda_{e_{k-r+1}}-1),\ldots,(\lambda_{e_k}-1)).$$
Here we note that $\mu \in P(n)$ and f_1 reverses the parity of parts; i.e., for λ with k even and m odd parts, we get $f_1(\lambda) = \mu$ with k odd and m even parts and $\mu \in P_o(n)$. Suppose for $\lambda' \neq \lambda''(\in G_{e,0}(n))$ with $\ell(\lambda') = \ell(\lambda'')$, we have $\mu' = f_1(\lambda') = f_1(\lambda'') = \mu''$. Then $\ell_e(\lambda') = \ell_e(\lambda'')$ and so, $\ell_o(\lambda') = \ell_o(\lambda'')$. Now, since λ' and λ'' are distinct, by the definition of f_1 we have at least a tuple $(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that $\mu'_i \neq \mu''_j$. Next, we consider the case when $\lambda' \neq \lambda''(\in G_{e,0}(n))$ with $\ell(\lambda') \neq \ell(\lambda'')$ and it is immediate that $\ell(\mu') \neq \ell(\mu'')$ and therefore, $\mu' \neq \mu''$. So, f_1 is an injective map.

For $\lambda \in G_{e,1}(n)$ with $\lambda_e = (\lambda_{e_1},\ldots,\lambda_{e_k})$ and $\lambda_o = (\lambda_{o_1},\ldots,\lambda_{o_m})$ where $k+m = \ell(\lambda) = 2r+1$ for some $r \in \mathbb{Z}_{>0}$, we note that, $k > r$ and $k - m \geq 3$. But $k - m = 1$ holds only when $a(\lambda)$ is odd, because $k = m+1$ and $a(\lambda)$ is even implies that $\lambda \in G_e^0(n)$. We exclude the condition $k = m+2$ as it contradicts that $k+m = 2r+1$.

We define $f_2 : G_{e,1}(n) \to P_o(n)$ with $f_2(\lambda) := \mu$, where

$$\mu_e = \begin{cases} ((\lambda_{o_1}+1),\ldots,(\lambda_{o_m}+1)) \cup (\lambda_{e_1}+2) & \text{if } a(\lambda) \text{ is even,} \\ ((\lambda_{o_2}+1),\ldots,(\lambda_{o_m}+1)) & \text{if } a(\lambda) \text{ is odd,} \end{cases} \tag{2.2}$$

and

$$\mu_o = \begin{cases} ((\lambda_{e_2}+1),\ldots,(\lambda_{e_{k-r-1}}+1),(\lambda_{e_{k-r}}-1),\ldots,(\lambda_{e_k}-1)) & \text{if } a(\lambda) \text{ is even,} \\ ((\lambda_{e_1}+1),\ldots,(\lambda_{e_{k-r-1}}+1),(\lambda_{e_{k-r}}-1),\ldots,(\lambda_{e_k}-1)) \cup (\lambda_{o_1}+2) & \text{if } a(\lambda) \text{ is odd.} \end{cases} \tag{2.3}$$

For $a(\lambda)$ even, $\ell_o(\mu) - \ell_e(\mu) = k-1-(m+1) = k-m-2 \geq 1$ and for $a(\lambda)$ odd, $\ell_o(\mu) - \ell_e(\mu) = k+1-(m-1) = k-m+2 \geq 3$. Hence, $\mu \in P_o(n)$ and by a similar argument as given above, one can show that f_2 is injective.

Next, for $\lambda \in \overline{G_e^0}(n)$ with $\lambda = (\lambda_1,\ldots,\lambda_\ell)$, we define $f_3 : \overline{G_e^0}(n) \to P_o(n)$ by

$$f_3(\lambda) = \mu = ((\lambda_1+1),\lambda_4,\ldots,\lambda_\ell) \cup ((\lambda_2-2),(\lambda_3-2)) \cup (2,1).$$

Independent of whether λ_2 and λ_3 are odd or even, we can observe that $\ell_o(\mu) - \ell_e(\mu) = 1$ and $a(\mu) = \lambda_1 + 1$ is odd. By definition, f_3 is an injective map.

We now show that the images of $\{f_i\}_{1 \leq i \leq 3}$ are mutually disjoint by considering the following cases

1. By definition of the maps given above, $f_1(G_{e,0}(n)) \subsetneq P_o^0(n)$ where
$$P_o^0(n) := \{\mu \in P_o(n) : \ell(\mu) \equiv 0 \pmod 2\}$$
and $f_2(G_{e,1}(n)) \subsetneq P_o^1(n)$ with
$$P_o^1(n) := \{\mu \in P_o(n) : \ell(\mu) \equiv 1 \pmod 2\}.$$
So, $f_1(G_{e,0}(n)) \cap f_2(G_{e,1}(n)) = \emptyset$.

2. For $\lambda \in G_{e,0}(n)$ with $\ell_e(\lambda) - \ell_o(\lambda) \geq 2$, we have $f_1(\lambda) = \mu \in P_o(n)$ with $\ell_o(\mu) - \ell_e(\mu) \geq 2$ and for $\lambda \in G_{e,0}(n)$ with $\ell_e(\lambda) - \ell_o(\lambda) = 1$, $f_1(\lambda) = \mu \in P_o(n)$ with $\ell_o(\mu) - \ell_e(\mu) = 1$ but then $a(\mu) = \lambda_1 + 1$ is even. Considering $\lambda \in \overline{G_e^0}(n)$, $f_3(\lambda) = \mu \in P_o(n)$ with $a(\mu) = \mu_1$ is odd and $\ell_o(\mu) - \ell_e(\mu) = 1$. Therefore, $f_1(G_{e,0}(n)) \cap f_3(\overline{G_e^0}(n)) = \emptyset$.

3. Let us consider $\lambda \in G_{e,1}(n)$ with $a(\lambda)$ even and $\ell_e(\lambda) - \ell_o(\lambda) = 3$. Then $f_2(\lambda) = \mu \in P_o(n)$ with $a(\mu)$ even and $\ell_o(\mu) - \ell_e(\mu) = 1$. For $\ell_e(\lambda) - \ell_o(\lambda) \geq 4$, we have $f_2(\lambda) = \mu$ with $\ell_o(\mu) - \ell_e(\mu) \geq 2$. Moreover, if $a(\lambda)$ is odd then it is immediate that $\ell_o(\mu) - \ell_e(\mu) \geq 3$ and consequently, $f_2(G_{e,1}(n)) \cap f_3(\overline{G_e^0}(n)) = \emptyset$. So, the map $f : \overline{G_e}(n) \to P_o(n)$ is injective.

For $\mu \in P_o(n)$ with its odd component $\mu_o = (\mu_{o_1}, \ldots, \mu_{o_s})$, we define

$$\overline{P_o}(n) := \{\mu \in P_o(n) : \ell_e(\mu) = 2 \text{ and } \mu_{o_i} = 1 \text{ for all } 1 \leq i \leq s\}.$$

By the definition of f, it is clear that $\overline{P_o}(n) \subsetneq P_o(n) \setminus f(\overline{G_e}(n))$. Now, it remains to show that $|\overline{P_o}(n)| > |P_e(n) \setminus \overline{G_e}(n)|$.

For n even and for $\lambda \in \overline{P_o}(n)$, we have $\ell_o(\lambda) = 2k + 2$ for some $k \in \mathbb{Z}_{>0}$. Here we observe that

$$|\{\lambda \in P(n) : \lambda_1 + \lambda_2 = n; \lambda_1, \lambda_2 \text{ both even}\}| = \left\lfloor \frac{n}{4} \right\rfloor \tag{2.4}$$

and

$$|\{\lambda \in P(n) : \lambda_1 + \lambda_2 = n; \lambda_1 \text{ even}, \lambda_2 \text{ odd and } \lambda_2 \in \mathbb{Z}_{\geq 3}\}| = \left\lfloor \frac{n-3}{4} \right\rfloor. \tag{2.5}$$

Since, $\lambda \in \overline{P_o}(n)$ with n an even positive integer and $\ell_o(\lambda) = 2k + 2$, for each $k \in \mathbb{Z}_{>0}$, then by (2.4),

$$|\{\lambda \in \overline{P_o}(n) : \lambda_1 + \lambda_2 + (2k + 2) \times 1 = n; \lambda_1, \lambda_2 \text{ both even}\}| = \left\lfloor \frac{n - 2k - 2}{4} \right\rfloor \tag{2.6}$$

and $1 \leq k \leq \dfrac{n-6}{2}$ because k maximizes only when both λ_1 and λ_2 minimizes; i.e., only the instance $2 + 2 + (2k + 2) \times 1 = n$ which implies $k = \dfrac{n-6}{2}$. Therefore we have,

$$|\overline{P_o}(n)| = \sum_{k=1}^{\frac{n-6}{2}} \left\lfloor \frac{n - 2k - 2}{4} \right\rfloor. \tag{2.7}$$

Similarly, for n odd, we have $\ell_o(\lambda) = 2k + 1$ with $1 \leq k \leq \dfrac{n-5}{2}$ and

$$|\{\lambda \in \overline{P_o}(n) : \lambda_1 + \lambda_2 + (2k + 1) \times 1 = n; \lambda_1, \lambda_2 \text{ even}\}| = \left\lfloor \frac{n - 2k - 1}{4} \right\rfloor. \tag{2.8}$$

Consequently,

$$|\overline{P_o}(n)| = \sum_{k=1}^{\frac{n-5}{2}} \left\lfloor \frac{n-2k-1}{4} \right\rfloor. \tag{2.9}$$

Now for n even we will show that

$$|P_e(n) \setminus \overline{G_e}(n)| = 1 + \sum_{k=1}^{\lfloor \frac{n-2}{6} \rfloor} \left\lfloor \frac{n-6k+2}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-6}{6} \rfloor} \left\lfloor \frac{n-6k-2}{4} \right\rfloor. \tag{2.10}$$

We interpret the set $P_e(n) \setminus \overline{G_e}(n)$ as a disjoint union of its three proper subsets given by $P_e(n) \setminus \overline{G_e}(n) = A_1 \cup A_2 \cup A_3$ where,

$$A_1 = \{\lambda \in P_e(n) \setminus \overline{G_e}(n) : 0 \le \lambda_3 \le 1\}, \quad A_2 = \bigcup_{k \ge 1} A_{2,k}, \quad \text{and} \quad A_3 = \bigcup_{k \ge 1} A_{3,k},$$

with

$$A_{2,k} = \{\lambda = (\lambda_1, \lambda_2, 2, \dots, 2, 1, \dots, 1) \vdash n : \lambda_1 \text{ and } \lambda_2 \text{ even,}$$
$$\text{mult}_\lambda(2) = 2k - 1, \text{mult}_\lambda(1) = 2k\},$$

and

$$A_{3,k} = \{\lambda = (\lambda_1, \lambda_2, 2, \dots, 2, 1, \dots, 1) \vdash n : \lambda_1 \text{ even}, \lambda_2 \text{ odd,}$$
$$\text{mult}_\lambda(2) = 2k, \text{mult}_\lambda(1) = 2k - 1\}.$$

Next, we explicitly describe the sets and derive their cardinality by separating into three cases.

Case 1(E): We observe that $|A_1| = 1$ because we have only one possibility $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 0, 0)$. We reject the other three possibilities; i.e., $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 0, 1)$ as $\lambda_2 \ge \lambda_3$, $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 1, 0)$ as n even and $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 1, 1)$ as $\lambda \in P_e(n) \setminus \overline{G_e}(n)$. Next, we look at the subset of A_1, say $A_{1, \ge 2} := \{\lambda \in A_1 : \lambda_2 \ge 2\}$ and note that $A_{1, \ge 2} = \varnothing$. This is because for $\lambda \in A_{1, \ge 2}$, there are altogether four possibilities for $\lambda_3 \in \{0, 1\}$.

For $\lambda_3 = 0$, the choice $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, \lambda_2, 0)$ and λ_2 is even is impossible as $\lambda \in P_e(n) \setminus \overline{G_e}(n)$ and if λ_2 is odd, is again an impossible option since n is even. Whereas for $\lambda_3 = 1$, the choice $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, \lambda_2, 1)$ and λ_2 is even is impossible as n is even and if λ_2 is odd, again an impossible option since $\lambda \in P_e(n) \setminus \overline{G_e}(n)$.

Case 2(E): By (2.4),

$$|A_{2,k}| = \left\lfloor \frac{n-6k+2}{4} \right\rfloor \tag{2.11}$$

and $1 \le k \le \lfloor \frac{n-2}{6} \rfloor$ because k maximizes only when both λ_1 and λ_2 minimizes; i.e., the instance $2 + 2 + (2k - 1) \times 2 + (2k) \times 1 = n$ which implies $k \le \lfloor \frac{n-2}{6} \rfloor$. By (2.11),

$$A_2 = \bigcup_{k=1}^{\lfloor \frac{n-2}{6} \rfloor} A_{2,k} \text{ and } |A_2| = \sum_{k=1}^{\lfloor \frac{n-2}{6} \rfloor} \left\lfloor \frac{n-6k+2}{4} \right\rfloor. \tag{2.12}$$

Case 3(E): From (2.5), it follows that

$$|A_{3,k}| = \left\lfloor \frac{(n-6k+1)-3}{4} \right\rfloor = \left\lfloor \frac{n-6k-2}{4} \right\rfloor \tag{2.13}$$

and $1 \leq k \leq \lfloor \frac{n-6}{6} \rfloor$ because k maximizes only when both λ_1 and λ_2 minimizes; i.e., the instance $4 + 3 + (2k) \times 2 + (2k-1) \times 1 = n$ which implies $k \leq \lfloor \frac{n-6}{6} \rfloor$. By (2.13),

$$A_3 = \bigcup_{k=1}^{\lfloor \frac{n-6}{6} \rfloor} A_{3,k} \text{ and } |A_3| = \sum_{k=1}^{\lfloor \frac{n-6}{6} \rfloor} \left\lfloor \frac{n-6k-2}{4} \right\rfloor. \tag{2.14}$$

By *Case 1(E)*, (2.12) and (2.14) we have (2.10).

For all odd integers, n with $n \geq 9$, we will show that

$$|P_e(n) \setminus \overline{G_e}(n)| = 1 + \left\lfloor \frac{n-4}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-5}{6} \rfloor} \left\lfloor \frac{n-6k-1}{4} \right\rfloor + \sum_{k=1}^{\lfloor \frac{n-9}{6} \rfloor} \left\lfloor \frac{n-6k-5}{4} \right\rfloor. \tag{2.15}$$

Similarly as before, we write $P_e(n) \setminus \overline{G_e}(n)$ as a disjoint union of its four proper subsets given by $P_e(n) \setminus \overline{G_e}(n) = B_0 \cup B_1 \cup B_2 \cup B_3$ where,

$$B_0 = \{\lambda = (\lambda_1, \lambda_2, 1) \in P_e(n) \setminus \overline{G_e}(n) : \lambda_2 \geq 4\}$$
$$B_1 = \{\lambda \in P_e(n) \setminus \overline{G_e}(n) : 0 \leq \lambda_2 \leq 2 \text{ and } 0 \leq \lambda_3 \leq 1\}$$
$$B_2 = \bigcup_{k \geq 1} B_{2,k}, \quad \text{and} \quad B_3 = \bigcup_{k \geq 1} B_{3,k},$$

with

$$B_{2,k} = \{\lambda = (\lambda_1, \lambda_2, 2, \ldots, 2, 1, \ldots, 1) \vdash n : \lambda_1 \text{ and } \lambda_2 \text{ even,}$$
$$\text{mult}_\lambda(2) = 2k, \text{mult}_\lambda(1) = 2k+1\},$$

$$B_{3,k} = \{\lambda = (\lambda_1, \lambda_2, 2, \ldots, 2, 1, \ldots, 1) \vdash n : \lambda_1 \text{ even}, \lambda_2 \text{ odd,}$$
$$\text{mult}_\lambda(2) = 2k+1, \text{mult}_\lambda(1) = 2k\}.$$

Case 1(O): For $\lambda = (\lambda_1, \lambda_2, 1) \in B_0$ and n odd, it follows that both λ_1 and λ_2 are even. Therefore minimal choice for n is 9, otherwise $\lambda_1 \geq \lambda_2 \geq 4$ with the constraint that both λ_1 and λ_2 even would be an impossibility. Moreover, we can observe that

$$|B_0| = \left\lfloor \frac{n-4}{4} \right\rfloor. \tag{2.16}$$

Case 2(O): We observe that $|B_1| = 1$ because we have only one possibility $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 2, 1)$. We reject the other three possibilities; i.e., $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 0, 1)$ as $\lambda_2 \geq \lambda_3$,

$(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 0, 0)$ and $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 2, 0)$ as n odd, $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 1, 0)$ and $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 1, 1)$ as $\lambda \in P_e(n) \setminus \overline{G_e}(n)$.

Case 3(O): By (2.4),

$$|B_{2,k}| = \left\lfloor \frac{n - 6k - 1}{4} \right\rfloor \tag{2.17}$$

and $1 \leq k \leq \lfloor \frac{n-5}{6} \rfloor$ because k maximizes only when both λ_1 and λ_2 minimizes; i.e., the instance $2 + 2 + (2k) \times 2 + (2k + 1) \times 1 = n$ which implies $k \leq \lfloor \frac{n-5}{6} \rfloor$. By (2.17),

$$B_2 = \bigcup_{k=1}^{\lfloor \frac{n-5}{6} \rfloor} B_{2,k} \text{ and } |B_2| = \sum_{k=1}^{\lfloor \frac{n-5}{6} \rfloor} \left\lfloor \frac{n - 6k - 1}{4} \right\rfloor. \tag{2.18}$$

Case 4(O): From (2.5), it follows that

$$|B_{3,k}| = \left\lfloor \frac{(n - 6k - 2) - 3}{4} \right\rfloor = \left\lfloor \frac{n - 6k - 5}{4} \right\rfloor \tag{2.19}$$

and $1 \leq k \leq \lfloor \frac{n-9}{6} \rfloor$ because k maximizes only when both λ_1 and λ_2 minimizes; i.e., the instance $4 + 3 + (2k + 1) \times 2 + (2k) \times 1 = n$ which implies $k \leq \lfloor \frac{n-9}{6} \rfloor$. By (2.19),

$$B_3 = \bigcup_{k=1}^{\lfloor \frac{n-9}{6} \rfloor} B_{3,k} \text{ and } |B_3| = \sum_{k=1}^{\lfloor \frac{n-9}{6} \rfloor} \left\lfloor \frac{n - 6k - 5}{4} \right\rfloor. \tag{2.20}$$

By *Case 2(O)*, (2.16), (2.18) and (2.20), we have (2.15).

Therefore, by Lemma 2.1, $|\overline{P_o}(n)| > |P_e(n) \setminus \overline{G_e}(n)|$ for all $n \geq \mathbb{Z}_{\geq 14} \cup \{11, 13\}$. To conclude the proof, it remains to check $n \in \{1, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$ which we did by numerically checking in Mathematica.

3   Proof of Theorem 1.4

Following the definitions in Section 2, set

$$H_e^0(n) := G_e^0(n) \cap D(n),$$
$$\overline{H_e^0}(n) := \{\lambda \in H_e^0(n) : \ell_o(\lambda) > 1\},$$
$$\text{and } H_e(n) := G_e(n) \cap D(n).$$

We split $H_e(n)$ into $H_{e,0}(n) = G_{e,0}(n) \cap D(n)$ and $H_{e,1}(n) = G_{e,1}(n) \cap D(n)$. Similarly, define the map $f : H_e(n) \to D_o(n)$ by $f|_{H_{e,0}(n)} = f_1$ and $f|_{H_{e,1}(n)} = f_2$. Since $H_e(n) \subsetneq G_e(n)$, we conclude that the map f is injective by (2.2) and (2.3).

Now we show that $d_o(n) - |f(H_e(n))| > d_e(n) - |H_e(n)|$ for all $n > 31$. The subset $D_o(n) \setminus f(H_e(n))$ contains different classes of partitions, one of which is

$$\overline{D_o}(n) := \{\lambda \in D_o(n) \setminus f(H_e(n)) : \ell_o(\lambda) - \ell_e(\lambda) = 1 \text{ and } a(\lambda) \equiv 1 \pmod{2}\}.$$

We note that $\overline{D_o}(n)$ may contain other classes of partitions depending on whether n is even or odd.

For a partition $\lambda \in \overline{H_e^0}(n)$, we split λ into its even component $\lambda_e = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_{m+1}})$ and odd component $\lambda_o = (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_m})$ for some $m \in \mathbb{Z}_{\geq 2}$. Now, me make a transformation of λ into λ^* with

$$\lambda^* = (\lambda_{e_1} + \lambda_{o_1}, \lambda_{e_2} + \lambda_{o_2}) \cup (\lambda_{e_3}, \lambda_{e_4}, \ldots, \lambda_{e_{m+1}}) \cup (\lambda_{o_3}, \lambda_{o_4}, \ldots, \lambda_{o_m}) \in \overline{D_o}(n).$$

We observe that two partitions, say $\lambda, \overline{\lambda} \in \overline{H_e^0}(n)$, where

$$\lambda = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_{m+1}}) \cup (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_m})$$
$$\text{and } \overline{\lambda} = (\overline{\lambda}_{e_1}, \overline{\lambda}_{e_2}, \ldots, \overline{\lambda}_{e_{m+1}}) \cup (\overline{\lambda}_{o_1}, \overline{\lambda}_{o_2}, \ldots, \overline{\lambda}_{o_m}),$$

transform to a same partition, say $\mu \in \overline{D_o}(n)$ if and only if

$$\lambda_{e_1} - \overline{\lambda}_{e_1} = \overline{\lambda}_{o_1} - \lambda_{o_1} \equiv 0 \pmod 2 \text{ or/and } \lambda_{e_2} - \overline{\lambda}_{e_2} = \overline{\lambda}_{o_2} - \lambda_{o_2} \equiv 0 \pmod 2.$$

If those cases arise we subtract some multiple of 2 from the greatest part of the resultant partition and add the multiple of 2 to the other even parts which are present in the partition, and continue this process till we have a repetition among the parts of the partition. This process is injective by its definition, and we denote it by g. For example, consider partitions $\lambda = (12, 10, 6, 2) \cup (7, 3, 1)$ and $\overline{\lambda} = (10, 8, 6, 2) \cup (9, 5, 1)$ in $\overline{H_e^0}(41)$, then both λ and $\overline{\lambda}$ maps to the same partition $\mu = (19, 13, 6, 2, 1) \in \overline{D_o}(41)$. Consequently, by the process g, finally $\lambda \mapsto (19, 13, 6, 2, 1)$, whereas $\overline{\lambda} \mapsto (17, 13, 8, 2, 1)$. As a trivial remark, $\overline{H_e^0}(n) = \varnothing$ for all positive even integers $n \leq 14$, since $6 + 4 + 2 + 3 + 1 = 16$ is the least possible option.

Depending on the parity of n, it remains to analyze the left over set

$$\widetilde{H_e^0}(n) := \{\lambda \in H_e^0(n) : \ell_e(\lambda) - \ell_o(\lambda) = 1, \ell_o(\lambda) \leq 1 \text{ and } a(\lambda) \equiv 0 (\text{mod } 2)\}, \qquad (3.1)$$

which is unmapped yet (after applying the map f and g).

For n to be an even positive integer, we observe that $\widetilde{H_e^0}(n)$ consists of only one partition (n). An even integer n can be expressed as a sum of two consecutive odd integers if and only if n is divisible by 4. If n is divisible by 4, then for some definite odd integer λ_{o_1} we get $(\lambda_{o_1}, \lambda_{o_1} - 2) \in D_o(n)$, which is not mapped yet. So we map (n) to $(\lambda_{o_1}, \lambda_{o_1} - 2)$. If n is not divisible by 4, then for some definite odd integer λ_{o_1} we get $(\lambda_{o_1}, \lambda_{o_1} - 2, 2) \in D_o(n)$, which is not mapped yet. So in this case we map (n) to $(\lambda_{o_1}, \lambda_{o_1} - 2, 2)$. Therefore, by some elementary observations we get that the theorem is true for all even integer $n > 6$.

Let n be odd. We rewrite (3.1) as

$$\widetilde{H_e^0}(n) := \{\lambda \in H_e^0(n) : \ell_e(\lambda) = 2, \ell_o(\lambda) = 1 \text{ and } a(\lambda) \equiv 0 (\text{mod } 2)\}.$$

Write a partition $\lambda \in \widetilde{H_e^0}(n)$ into its even component $\lambda_e = (\lambda_{e_1}, \lambda_{e_2})$ and odd component $\lambda_o = (\lambda_{o_1})$. We split $\widetilde{H_e^0}(n)$ into following three classes:

1. $\widetilde{H}^0_{e,1}(n) := \{\lambda \in \widetilde{H}^0_e(n) : \lambda_{e_2} = 2\}$,

2. $\widetilde{H}^0_{e,2}(n) := \{\lambda \in \widetilde{H}^0_e(n) : \lambda_{e_2} \geq 6\}$, and

3. $\widetilde{H}^0_{e,3}(n) := \{\lambda \in \widetilde{H}^0_e(n) : \lambda_{e_2} = 4\}$.

Now we consider the following three classes of partitions from the set of partitions, say $\widetilde{D}_o(n) \subsetneq D_o(n)$ which have no preimage yet:

1. $\widetilde{D}_{o,1}(n) := \{\pi \in \widetilde{D}_o(n) : \ell(\pi) = 4 \text{ and } \pi_{o_1} - \pi_{o_2} = 2\}$,

2. $\widetilde{D}_{o,2}(n) := \{\pi \in \widetilde{D}_o(n) : \ell_o(\pi) = 3 \text{ and } \pi_{o_1} - \pi_{o_2} = 2\}$, and

3. $\widetilde{D}_{o,3}(n) := \{\pi \in \widetilde{D}_o(n) : \ell_o(\pi) - \ell_e(\pi) = 1, a(\pi) \equiv 0 (\text{mod } 2) \text{ and } \pi_{e_1} - \pi_{o_1} = 1 \text{ or } 3\}$.

Now we construct an injective map from $\widetilde{H}^0_{e,1}(n)$ to $\widetilde{D}_{o,1}(n)$. Let $\lambda = (\lambda_{e_1}, 2) \cup (\lambda_{o_1}) \in \widetilde{H}^0_{e,1}(n)$. Define a transformation S such that $S(\lambda) = (\lambda_{e_1}) \cup (\lambda_{o_1} + 1, 1)$. Now define S^* such that

$$S^*(S(\lambda)) = \begin{cases} S(\lambda) & \text{if } \lambda_{e_1} \equiv 0 (\text{mod } 4), \\ (\lambda_{e_1} - 2) \cup (\lambda_{o_1} + 1, 3) & \text{if } \lambda_{e_1} \equiv 2 (\text{mod } 4). \end{cases}$$

Now define S^{**} such that $S^{**}(S^*(S(\lambda))) = (\lambda_{o_2}, \lambda_{o_2} - 2) \cup (\lambda_{o_3}) \cup (\lambda_{o_1} + 1)$, where $\lambda_{o_2} + \lambda_{o_2} - 2 = \lambda_{e_1}$ or $\lambda_{e_1} - 2$, and $\lambda_{o_3} = 1$ or 3 accordingly. For example: $(24, 5, 2)$ maps to $(13, 11, 6, 1)$ and $(22, 7, 2)$ maps to $(11, 9, 8, 3)$. This process is injective.

Our next objective is to embed the set $\widetilde{H}^0_{e,2}(n)$ into a subset of $\widetilde{D}_o(n)$ which is not mapped till now. Define a transformation U such that $U(\lambda) = ((\lambda_{e_1-3}, 3) \cup (\lambda_{o_1})) \cup (\lambda_{e_2}, 2)$ for $\lambda \in \widetilde{H}^0_{e,2}(n)$. Associated with U, let us define U^* in such a way that

$$U^*(U(\lambda)) = \begin{cases} U(\lambda) & \text{if } \lambda_{o_1} \neq 3, \lambda_{e_1} - 3 \neq \lambda_{o_1}, \\ (\lambda_{e_1} - 3, 5, 1) \cup (\lambda_{e_2} - 2, 2) & \text{if } \lambda_{o_1} = 3, \\ (\text{this transformation is impossible for } n \leq 17) \\ (\text{e.g. } (10, 6, 3) \mapsto (7, 5, 4, 2, 1)) \\ ((\lambda_{e_1} - 3, \lambda_{o_1} - 2) \cup (5)) \cup (\lambda_{e_2} - 2, 2) & \text{if } \lambda_{e_1} - 3 = \lambda_{o_1}, \\ (\text{this transformation is impossible for } n \leq 25) \\ (\text{e.g. } (12, 9, 6) \mapsto (9, 7, 5, 4, 2)) \\ ((\lambda_{e_1} - 3, \lambda_{o_1} - 4) \cup (5)) \cup (\lambda_{e_2} - 2, 4) & \text{if } \lambda_{e_1} - 1 = \lambda_{o_1}, \lambda_{e_2} \neq 6, \\ (\text{this transformation is impossible for } n \leq 29) \\ (\text{e.g. } (12, 11, 8) \mapsto (9, 7, 6, 5, 4)) \\ ((\lambda_{e_1} - 3, \lambda_{o_1} - 4) \cup (3)) \cup (6, 4) & \text{if } \lambda_{e_1} - 1 = \lambda_{o_1}, \lambda_{e_2} = 6. \\ (\text{this transformation is impossible for } n \leq 23) \\ (\text{e.g. } (10, 9, 6) \mapsto (7, 6, 5, 4, 3)) \end{cases}$$

Denote the resulting transformation U^*U by $\widetilde{U}$. Note that $\ell(\widetilde{U}(\lambda)) = 5$, where the resulting partitions; i.e., images under $\widetilde{U}$, contains parts from $\{3,5\}$ and its smallest even part from $\{2,4\}$. For a partition $\lambda \in H_e^0(n)$ with $\ell(\lambda) = 7$, $\ell(g(\lambda)) = 5$. Now, assume $\widetilde{U}(\lambda) = g(\mu)$ for some partition μ with $\ell(\mu) = 7$. In the map g, after applying the first transformation, the other transformations are applied on the even parts only (if it is necessary). If $\widetilde{U}(\lambda) = g(\mu)$, then we remove one of 2, 3, 4, or 5 (which may exist in $g(\mu)$) from $g(\mu)$ by applying a similar transformation.

Now we compare the number of partitions in $\widetilde{H}_{e,3}^0(n)$ with $\widetilde{D}_{o,2}(n)$ and $\widetilde{D}_{o,3}(n)$,

$$\left|\widetilde{H}_{e,3}^0(n)\right| = \left\lfloor \frac{n-3}{4} \right\rfloor.$$

Let $\pi = (\pi_{o_1}, \pi_{o_1} - 2, \pi_{o_3}) \in \widetilde{D}_{o,2}(n)$, and $\pi_{o_1} = 2k+1$. Then the least possible value of k is given by $(2k+1) + (2k-1) \geq 2\{n - (2k+1) - (2k-1)\} + 6$, which implies $k = \left\lceil \frac{n+3}{6} \right\rceil$. So the total number of partitions in $\widetilde{D}_{o,2}(n)$ is at least $\left\lceil \frac{n-4k}{4} \right\rceil$, which is equal to $\left\lceil \frac{n}{4} - \left\lceil \frac{n+3}{6} \right\rceil \right\rceil$. Any odd integer n is of the form $12\ell + r$, where $r = 1, 3, 5, 7, 9$, or 11. Calculating for all the six cases, we get the total number of partitions in this class is

$$= \begin{cases} \left\lfloor \frac{n}{12} \right\rfloor, & \text{if } n \neq 12\ell + 9, \\ \left\lfloor \frac{n}{12} \right\rfloor + 1, & \text{if } n = 12\ell + 9. \end{cases}$$

Now, $\left\lfloor \frac{n-3}{4} \right\rfloor - \begin{cases} \left\lfloor \frac{n}{12} \right\rfloor, & \text{if } n \neq 12\ell + 9, \\ \left\lfloor \frac{n}{12} \right\rfloor + 1, & \text{if } n = 12\ell + 9. \end{cases} = \left\lfloor \frac{n}{6} \right\rfloor - 1, \left\lfloor \frac{n}{6} \right\rfloor, \text{ or } \left\lfloor \frac{n}{6} \right\rfloor + 1.$

It remains to estimate a lower bound of $\left|\widetilde{D}_{o,3}(n)\right|$. Let $\pi = (\pi_{o_1}, \pi_{o_2}, \pi_{o_3}) \cup (\pi_{e_1}, \pi_{e_2}) \in \widetilde{D}_{o,3}(n)$. The greatest odd part π_{o_1} is the largest possible value if π contains 3, 1 and 2. So the largest possible value of π_{o_1} is $\dfrac{n-9}{2}$ or $\dfrac{n-7}{2}$ if $n \equiv 3 \pmod 4$ or $n \equiv 1 \pmod 4$ respectively. The smallest possible value of π_{o_1} is greater than $\left\lfloor \frac{n}{5} \right\rfloor$. When π_{o_1} is not the largest or the smallest possible value, then for each possible value of π_{o_1}, we get at least 6 partitions. So total number of partitions is greater than

$$6 \times \frac{1}{2} \times \left\{ \frac{n-9}{2} - 1 - \left(\left\lfloor \frac{n}{5} \right\rfloor + 1 + 1 \right) \right\} \geq 3 \times \left\{ \frac{n-9}{2} - \frac{n}{5} - 3 \right\} = \frac{9(n-25)}{10}.$$

Now, $\left\lfloor \frac{9(n-25)}{10} \right\rfloor > \left\lfloor \frac{n}{6} \right\rfloor + 1$ for all odd positive integers $n > 31$.

This proves the theorem for all $n > 31$. We can verify the result numerically for $19 < n \leq 31$.

4   Other Results

Our method can be amended to prove other results where biases in parity are found for restricted partitions. If $q_o(n)$ (resp. $q_e(n)$) denotes the number of partitions of n with

more odd parts (resp. even parts) than even parts (resp. odd parts) where the smallest part is at least 2, then we have the following theorem.

Theorem 4.1. *For all positive integers $n > 7$, we have*

$$q_o(n) < q_e(n).$$

These parity biases seems to also occur for more restricted partition functions. For a nonempty set $S \subsetneq \mathbb{Z}_{\geq 0}$, define $P_e^S(n) := \{\lambda \in P_e(n) : \lambda_i \notin S\}$ and $P_o^S(n) := \{\lambda \in P_o(n) : \lambda_i \notin S\}$. Consequently, denote the number of partitions in $P_e^S(n)$ (resp. $P_o^S(n)$) by $p_e^S(n)$ (resp. $p_o^S(n)$). The above definition leads us to the following results that describes not only the parity of parts but also its arithmetic by putting a constraint on its support.

Theorem 4.2. *For all $n \geq 1$ we have*

$$p_o^{\{2\}}(n) > p_e^{\{2\}}(n).$$

Theorem 4.3. *If $S = \{1, 2\}$, then for all integers $n > 8$, we have*

$$p_o^S(n) > p_e^S(n).$$

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