EXTENSIONS OF SOME RESULTS OF VLADETA AND DHAR

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Abstract. We look at extensions of formulas given by Vladeta and recently proved by Dhar on integer partitions where the smallest part occurs at least m times and on integer partitions with fixed differences between the largest and smallest parts where the smallest part occurs at least k times. Our results extend Dhar's results for the $m=2$ and $k=1$ cases to the general cases for arbitrary m and k. We also look at analogous results for overpartitions and ℓ-regular partitions.

1. Introduction

A partition λ of n is a non-increasing sequence of natural numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, such that $\sum_{i=1}^{k} \lambda_i = n$, where λ_i's are called the parts of the partition and k is called the length of the partition. Here, λ_1 and λ_k are called the largest and smallest parts of the partition respectively. It is customary to denote by $p(n)$ the number of partitions of n. Several interesting statistics on ordinary partitions and other generalizations have been studied since a long time. Andrews [And98] wrote the definitive book on partitions to which we refer the reader for more details.

Recently, Dhar [Dha21] studied the statistic $a_m(n)$ (A117989), which counts the number of partitions of n where the smallest part occurs at least m times, for $m=2$. He proved the following proposition using both analytic and combinatorial methods.

Proposition 1.1 (Formula 1, [Dha21])**.** *For all natural numbers n, we have*

$$a_2(n) = 2p(n) - p(n+1).$$

He further proved the following result combinatorially.

Proposition 1.2 (Formula 2, [Dha21])**.** *For all natural numbers n, we have*

$$a_2(n) = p(2n, n),$$

where $p(m,n)$ is the number of partitions of m with fixed difference between the largest and smallest parts equal to n.

The function $p(n,t)$ was studied by Andrews, Beck and Robbins [ABR15] who gave a generating function for it.

In this paper we present two generalizations of Proposition 1.1 to $a_m(n)$, from which Proposition 1.1 follows as corollaries. The first generalization is proved combinatorially and is given below.

Theorem 1.3. *For all $n \geq 1$ and $m \geq 2$, we have*

$$a_m(n) = 2p(n) - p(n+1) - p(n-2) + p(n-m) - \sum_{\ell=2}^{m-1} \sum_{k=3}^{\left\lfloor \frac{n}{\ell} \right\rfloor + 1} \mathcal{Q}_{\ell,k}(n),$$

where $\mathcal{Q}_{\ell,k}(n)$ is the number of partitions of $n - \ell(k-1)$ with smallest part k.

Remark 1.4. *Clearly Proposition 1.1 is a corollary of this result.*

Remark 1.5. *All empty sums are taken to be* 0 *and all empty products are taken to be* 1 *in this paper.*

The second generalization is proved analytically (Theorem 3.3), and we wait until Section 3 to state and prove it in its full generality. This generalization answers the question that Dhar [Dha21] asked in his paper about a closed form of the generating function of $a_m(n)$. We state only two simple cases of the result.

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Theorem 1.6. *We have*

$$a_3(n) = 3p(n) - p(n+1) - 2p(n+2) + p(n+3),$$

for all natural numbers n.

Theorem 1.7. *We have*

$$a_4(n) = 4p(n) - p(n+1) - 2p(n+2) - 2p(n+3) + p(n+4) + 2p(n+5) - p(n+6),$$

for all natural numbers n.

With regards to Proposition 1.2, if we denote by $a_m(n,k)$ the total number of partitions of n where the smallest part occurs at least m times and the difference between the largest and smallest parts is k, then the following result is obvious (if the smallest part is k and the largest part is $n+k$ for a partition counted by $a_{m-1}(2n,n)$, then just remove the part $n+k$ and add a part of size k to get a partition counted by $a_m(n)$).

Proposition 1.8. *For all natural number n and $m \geq 2$ we have,*

$$a_m(n) = a_{m-1}(2n,n).$$

Remark 1.9. *Clearly $a_1(2n,n) = p(2n,n)$ and hence Proposition 1.2 follows as a corollary of Proposition 1.8.*

In Section 4 we will find the generating function of $a_m(n,\ell)$ for $\ell > 1$. Specializing $m = 1$ will give us the main result of Andrews, Beck and Robbins [ABR15].

After the work of Andrews, Beck and Robbins [ABR15], other authors such as Chern [Che17], Chern and Yee [CY18] and Lin and Xheng [LZ21] looked at similar results for other classes of partitions, such as overpartitions and ℓ-regular partitions. In a similar spirit, all of the statistics defined in this section can also be suitably modified and defined for other classes of partitions, such as overpartitions, ℓ-regular partitions, etc. We discuss these briefly towards the end of the paper. The paper is organized as follows: in Section 2 we prove Theorem 1.3, in Section 3 we prove Theorems 1.6 and 1.7 and generalize them for $a_m(n)$, in Section 4 we find the generating function of $a_m(n,\ell)$ for $\ell > 1$, in Section 5 we look at analogous results for overpartitions and ℓ-regular partitions, and finally we end the paper with some concluding remarks and questions in Section 6.

2. Proof of Theorem 1.3

We define the following sets

$$P(n) := \{\lambda | \lambda \text{ is a partitions of } n\},$$

and

$$A_m(n) := \{\lambda | \lambda \in P(n), \text{ the smallest part occurs at least } m \text{ times}\}.$$

The set $P(n) - A_3(n)$ contains all partitions of n in which the least part occurs exactly once or twice. Now we have three cases.

Case 1. (The least part occurs exactly once) Here we have $p(n+1) - p(n)$ partitions. (Although the proof is clear for this assertion, the interested reader can see Section 4 of Dhar's paper for more details [Dha21].)

Case 2. (The least parts are 1 and 1) If we remove the 1's, then they become the partitions of $n-2$ in which there is no 1 as a part. So, here we have $p(n-2) - p(n-3)$ partitions.

Case 3. (The least parts are greater than 1) Let the least parts of a partition be $k-1$ and $k-1$. If we remove both the parts, then the partition of n becomes a partition of $n - 2(k-1)$ with the least part greater than or equal to k. If the least part of such partitions of n is $k-1$, then here we have $\mathcal{Q}_k(n)$ partitions of $n - 2(k-1)$, where $\mathcal{Q}_k(n)$ is the number of partitions of $n - 2(k-1)$ with lowest part at least k. Therefore,

$$p(n) - a_3(n) = p(n+1) - p(n) + p(n-2) - p(n-3) + \sum_{k=3}^{\lfloor \frac{n}{2} \rfloor + 1} \mathcal{Q}_k(n).$$

Hence,

$$a_3(n) = 2p(n) - p(n+1) - p(n-2) + p(n-3) - \sum_{k=3}^{\lfloor \frac{n}{2} \rfloor + 1} \mathcal{Q}_k(n).$$

For $a_4(n)$ two more cases arise.

Case 4. (The least parts are $1, 1,$ **and** 1**)** Similar to Case 2, here we have $p(n-3) - p(n-4)$ partitions.

Case 5. (There are exactly three least parts and they are greater than 1**)** Here we have $\mathcal{Q}_{3,k}(n)$ partitions, corresponding to the partitions of n with the least parts $k-1(>1)$, where $\mathcal{Q}_{\ell,k}(n)$ denote the number of partitions of $n - \ell(k-1)$ with smallest part k. Therefore,

$$p(n) - a_4(n) = p(n+1) - p(n) + p(n-2) - p(n-3) + p(n-3) - p(n-4) + \sum_{k=3}^{\lfloor \frac{n}{2} \rfloor + 1} \mathcal{Q}_k(n) + \sum_{k=3}^{\lfloor \frac{n}{3} \rfloor + 1} \mathcal{Q}_{3,k}(n).$$

Hence,

$$a_4(n) = 2p(n) - p(n+1) - p(n-2) + p(n-4) - \sum_{k=3}^{\lfloor \frac{n}{2} \rfloor + 1} \mathcal{Q}_{2,k}(n) - \sum_{k=3}^{\lfloor \frac{n}{3} \rfloor + 1} \mathcal{Q}_{3,k}(n).$$

Proceeding in this way we get Theorem 1.3.

3. Generating Function of $a_m(n)$

In this section we generalize Proposition 1.1 by finding an explicit form of the generating function of $a_m(n)$. We use the standard notations

$$(a)_n = (a;q)_n := \prod_{i=o}^{n-1} (1 - aq^i),$$

and

$$(a)_\infty = (a;q)_\infty := \lim_{n \to \infty} (a;q)_n.$$

Before proceeding further, we state two results which we will need in our analysis.

Proposition 3.1 (Cauchy's identity, [And98], p.17, Theorem 2.1)**.** *If* $|q| < 1, |t| < 1,$ *then*

$$\sum_{k=0}^{\infty} \frac{(a)_k t^k}{(q)_k} = \frac{(at)_\infty}{(t)_\infty}. \tag{3.1}$$

By replacing $a = 0$ in (3.1), we get

$$\sum_{k=0}^{\infty} \frac{t^k}{(q)_k} = \frac{1}{(t)_\infty}. \tag{3.2}$$

We will use this identity by replacing $t = q, q^2$, and so on, without commentary.

Proposition 3.2 (Heine's transformation, [And98], p.19, Corollary 2.3)**.** *If* $|q|, |t|, |b| < 1,$ *then*

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k t^k}{(q)_k (c)_k} = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{k=0}^{\infty} \frac{(c/b)_k (t)_k b^k}{(q)_k (at)_k}.$$

The generating function of $a_m(n)$ is given by

$$\begin{aligned}
\sum_{n=1}^{\infty} a_m(n)q^n &= \sum_{k=1}^{\infty} q^{\overbrace{k+k+\cdots+k}^{m}}(1+q^k+q^{2k}+\cdots)(1+q^{k+1}+q^{2(k+1)}+\cdots)\cdots\\
&= \sum_{k=1}^{\infty}\frac{q^{mk}}{(q^k)_\infty} = \frac{1}{(q)_\infty}\sum_{k=1}^{\infty}(q)_{k-1}q^{mk} = \frac{q^m}{(q)_\infty}\sum_{k=0}^{\infty}(q)_k q^{mk} = \frac{q^m}{(q)_\infty}\sum_{k=0}^{\infty}\frac{(q)_k(q)_k q^{mk}}{(q)_k}\\
&= \frac{q^m(q^{m+1})_\infty}{(q^m)_\infty}\sum_{k=0}^{\infty}\frac{(q^m)_k q^k}{(q)_k(q^{m+1})_k}\\
&\qquad \text{(by replacing } a=q, b=q, c=0, \text{ and } t=q^m \text{ in Heine's transformation)}\\
&= \sum_{k=0}^{\infty}\frac{q^{k+m}}{(q)_k(1-q^{k+m})}.
\end{aligned}$$

Thus, we get

$$\sum_{n=1}^{\infty} a_m(n)q^n = \sum_{k=0}^{\infty}\frac{q^{k+m}}{(q)_{k+m}}(1-q^{k+1})(1-q^{k+2})\cdots(1-q^{k+m-1}). \tag{3.3}$$

We are now ready to prove Theorems 1.6 and 1.7.

Proof of Theorem 1.6. Replacing $m=3$ in equation (3.3), we get

$$\begin{aligned}
\sum_{n=1}^{\infty} a_3(n)q^n &= \sum_{k=0}^{\infty}\frac{q^{k+3}}{(q)_{k+3}}(1-q^{k+1})(1-q^{k+2}) = \sum_{k=0}^{\infty}\frac{q^{k+3}}{(q)_{k+3}}(1-(q^{k+1}+q^{k+2})+q^{k+1+k+2})\\
&= \sum_{k=0}^{\infty}\frac{q^{k+3}}{(q)_{k+3}} - \left(\frac{1+q}{q^2}\right)\sum_{k=0}^{\infty}\frac{(q^2)^{k+3}}{(q)_{k+3}} + \frac{1}{q^{2+1}}\sum_{k=0}^{\infty}\frac{(q^3)^{k+3}}{(q)_{k+3}}\\
&= \frac{1}{(q)_\infty} - \frac{q^2}{(1-q)(1-q^2)} - \frac{q}{1-q} - 1\\
&\quad - \left(\frac{1+q}{q^2}\right)\left(\frac{1}{(q^2)_\infty} - \frac{q^4}{(1-q)(1-q^2)} - \frac{q^2}{1-q} - 1\right)\\
&\quad\quad + \frac{1}{q^{2+1}}\left(\frac{1}{(q^3)_\infty} - \frac{q^6}{(1-q)(1-q^2)} - \frac{q^3}{1-q} - 1\right)\\
&= \frac{1}{(q)_\infty} - \frac{1+q}{q^2}\cdot\frac{1-q}{(q)_\infty} + \frac{1}{q^{2+1}}\cdot\frac{(1-q)(1-q^2)}{(q)_\infty} - \frac{q^3-q^2-q+1}{q^3}\\
&= \left(3-\frac{1}{q}-\frac{2}{q^2}+\frac{1}{q^3}\right)\frac{1}{(q)_\infty} - \frac{q^3-q^2-q+1}{q^3}\\
&= \frac{3}{(q)_\infty} - \frac{1}{q(q)_\infty} - \frac{2}{q^2(q)_\infty} + \frac{1}{q^3(q)_\infty} - \frac{q^3-q^2-q+1}{q^3}.
\end{aligned}$$

Clearly the last quantity is the generating function of $3p(n)-p(n+1)-2p(n+2)+p(n+3)$ for $n\geq 1$. This proves the result. $\square$

Proof of Theorem 1.7. Replacing $m = 4$ in equation (3.3) and following a similar process as in the proof of Theorem 1.6, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_4(n)q^n \\
=&\frac{1}{(q)_\infty} - \frac{q^3}{(1-q)(1-q^2)(1-q^3)} - \frac{q^2}{(1-q)(1-q^2)} - \frac{q}{1-q} - 1 \\
&- \left(\frac{1+q+q^2}{q^3}\right) \times \left(\frac{1}{(q^2)_\infty} - \frac{q^6}{(1-q)(1-q^2)(1-q^3)} - \frac{q^4}{(1-q)(1-q^2)} - \frac{q^2}{1-q} - 1\right) \\
&+ \left(\frac{1+q+q^2}{q^{3+2}}\right) \times \left(\frac{1}{(q^3)_\infty} - \frac{q^9}{(1-q)(1-q^2)(1-q^3)} - \frac{q^6}{(1-q)(1-q^2)} - \frac{q^3}{1-q} - 1\right) \\
&- \frac{1}{q^{3+2+1}} \times \left(\frac{1}{(q^4)_\infty} - \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} - \frac{q^8}{(1-q)(1-q^2)} - \frac{q^4}{1-q} - 1\right) \\
=&\frac{1}{(q)_\infty} - \left(\frac{1+q+q^2}{q^3}\right) \times \frac{1}{(q^2)_\infty} + \left(\frac{1+q+q^2}{q^{3+2}}\right) \times \frac{1}{(q^3)_\infty} - \frac{1}{q^{3+2+1}} \times \frac{1}{(q^4)_\infty} + \mathcal{A},
\end{aligned}$$

where

$$\mathcal{A} = -\frac{q^6 - q^5 - q^4 + q^2 + q - 1}{q^6}.$$

Simplifying this further, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_4(n)q^n \\
=&\frac{1}{(q)_\infty} - \frac{(1-q^3)}{q^3} \times \frac{1}{(q)_\infty} + \frac{(1-q^3)(1-q^2)}{q^{3+2}} \times \frac{1}{(q)_\infty} - \frac{(1-q^3)(1-q^2)(1-q)}{q^{3+2+1}} \times \frac{1}{(q)_\infty} + \mathcal{A} \\
=&\left(1 - \left(\frac{1}{q^3} - 1\right) + \left(\frac{1}{q^3} - 1\right)\left(\frac{1}{q^2} - 1\right) - \left(\frac{1}{q^3} - 1\right)\left(\frac{1}{q^2} - 1\right)\left(\frac{1}{q} - 1\right)\right)\frac{1}{(q)_\infty} + \mathcal{A} \\
=&\left(4 - \frac{1}{q} - \frac{2}{q^2} - \frac{2}{q^3} + \frac{1}{q^4} + \frac{2}{q^5} - \frac{1}{q^6}\right)\frac{1}{(q)_\infty} + \mathcal{A} \\
=&\sum_{n=0}^{\infty}(4p(n) - p(n+1) - 2p(n+2) - 2p(n+3) + p(n+4) + 2p(n+5) - p(n+6))q^n \\
&+ \mathcal{B} + \mathcal{A}
\end{aligned}$$

(Since, $\displaystyle\frac{1}{q^k(q)_\infty} = \sum_{n=0}^{\infty} p(n+k)q^n + \sum_{i=0}^{k-1} \frac{p(i)}{q^{k-i}}$, we take $\mathcal{B}$ as the sum of the remaining terms.)

$$\begin{aligned}
=&\sum_{n=1}^{\infty} \left(4p(n) - p(n+1) - 2p(n+2) - 2p(n+3) + p(n+4) + 2p(n+5) - p(n+6)\right) q^n \\
&+ \mathcal{C} + \mathcal{B} + \mathcal{A},
\end{aligned}$$

where

$$\mathcal{C} = 4p(0) - p(1) - 2p(2) - 2p(3) + p(4) + 2p(5) - p(6),$$

that is, the coefficient of q^0 in the summand which is missing. We can now check that $\mathcal{C} + \mathcal{B} + \mathcal{A} = 0$, which proves the result. □

As promised in Section 1, proceeding in an exactly similar way, we get a more general result.

Theorem 3.3. *For $n, m > 0$, we have*

$$\sum_{n=1}^{\infty} a_m(n)q^n = \frac{1}{(q)_\infty}\left(1 + \sum_{k=1}^{m-1}(-1)^k \prod_{i=0}^{k-1}\left(\frac{1}{q^{m-1-i}} - 1\right)\right) - \mathcal{D}_m,$$

where $\mathcal{D}_m$ is the sum of the terms with power of q less than or equal to 0 in the expansion of

$$1+\sum_{k=1}^{m-1}(-1)^k\prod_{i=0}^{k-1}\left(\frac{1}{q^{m-1-i}}-1\right).$$

Remark 3.4. *Putting $m=2$ in Theorem 3.3, we get Proposition 1.1.*

Remark 3.5. *Putting $m=3$ in Theorem 3.3, we get*

$$\begin{aligned}\sum_{n=1}^{\infty}a_3(n)q^n =&\frac{1}{(q)_\infty}\left(1-\left(\frac{1}{q^2}-1\right)+\left(\frac{1}{q^2}-1\right)\left(\frac{1}{q}-1\right)\right)-\mathcal{D}_3\\ =&\frac{1}{(q)_\infty}\left(3-\frac{1}{q}-\frac{2}{q^2}+\frac{1}{q^3}\right)-\mathcal{D}_3.\end{aligned}$$

This was shown in the proof of Theorem 1.6, which now follows as a corollary of Theorem 3.3.

4. Generating Functions of $a_m(n,\ell)$

We follow the method used by Andrews, Beck and Robbins [ABR15] to find the generating function of $a_m(n,\ell)$ below.

Theorem 4.1. *For $\ell>1$ we have*

$$\sum_{n=1}^{\infty}a_m(n,\ell)q^n=\frac{q^{\ell+m+1}(q)_m(q)_{\ell-m+1}}{((q)_\ell)^2}(-1)^{-m-1}q^{(-m^2-3m-2)/2}\left((q)_\ell-\sum_{j=0}^{m}\binom{\ell}{j}_q(-1)^jq^{j+j(j-1)/2}\right).$$

Proof. Clearly the generating function of $a_m(n,\ell)$ is given by

$$\begin{aligned}\sum_{n=1}^{\infty}a_m(n,\ell)q^n&=\sum_{k=1}^{\infty}q^{\overbrace{k+k+\cdots+k}^{m}}q^{k+\ell}\prod_{i=0}^{\ell}(1+q^{k+i}+q^{2(k+i)}+\cdots)\\ &=q^\ell\sum_{k\geq1}\frac{q^{(m+1)k}(q)_{k-1}}{(q)_{k+\ell}}=q^{\ell+m+1}\sum_{k\geq0}\frac{q^{(m+1)k}(q)_k}{(q)_{k+\ell+1}}\\ &=\frac{q^{\ell+m+1}}{(q)_{\ell+1}}\sum_{k\geq0}\frac{(q)_k(q)_kq^{(m+1)k}}{(q)_k(q^{\ell+2})_k}.\end{aligned}$$

We now use the following transformation [And98, p. 38]

$$\sum_{k\geq0}\frac{(a)_k(b)_kz^k}{(q)_k(c)_k}=\frac{(c/b)_\infty(bz)_\infty}{(c)_\infty(z)_\infty}\sum_{j\geq0}\frac{(abz/c)_j(b)_j(c/b)^j}{(q)_j(bz)_j},$$

to get

$$\begin{aligned}\sum_{n=1}^{\infty}a_m(n,\ell)q^n&=\frac{q^{\ell+m+1}(q^{\ell+1})_\infty(q^{m+2})_\infty}{(q)_{\ell+1}(q^{\ell+2})_\infty(q^{m+1})_\infty}\sum_{j\geq0}\frac{(q^{m+1-\ell})_j(q)_jq^{(\ell+1)j}}{(q)_j(q^{m+2})_j}\\ &=\frac{q^{\ell+m+1}}{(q)_\ell}\sum_{j=0}^{\ell-m-1}\frac{(q^{m+1-\ell})_jq^{(\ell+1)j}}{(q^{m+1})_{j+1}}\\ &=\frac{q^{\ell+m+1}}{(q)_\ell}\sum_{j=0}^{\ell-m-1}\frac{(1-q^{\ell-m-1})(1-q^{\ell-m-2})\cdots(1-q^{\ell-m-j})(-1)^jq^{(m+1)j+\binom{j+1}{2}}}{(q^{m+1})_{j+1}}\\ &=\frac{q^{\ell+m+1}(1-q)\cdots(1-q^m)}{(1-q^{\ell-m})\cdots(1-q^\ell)}\sum_{j=0}^{\ell-m-1}\frac{(-1)^jq^{(m+1)j+\binom{j+1}{2}}}{(q)_{m+j+1}(q)_{\ell-m-j-1}}\\ &=\frac{q^{\ell+m+1}(q)_m(q)_{\ell-m+1}}{(q)_\ell(q)_\ell}\sum_{j=0}^{\ell-m-1}\binom{\ell}{m+j+1}_q(-1)^jq^{(m+1)j+\binom{j+1}{2}},\end{aligned}$$

where

$$\binom{a}{b}_q := \frac{(q)_a}{(q)_b(q)_{a-b}}.$$

We use the q-binomial theorem [And98, p. 36]

$$(z)_n = \sum_{j=0}^{n} \binom{n}{j}_q (-1)^j z^j q^{j(j-1)/2},$$

to get

$$\begin{aligned}\sum_{n=1}^{\infty} a_m(n,\ell)q^n &= \frac{q^{\ell+m+1}(q)_m(q)_{\ell-m+1}}{((q)_\ell)^2} \sum_{j=m+1}^{\ell} \binom{\ell}{j}_q (-1)^{j-m-1} q^{(m+1)(j-m-1)+\binom{j-m}{2}} \\ &= \frac{q^{\ell+m+1}(q)_m(q)_{\ell-m+1}}{((q)_\ell)^2} (-1)^{-m-1} q^{(-m^2-3m-2)/2} \sum_{j=m+1}^{\ell} \binom{\ell}{j}_q (-1)^j q^{j+j(j-1)/2} \\ &= \frac{q^{\ell+m+1}(q)_m(q)_{\ell-m+1}}{((q)_\ell)^2} (-1)^{-m-1} q^{(-m^2-3m-2)/2} \left((q)_\ell - \sum_{j=0}^{m} \binom{\ell}{j}_q (-1)^j q^{j+j(j-1)/2} \right).\end{aligned}$$

□

If we replace $m = 1$ in Theorem 4.1 we get back the result of Andrews, Beck and Robbins [ABR15, Theorem 1].

5. Other Classes of Partitions

In this section we briefly look at overpartitions and ℓ-regular partitions and prove results analogous to those stated in Section 1.

5.1. **Overpatitions.** Overpartitions of n are the partitions of n in which the first occurrence (equivalently, the last occurrence) of a part may be overlined. The number of overpartitions of n are denoted by $\bar{p}(n)$. For example, $\bar{p}(3) = 8$, and the overpartitions of 3 are

$$3, \bar{3}, 2+1, \bar{2}+1, 2+\bar{1}, \bar{2}+\bar{1}, 1+1+1, \bar{1}+1+1.$$

We have the generating function

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots.$$

We define $\bar{a}_m(n)$ to be the number of overpartitions of n where the smallest part occurs at least m times. We get the generating function of $\bar{a}_m(n)$ as follows

$$\sum_{n=1}^{\infty} \bar{a}_m(n)q^n = \sum_{k=1}^{\infty} \frac{2q^{mk}(-q^k)_\infty}{(1+q^k)(q^k)_\infty}.$$

Here, a overlined part is equal to a non-overlined part if their value is equal. For example $\bar{1} = 1$. The terms q^{mk}, $(-q^k)_\infty$ and $(q^k)_\infty$ are similarly explained as in finding the generating function for $a_m(n)$. We have the factor of 2 because the smallest part can be either overlined or non-overlined and finally we need to divide by $(1+q^k)$ because if k is the smallest part then it would not be generated by $(-q^k)_\infty$.

Analogous to Proposition 1.1, we now have the following result.

Theorem 5.1. *For all $n \geq 1$, we have*

$$\bar{a}_2(n) = 2\bar{p}(n) - \bar{p}(n+1) + \bar{u}(n+1),$$

where $\bar{u}(n)$ is the number of overpartitions of n with the following conditions:

(1) *1 is not a part.*
(2) *Smallest part must be overlined, and no other part of that value is present. (For example, $7+7+\bar{2}$ is included, but $7+7+\bar{2}+2$ is not.)*

(3) *The value of the first and second greatest parts are equal or consecutive, but if they are consecutive then the second greatest part must be overlined. (For example, $7+\bar{6}+\bar{2}$ is included, but $7+6+\bar{2}$ is not.)*

Proof. Here we get, $\bar{p}(n)-\bar{a}_2(n)$ counts the overpartitions of n where the least part occurs exactly once. And $\bar{p}(n+1)-\bar{p}(n)$ counts the overpartitions of $n+1$ where there is no 1 as a part. Let us denote the sets by $\bar{A}$ and $\bar{B}$, respectively, whose cardinalities are counted by $\bar{p}(n)-\bar{a}_2(n)$ and $\bar{p}(n+1)-\bar{p}(n)$ respectively.

We divide $\bar{A}$ into two classes:

(a) **The least part is not overlined.** In this case we add 1 to the least part. Then for each of these overpatitions there corresponds a unique overpartition in $\bar{B}$. The least part of these overpartitions in $\bar{B}$ are not overlined. Conversely, for each overpartition in $\bar{B}$ with the least part not overlined, if we subtract 1 form the least part, then there corresponds a unique overpartition in $\bar{A}$. Let us denote this class of $\bar{B}$ by $\bar{B}_1$.

(b) **The least part is overlined.** In this case we add 1 to the largest part. Then for each of these overpatitions there corresponds a unique overpartition in $\bar{B}$. Let us denote this class of $\bar{B}$ by $\bar{B}_2$. Conversely, we subtract 1 from the largest part of the overpatitions in $\bar{B}_2$.

So, in $\bar{B}-\bar{B}_1\bigcup\bar{B}_2$, we are left with the overpartitions, where,

(1) the least part is overlined,

(2) the first and second largest parts are equal, or they are consecutive with the property that the second largest part is overlined. (Note that, the later case is necessary because of the converse part of the above Case (b). Since, if we subtract 1 from the largest part, then the overpartitions become of the form $a+\bar{a}+\cdots$ or $\bar{a}+\bar{a}+\cdots$.)

Therefore, $|\bar{B}-\bar{B}_1\bigcup\bar{B}_2|=\bar{u}(n+1)$. Hence, $\bar{p}(n)-\bar{a}_2(n)=\bar{p}(n+1)-\bar{p}(n)-\bar{u}(n+1)$. □

For the sake of completeness, the generating function for $\bar{u}(n)$ is

$$\begin{aligned}\sum_{n=1}^{\infty}\bar{u}(n)q^n =&\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\frac{q^k(1+q^{k+1})(1+q^{k+2})\cdots(1+q^{k+t-1})(q^{2(k+t)}+q^{2(k+t)})}{(1-q^{k+1})(1-q^{k+2})\cdots(1-q^{k+t-1})(1-q^{k+t})}\\ &+\sum_{k=1}^{\infty}\sum_{t=2}^{\infty}\frac{q^k(1+q^{k+1})(1+q^{k+2})\cdots(1+q^{k+t-2})q^{k+t-1}(q^{k+t}+q^{k+t})}{(1-q^{k+1})(1-q^{k+2})\cdots(1-q^{k+t-1})}\\ &+\sum_{k=1}^{\infty}q^k(q^{k+1}+q^{k+1})\\ =&\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\frac{2q^{3k+2t}(-q^{k+1};q)_{t-1}}{(q^{k+1};q)_t}+\sum_{k=1}^{\infty}\sum_{t=2}^{\infty}\frac{2q^{3k+2t-1}(-q^{k+1};q)_{t-2}}{(q^{k+1};q)_{t-1}}+\sum_{k=1}^{\infty}2q^{2k+1}\\ =&2\sum_{k=1}^{\infty}\left(\frac{q^{2k+1}}{(q^{k+1};q)_1}+\sum_{t=2}^{\infty}q^{3k+2t-1}(1+q)\frac{(-q^{k+1};q)_{t-2}}{(q^{k+1};q)_t}\right).\end{aligned}$$

Let us now define by $\bar{p}(n,t)$ to be the number of overpartitions of n where the difference between the largest and smallest parts equal t. Analogous to Proposition 1.2 we have the following result.

Theorem 5.2. *For all $n\geq 1$, we have*

$$2\bar{a}_2(n)=\bar{p}(2n,n).$$

Proof. Following a similar method as Dhar [Dha21], here we get a one-to-two correspondence between the overpartitions counted by $\bar{a}_2(n)$ and the overpatitions counted by $\bar{p}(2n,n)$.

We add n to the rightmost smallest part of each overpartition in the set counted by $\bar{a}_2(n)$. For example, if $\bar{k}+k$ is a part in a partition in this set, then we add n to k. Then the new overpartition belongs to the set counted by $\bar{p}(2n,n)$ and its largest part is not overlined, which is greater than all other parts. So corresponding to this overpartition there is another unique overpatition in the second set, all of whose parts are same except now the largest part is overlined. □

If we define $\bar{a}_m(n,\ell)$ to be the number of overpartitions of n where the smallest part occurs at least m times and the largest part minus the smallest part is ℓ, then using the above method we get the following theorem, analogous to Proposition 1.8.

Theorem 5.3. *For all $n \geq 1$ and $m \geq 2$, we have*

$$2\bar{a}_m(n) = \bar{a}_{m-1}(2n,n).$$

Theorem 5.2 follows from this when $m = 2$.

5.2. **ℓ-regular partitions.** The partitions of n with no parts divisible by ℓ are called ℓ-regular partitions, and the total number of such partitions is denoted by $b_\ell(n)$. We have the following generating function

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \prod_{n=1}^{\infty} \frac{1-q^{\ell n}}{1-q^n} = \frac{(q^\ell;q^\ell)_\infty}{(q)_\infty} = \frac{1}{(q,q^2,\ldots,q^{\ell-1};q^\ell)_\infty},$$

where

$$(a_1,a_2,\ldots,a_k;q)_\infty := \prod_{i=1}^{k}(a_i;q)_\infty.$$

Let us define $a_{m(\ell)}(n)$ to be the number of ℓ-regular partitions of n where the smallest part occurs at least m times. We get the generating function of $a_{m(\ell)}(n)$ as follows

$$\begin{aligned}\sum_{n=1}^{\infty} a_{m(\ell)}(n)q^n &= \sum_{k=0}^{\infty}\sum_{t=1}^{\ell-1} \frac{q^{(\ell k+t)m}}{\prod_{r=1}^{t-1}(q^{\ell(k+1)+r};q^\ell)_\infty \times \prod_{r=t}^{\ell-1}(q^{\ell k+r};q^\ell)_\infty}\\ &= \sum_{k=0}^{\infty}\sum_{t=1}^{\ell-1} \frac{q^{(\ell k+t)m}(q^{\ell k+1};q)_{t-1}}{(q^{\ell k+1},q^{\ell k+2},\ldots,q^{\ell k+\ell-1};q^\ell)_\infty}.\end{aligned}$$

If $\ell = 2$, then the partitions contain no even parts. In this case, we get

$$\sum_{n=1}^{\infty} a_{m(2)}(n)q^n = \frac{q^m}{(q;q^2)_\infty}\sum_{k=0}^{\infty}(q;q^2)_k q^{2km}.$$

Analogous to Proposition 1.1, we now have the following result.

Theorem 5.4. *For all $n \geq 1$, we have*

$$a_{2(2)}(n) = b_2(n) + b_2(n+1) - b_2(n+2).$$

Proof. The quantity $b_2(n) - a_{2(2)}(n)$ counts the 2-regular partitions of n in which the least part occurs exactly once. Again, $b_2(n+2) - b_2(n+1)$ counts the 2-regular partitions of $n+2$ in which there is no 1 as a part. These two quantities are equal. We add 2 to the least part of the 2-regular partitions counted by $b_2(n) - a_{2(2)}(n)$. And conversely we subtract 2 from the least part of the 2-regular partitions given by $b_2(n+2) - b_2(n+1)$. □

Let us now define $a_{m(\ell)}(n,k)$ to be the number of ℓ-regular partitions of n where the smallest part occurs at least m times and the largest part minus the smallest part is k, then like before we get the following theorem, analogous to Propositions 1.2 and 1.8.

Theorem 5.5. *If $m,\ell \geq 2$, and n is divisible by ℓ, then*

$$a_{2(\ell)}(n) = b_\ell(2n,n).$$

More generally,

$$a_{m(\ell)}(n) = a_{m-1(\ell)}(2n,n).$$

This result is not true if n is not divisible by ℓ, since the sum of the smallest part and n may be divisible by ℓ. For example, $b_2(2n,n) = 0$ if n is odd. But, we get the following more general result in this case.

Theorem 5.6. *If n is odd, then*

$$a_{2(2)}(n) = b_2(2n+1, n+1).$$

More generally, let $m, \ell \geq 2$, and n is not divisible by ℓ. If $n = \ell k + r$, for some interger k and $1 \leq r < \ell$, then

$$a_{m(\ell)}(n) = a_{(m-1)(\ell)}(2n + \ell - r, n + \ell - r).$$

6. Concluding Remarks

Several natural questions arise from the work we have described so far. Some of these we list below.

(1) Proposition 1.8 is a very simple generalization. Is it possible to find $a_m(n)$ in terms of partition functions such as $p(an, bn)$, where a, b are positive integers?
(2) Is it possible to find generating functions of analogues of $a_m(n)$ (and other statistics defined here) for other partition functions such as (ℓ, m)-regular partitions, t-core partitions, partition with designated summands, k-colored partitions, etc.?
(3) Dhar [Dha21] pointed out that the generating function of $p(2n, n)$ is still not found in a 'nice' closed form. Similarly, the generating function of $a_m(2n, n)$ is also not found here. Can we find these?
(4) Is it possible to find q-series proofs of the results in Section 5 which are proved combinatorially?
(5) Is it possible to find $a_{m(2)}(n)$, $a_{2(\ell)}(n)$, and $a_{m(\ell)}(n)$ for $m, \ell > 2$?
(6) If an overlined part is not equal to a non-overlined part, even if their value is equal (for example $\bar{1} \neq 1$), then the following is the generating function of $\bar{a}_m(n)$

$$\sum_{n=1}^{\infty} \bar{a}_m(n) q^n = \sum_{k=1}^{\infty} \frac{q^{mk}(-q^k)_\infty}{(q^k)_\infty}.$$

Then, can we find $\bar{a}_2(n)$ and $\bar{a}_m(n)$?
(7) We can prove for $n \geq 1$,

$$p(2n, n) = 1 + p(n-2) + \sum_{m=2}^{\lfloor \frac{n}{3} \rfloor} p_m^\star(n - 2m),$$

where $p_m^\star(n-2m)$ is the number of partitions of $n-2m$ with the least part greater than or equal to m. Are there any interesting properties of $a_m(n, \ell)$ that can be proved using this relation?
(8) Andrews, Beck and Robbins [ABR15, Theorem 2] also give a generalization of Theorem 4.1 ($m = 1$) to partitions with a set of specified distances. It would be interesting to explore this direction with some of the statistics defined in this paper.
(9) Breuer and Kronholm [BK16] extended the result of Andrews, Beck and Robbins [ABR15] to partitions where the fixed difference between the largest and smallest parts is at most a fixed integer. Chapman [Cha16] gave a combinatorial proof of this result. It would be interesting to extend this setting for the statistics defined in this paper.

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