ON SOME CONJECTURES OF PAUDEL, SELLERS, AND WANG

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ABSTRACT. We prove some conjectured congruences of Paudel, Sellers, and Wang (2025) about ℓ-regular partition k-tuples, using the theory of modular forms.

An (integer) *partition* of a natural number n is a non-increasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, such that $\sum_{i=1}^{k} \lambda_i = n$. We denote by $p(n)$, the number of partitions of n. For instance

$$(5),\ (4,1),\ (3,2),\ (3,1,1),\ (2,2,1),\ (2,1,1,1),\ (1,1,1,1,1),$$

are the seven partitions of 5, which gives us $p(5) = 7$. It is well-known that their generating function is given by

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)} = \frac{1}{(q;q)_\infty} = \frac{1}{f_1},$$

where we have used the shorthand notations

$$(a;q)_\infty := \prod_{n\geq 0}(1-aq^n) \quad \text{and} \quad f_k := (q^k;q^k)_\infty.$$

Arithmetic properties of partitions, more specifically, congruences have been an object of intense study by mathematicians over the last century. Ramanujan [Ram20] gave the following intriguing congruences satisfied by the partition function: for all $n \geq 0$, we have

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5,\\ p(7n+5) &\equiv 0 \pmod 7,\\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

These congruences have been the inspiration over the years for mathematicians to search for such "Ramanujan-type congruences" satisfied by the partition function or its generalizations.

In this paper, we are interested in tuples of partitions: consider positive integers n_i $(1 \leq i \leq k)$ such that $\sum_{i=1}^{k} n_i = n$ and partitions λ_i of n_i; then $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is called a partition k-tuple of n. For instance, $((2,1,1),(4,2),(1))$ is a partition 3-tuple of 11. In addition, if none of the parts in the partitions λ_i $(1 \leq i \leq k)$ are divisible by ℓ, then the partition k-tuple is also called ℓ-regular. For instance, the tuple above is 3-regular. Recently, there has been some interest in studying ℓ-regular partition k-tuple, for instance, see the work of Nadji & Ahmia [NA24], Nath, Saikia, & Sarma [NSS24], and Paudel, Sellers, & Wang [PSW25] and the references therein.

Let us denote the number of ℓ-regular partition k-tuples of n by $T_{\ell,k}(n)$. It is easy to see that its generating function is given by

$$\sum_{n\geq 0} T_{\ell,k}(n)q^n = \frac{f_\ell^k}{f_1^k}.$$

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When $k=3$, we use the notation $T_{\ell,3}(n) = T_\ell(n)$. So, we get

$$\sum_{n\geq 0} T_\ell(n) q^n = \frac{f_\ell^3}{f_1^3}. \tag{1}$$

In their recent work, Paudel, Sellers, and Wang [PSW25] proved some infinite family of congruences for $T_2(n)$ and stated the following conjectures: for all $n \geq 0$, we have

$$T_2(25n+r) \equiv 0 \pmod{2^5} \quad r \in \{8, 13, 18, 23\},$$
$$T_2(49n+r) \equiv 0 \pmod{2^5} \quad r \in \{13, 20, 27, 34, 41, 48\}.$$

In this short note, we prove their conjectures. We have better moduli as given in the following theorem.

Theorem 1. *For all $n \geq 0$, we have*

$$T_2(25n+r) \equiv 0 \pmod{2^6} \quad r \in \{8, 13, 18, 23\}, \tag{2}$$
$$T_2(49n+r) \equiv 0 \pmod{2^9} \quad r \in \{13, 20, 27, 34, 41, 48\}. \tag{3}$$

We recall some aspects of modular forms that will be used in the proof of Theorem 1. Let $\mathbb{H}$ be the complex upper half-plane. For a positive integer N, we define the following matrix groups:

$$\Gamma := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$
$$\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in \Gamma : n \in \mathbb{Z} \right\}.$$

and

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$

We need some preliminary results, which describe an algorithmic approach to proving partition concurrences, developed by Radu [Rad09, Rad15]. For integers $M \geq 1$, suppose that $R(M)$ is the set of all the integer sequences

$$(r_\delta) := (r_{\delta_1}, r_{\delta_2}, r_{\delta_3}, \ldots, r_{\delta_k})$$

indexed by all the positive divisors δ of M, where $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$. For integers $m \geq 1$, $(r_\delta) \in R(M)$, and $t \in \{0, 1, 2, \ldots, m-1\}$, we define the set $P(t)$ as

$$P(t) := \Bigg\{ t' \in \{0, 1, 2, \ldots, m-1\} : t' \equiv ts + \frac{s-1}{24} \sum_{\delta | M} \delta r_\delta \pmod{m}$$
$$\text{for some } [s]_{24m} \in \mathbb{S}_{24m} \Bigg\}. \tag{4}$$

For integers $N \geq 1$, $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $(r_\delta) \in R(M)$, and $(r'_\delta) \in R(N)$, we also define

$$p(\gamma) := \min_{\lambda \in \{0, 1, \ldots, m-1\}} \frac{1}{24} \sum_{\delta | M} r_\delta \frac{\gcd(\delta(a + k\lambda c), mc)^2}{\delta m},$$
$$p'(\gamma) := \frac{1}{24} \sum_{\delta | N} r'_\delta \frac{\gcd(\delta, c)^2}{\delta}.$$

For integers $m \geq 1$; $M \geq 1$, $N \geq 1$, $t \in \{0, 1, 2, \ldots, m-1\}$, $k := \gcd(m^2 - 1, 24)$, and $(r_\delta) \in R(M)$, define Δ^* to be the set of all tuples $(m, M, N, t, (r_\delta))$ such that all of the following conditions are satisfied

1. Prime divisors of m are also prime divisors of N;

2. If $\delta \mid M$, then $\delta \mid mN$ for all $\delta \geq 1$ with $r_\delta \neq 0$;
3. $24 \mid kN \sum_{\delta|M} \frac{r_\delta mN}{\delta}$;
4. $8 \mid kN \sum_{\delta|M} r_\delta$;
5. $\frac{24m}{\left(-24kt - k\sum_{\delta|M}\delta r_\delta, 24m\right)} \mid N$;
6. If $2|m$ then either $4|kN$ and $8|\delta N$ or $2|s$ and $8|(1-j)N$, where $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s \cdot j$.

We now state a result of Radu [Rad09], which we use in completing the proof of Theorem 1.

Lemma 2. [Rad09, Lemma 4.5] *Suppose that* $(m, M, N, t, (r_\delta)) \in \Delta^*$, $(r'_\delta) := (r'_\delta)_{\delta|N} \in R(N)$, $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ *is a complete set of representatives of the double cosets of* $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$, *and* $t_{\min} := \min_{t' \in P(t)} t'$,

$$\nu := \frac{1}{24}\left(\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N}\delta r'_\delta - \frac{1}{m}\sum_{\delta|M}\delta r_\delta\right) - \frac{t_{min}}{m}, \tag{5}$$

$p(\gamma_j) + p'(\gamma_j) \geq 0$ *for all* $1 \leq j \leq n$, *and* $\sum_{n=0}^{\infty} A(n)q^n := \prod_{\delta|M} f_\delta^{r_\delta}$. *If for some integers* $u \geq 1$, *all* $t' \in P(t)$, *and* $0 \leq n \leq \lfloor \nu \rfloor$, $A(mn + t') \equiv 0 \pmod{u}$ *is true, then for integers* $n \geq 0$ *and all* $t' \in P(t)$, *we have* $A(mn + t') \equiv 0 \pmod{u}$.

The following lemma supports Lemma 2 in the proof of Theorem 1.

Lemma 3. [RS11, Lemma 2.6] *Let* N *or* $\frac{N}{2}$ *be a square-free integer, then we have*

$$\bigcup_{\delta|N} \Gamma_0(N)\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\Gamma_\infty = \Gamma.$$

Proof of Theorem 1. Since the proofs of the congruences are similar, we only prove the $r = 8, 18$ cases of (2). We choose $(m, M, N, t, (r_\delta)) = (25, 10, 10, 8, (-3, 3, 0, 0))$. It easy to verify that this choice of $(m, M, N, t, (r_\delta))$ satisfies the Δ^* conditions and from equation (4) we see that $P(t) = \{8, 23\}$. By Lemma 3, we know that $\left\{\begin{bmatrix}1 & 0\\ \delta & 1\end{bmatrix} : \delta|10\right\}$ forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Let $\gamma_\delta = \begin{bmatrix}1 & 0\\ \delta & 1\end{bmatrix}$, and for the choice of $(r'_\delta) = (38, 0, 0, 0)$ we see that

$$p(\gamma_\delta) + p'(\gamma_\delta) \geq 0, \quad \text{for all } \delta|N.$$

By Lemma 2, we compute $\lfloor \nu \rfloor = 26$. Using Mathematica we verify the $r = 8, 23$ cases of (2) for $n \leq 26$, which proves the congruences.

We mention the relevant details required to verify the rest of the congruences in the table below.

Congruences	$(m, M, N, t, (r_\delta))$ and (r'_δ)	$P(t)$	$\lfloor \nu \rfloor$
$r = 13, 18$ cases of (2)	$(25, 10, 10, 13, (-3, 3, 0, 0))$ and $(38, 0, 0, 0)$	{13, 18}	26
$r = 13, 20$ cases of (3)	$(49, 14, 14, 13, (-3, 3, 0, 0))$ and $(74, 0, 0, 0)$	{13, 20}	70
$r = 27, 41$ cases of (3)	$(49, 14, 14, 27, (-3, 3, 0, 0))$ and $(74, 0, 0, 0)$	{27, 41}	70
$r = 34, 48$ case of (3)	$(49, 14, 14, 41, (-3, 3, 0, 0))$ and $(74, 0, 0, 0)$	{34, 48}	70

□

Some concluding remarks are in order:

(1) It is desirable to have an elementary proof of Theorem 1, but it seems challenging.
(2) It seems $T_2(p^2n + r)$ for primes $p > 7$ doesn't seem to be divisible by very high powers of 2 unlike the cases $p = 3, 5$.

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