CONGRUENCES AND CONJECTURES FOR REGULAR BIPARTITIONS AND BIPARTITION TUPLES

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ABSTRACT. We study some classes of partitions, namely regular partitions, bipartitions into distinct parts, and bipartition tuples. First, we prove a parity theorem for (j,k)-regular bipartitions into distinct parts: for all positive integers j and k, the coefficient of every odd power of q is even. Next, for the bipartition tuple function we prove two elementary infinite families of congruences. We then record stronger congruences modulo higher powers of 2 and 3, and formulate open problems directed toward infinite families.

1. INTRODUCTION

One of the most widely studied class of objects in all of mathematics is integer partitions. An integer partition (or, just a partition) λ of n is a non-increasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that the λ_i's sum up to n. The λ_i's are called parts of the partition λ. We denote the number of partitions of n by $p(n)$, and from the work of Euler we know that the following generating function holds

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{f_1},$$

where we use the following notation

$$f_r := (q^r;q^r)_\infty = \prod_{n\geq 1}(1-q^{rn}).$$

Partitions have been studied since the time of Euler, and one direction of study is to find arithmetic properties related to them. For instance, three of the most celebrated congruences that the partition function satisfy are due to Ramanujan, who proved that, for all $n \geq 0$, we have

$$p(5n+4)\equiv 0 \pmod 5, \quad p(7n+5)\equiv 0 \pmod 7, \quad p(11n+6)\equiv 0 \pmod{11}.$$

Numerous generalizations of the concept of partitions are known and studied. For instance, the (ℓ,k)-regular partition function $b_{\ell,k}(n)$ is defined by

$$\sum_{n\geq 0} b_{\ell,k}(n)q^n = \frac{f_\ell f_k}{f_1 f_{\ell k}}. \tag{1}$$

It is easy to see that $b_{\ell,k}(n)$ counts the number of partitions of n such that no part is divisible by multiples of ℓ or k. This paper concerns three closely related families of partitions. The first is (1). The second is the generating function for (j,k)-regular bipartitions into distinct parts, denoted by $\widetilde{B}_{j,k}(n)$, namely

$$\sum_{n\geq 0} \widetilde{B}_{j,k}(n)q^n = \frac{f_2^2 f_{2jk}^2 f_j^2 f_k^2}{f_1^2 f_{2j}^2 f_{2k}^2 f_{jk}^2}. \tag{2}$$

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The third is the family of (ℓ_1, ℓ_2) bipartition k-tuples,

$$\sum_{n\geq 0} BP^k_{\ell_1,\ell_2}(n)q^n = \frac{f^k_{\ell_1} f^k_{\ell_2}}{f_1^{2k}}. \tag{3}$$

Prasad and Prasad [PP20], and Drema and Saikia [DS22], studied the coefficients $\widetilde{B}_{j,k}(n)$. Puneeth and Roy [PR22] proved, among other congruences,

$$BP^3_{2,3}(3n+2) \equiv 0 \pmod 8, \qquad BP^3_{2,3}(9n+6) \equiv 0 \pmod 8.$$

The purpose of the present note is twofold. First, we isolate several congruences that can be proved using elementary means. Second, we formulate higher-power congruences suggested by computation that possibly needs some other techniques than what we have here in this paper.

Our first result is a general parity theorem for (2).

Theorem 1.1. *For all positive integers j, k and all $n \geq 0$,*

$$\widetilde{B}_{j,k}(2n+1) \equiv 0 \pmod 2.$$

For the bipartition tuples, we prove the following infinite family.

Theorem 1.2. *For every positive integer m and every $n \geq 0$,*

$$BP^3_{2,3m}(3n+2) \equiv 0 \pmod 8.$$

We also prove a congruence modulo 3.

Theorem 1.3. *For all positive integers a, b and all $n \geq 0$,*

$$BP^3_{3a,3b}(3n+2) \equiv 0 \pmod 3.$$

After some preliminaries in the next section, we prove Theorems 1.1–1.3 in Section 3 using only one very simple observation and consequences of Jacobi's triple product identity. However, the main goal is to raise several conjectures, which we do in Section 4.

2. Preliminaries

We need the following lemma easily proved using the binomial theorem.

Lemma 2.1. *Let p be prime, $\alpha \geq 1$, and $r \geq 1$. Then*

$$f_r^{p^\alpha} \equiv f_{pr}^{p^{\alpha-1}} \pmod{p^\alpha}.$$

In particular,

$$f_r^2 \equiv f_{2r} \pmod 2, \qquad f_r^8 \equiv f_{2r}^4 \pmod 8, \qquad f_r^9 \equiv f_{3r}^3 \pmod 3.$$

We shall also use two classical consequences of Jacobi's triple product identity.

Lemma 2.2. *We have*

$$\frac{f_1^2}{f_2} = \sum_{r=-\infty}^{\infty} (-1)^r q^{r^2}, \tag{4}$$

$$f_t^3 = \sum_{s=0}^{\infty} (-1)^s (2s+1) q^{ts(s+1)/2} \qquad (t \geq 1). \tag{5}$$

Proof. (4) is [Joh20, Eq. (5.1.14)] and (5) is [Joh20, Eq. (5.2.8)]. □

Lemma 2.3. *Modulo 3, the square residues are 0 and 1, and the triangular residues $s(s+1)/2$ are also 0 and 1. Consequently, neither a square nor a triangular number is congruent to 2 modulo 3.*

Proof. The assertion follows immediately by checking residues $0, 1, 2$ modulo 3. □

3. Proofs of our Main Results

Proof of Theorem 1.1. By Lemma 2.1, $f_r^2 \equiv f_{2r} \pmod 2$ for every $r \geq 1$. Applying this to the numerator and denominator of (2), gives us

$$\sum_{n\geq 0} \widetilde{B}_{j,k}(n)q^n \equiv \frac{f_4 f_{4jk} f_{2j} f_{2k}}{f_2 f_{4j} f_{4k} f_{2jk}} \pmod 2.$$

Every factor on the right is a product in powers of q^2. Therefore the coefficient of q^{2n+1} on the left is even for every $n \geq 0$. □

Proof of Theorem 1.2. Using Lemma 2.1, we have $f_1^8 \equiv f_2^4 \pmod 8$. Hence,

$$\sum_{n\geq 0} BP^3_{2,3m}(n)q^n = \frac{f_2^3 f_{3m}^3}{f_1^6} = \frac{f_2^3 f_{3m}^3 f_1^2}{f_1^8} \equiv \frac{f_1^2}{f_2} f_{3m}^3 \pmod 8.$$

By Lemma 2.2, we have

$$\frac{f_1^2}{f_2} f_{3m}^3 = \left(\sum_{r=-\infty}^{\infty} (-1)^r q^{r^2}\right)\left(\sum_{s=0}^{\infty} (-1)^s (2s+1) q^{3ms(s+1)/2}\right).$$

Every exponent occurring on the right is congruent modulo 3 to r^2, since the second factor contributes a multiple of 3. By Lemma 2.3, $r^2 \not\equiv 2 \pmod 3$. Thus the coefficient of q^{3n+2} in the right-hand side is 0, and the coefficient of q^{3n+2} in $\sum_{n\geq 0} BP^3_{2,3m}(n)q^n$ is divisible by 8. □

Proof of Theorem 1.3. Since $f_1^9 \equiv f_3^3 \pmod 3$, we have

$$\sum_{n\geq 0} BP^3_{3a,3b}(n)q^n = \frac{f_{3a}^3 f_{3b}^3}{f_1^6} = \frac{f_{3a}^3 f_{3b}^3 f_1^3}{f_1^9} \equiv f_1^3 \frac{f_{3a}^3 f_{3b}^3}{f_3^3} \pmod 3.$$

Meanwhile, by Lemma 2.2,

$$f_1^3 = \sum_{s=0}^{\infty} (-1)^s (2s+1) q^{s(s+1)/2}.$$

The exponent $s(s+1)/2$ is never congruent to 2 modulo 3, by Lemma 2.3. Therefore the coefficient of q^{3n+2} in $\sum_{n\geq 0} BP^3_{3a,3b}(n)q^n$ is divisible by 3 for all $n \geq 0$. □

4. Conjectured Stronger Congruences

The following congruence refine Theorem 1.1 for the first two pairs of parameters.

Conjecture 4.1. *For all $n \geq 0$,*

(6) $$\widetilde{B}_{2,3}(4n+3) \equiv 0 \pmod 4.$$

Conjecture 4.2. *For all $n \geq 0$,*

(7) $$\widetilde{B}_{3,4}(4n+3) \equiv 0 \pmod 4,$$

(8) $$\widetilde{B}_{3,4}(6n+r) \equiv 0 \pmod 4 \quad (r \in \{3,5\}),$$

(9) $$\widetilde{B}_{3,4}(10n+r) \equiv 0 \pmod 4 \quad (r \in \{5,9\}),$$

(10) $$\widetilde{B}_{3,4}(12n+11) \equiv 0 \pmod 8,$$

(11) $$\widetilde{B}_{3,4}(6n+r) \equiv 0 \pmod 3 \quad (r \in \{2,4\}),$$

(12) $$\widetilde{B}_{3,4}(9n+4) \equiv 0 \pmod 3.$$

Remark 4.1. *The congruences modulo powers of 2 in Conjectures 4.1 and 4.2 do not appear to extend uniformly to all pairs (j,k).*

Theorem 1.2 proves a uniform congruence modulo 8 for $BP^3_{2,3m}(3n+2)$. The computations that motivated this note suggest several higher-power refinements.

Conjecture 4.3. *For all $n \geq 0$,*

$$BP^3_{2,3}(9n+6) \equiv 0 \pmod{32}, \tag{13}$$

$$BP^3_{2,3}(81n+57) \equiv 0 \pmod{64}. \tag{14}$$

Conjecture 4.4. *For all $n \geq 0$,*

$$BP^3_{2,6}(9n+3) \equiv 0 \pmod{16}. \tag{15}$$

Conjecture 4.5. *For all $n \geq 0$,*

$$BP^3_{3,3}(9n+7) \equiv 0 \pmod{81}, \tag{16}$$

$$BP^3_{3,3}(27n+22) \equiv 0 \pmod{243}. \tag{17}$$

Our computations raise several natural questions.

Problem 4.1. *Determine whether there exist sequences A_s, R_s, and $c_s \to \infty$ such that*

$$BP^3_{2,3}(A_s n + R_s) \equiv 0 \pmod{2^{c_s}}$$

for every $s \geq 1$.

Problem 4.2. *Systematically study congruences modulo powers of 3 for $BP^3_{\ell_1,\ell_2}(n)$ when at least one of ℓ_1 and ℓ_2 is divisible by 3. Theorem 1.3 treats the case in which both are divisible by 3.*

Problem 4.3. *By work of Nayaka [Nay22] and the author [Sai24], $BP^4_{4,8}(n)$ satisfies many congruences modulo powers of 2. Develop infinite families for $BP^4_{4,8}(n)$ and, more generally, for $BP^4_{\ell_1,\ell_2}(n)$ when at least one of ℓ_1 and ℓ_2 is a power of 2.*

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