

THE SIGNED ROMAN DOMINATION NUMBER OF LADDER GRAPHS, CIRCULAR LADDER GRAPHS AND THEIR COMPLEMENTS

DILBAK HAJE, DELBRIN AHMED, HASSAN IZANLOO, AND MANJIL SAIKIA

ABSTRACT. Let $G = (V, E)$ be a finite connected simple graph with vertex set V and edge set E . A signed Roman dominating function (SRDF) on a graph G is a function $f : V \rightarrow \{-1, 1, 2\}$ that satisfies two conditions: (i) $\sum_{y \in N[x]} f(y) \geq 1$ for each $x \in V$, where the set $N[x]$ is the closed neighborhood of x consisting of x and vertices of V that are adjacent to x , and (ii) each vertex $x \in V$ where $f(x) = -1$ is adjacent to at least one vertex $y \in V$ where $f(y) = 2$. The weight of a SRDF is the sum of its function values over all vertices. The signed Roman domination number of G , denoted by $\gamma_{SR}(G)$, is the minimum weight of a SRDF on G . In this paper, we investigate the signed Roman domination number of the Ladder graph LG_n , the circular Ladder graph CL_n and their complements.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a connected graph of order $n = |V(G)|$ and size $m = |E(G)|$. In this paper, only (non trivial) simple graphs, i.e., finite, undirected graphs without loops or multiple edges are considered. When u is a vertex of G , then the open neighbourhood of u in G is the set $N_G(u) = \{v : \{u, v\} \in E(G)\}$ and the closed neighbourhood of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. The degree of vertex u is the number of edges adjacent to u and is denoted by $deg_G(u)$. A graph is said to be regular if all of its vertices have the same degree. The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph is called k -regular if each vertex of the graph has degree k . We write K_n for the complete graph of order n , P_n for the path graph of order n and C_n for a cycle of length n . The complement of a graph $G = (V, E)$ is a graph $G^C = (V, E^C)$ with the same vertex set V with two vertices $uv \in E^C$ if and only if $\{u, v\} \notin E$, for all pairs $u \neq v \in V$.

A set $D \subseteq V(G)$ is called a *dominating set* of G if each vertex outside D has at least one neighbour in D . The minimum cardinality of a dominating set of G is the domination number of G and is denoted by $\gamma(G)$. For example, the domination numbers of the n -vertex complete graph, path, and cycle are given by $\gamma(K_n) = 1$, $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$ and $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$, respectively (see [HHS98]). Domination is a rapidly developing area of research in graph theory, and has various applications to several other practical areas. The concept of domination has existed and was studied for a long time and early discussions on the topic can be found in the works of Ore [Ore62] and Berge [Ber73]. Garey and Johnson [GJ79] have shown that determining the domination number of an arbitrary graph is an NP-complete problem.

The domination number can be defined equivalently by means of a function, which can be considered as a characteristic function of a dominating set (see [HHS98]). A function $f : V(G) \rightarrow \{0, 1\}$ is called a *domination function* on G if for each vertex $u \in V(G)$, $\sum_{v \in N_G[u]} f(v) \geq 1$. The value $w(f) = \sum_{u \in V(G)} f(u)$ is called the *weight* of f . Now, the domination number of G can be defined as

$$\gamma(G) = \min\{w(f) : f \text{ is a domination function of } G\}.$$

2020 *Mathematics Subject Classification.* 05C69, 05C78, 90C27.

Key words and phrases. Signed Roman Domination number, Domination function.

Analogously, we define a *signed domination function* of G to be a labelling of the vertices of G with $+1$ and -1 such that the closed neighbourhood of each vertex contains more $+1$'s than -1 's. The signed domination number of G is the minimum value of the sum of vertex labels that is taken over all *signed domination function* of G . In this paper, we concentrate on a related function, called the signed Roman domination function, which we define now.

Definition 1. Let $G = (V, E)$ be a graph. A *signed Roman domination function* (SRDF) on the graph G is a function $f : V(G) \rightarrow \{-1, 1, 2\}$ which satisfies the following conditions:

- (i) For each $u \in V(G)$, $\sum_{v \in N_G[u]} f(v) \geq 1$, and
- (ii) Each vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$.

The value $f(V) = \sum_{u \in V(G)} f(u)$ is called the *weight* of the function f and is denoted by $w(f)$. The *signed Roman domination number* of G , $\gamma_{SR}(G)$, is the minimum weight of a SRDF on G .

The concept of a SRDF was introduced by Ahangar et al. [AAHL⁺14]. They described the usefulness of these concepts in various applicable areas (see [AAHL⁺14], [HH03] and [Ste99] for more details). It is obvious that for every graph G of order n we have $\gamma_{SR}(G) \leq n$, because assigning $+1$ to each vertex yields a SRDF. In [AAHL⁺14] Ahangar et al. presented various lower and upper bounds on the signed Roman domination number of a graph in terms of its order, size and vertex degrees. For instance, they showed that for a graph G with n vertices, we have

$$(1.1) \quad \gamma_{SR}(G) \geq \left(\frac{-2\Delta^2 + 2\Delta\delta + \Delta + 2\delta + 3}{(\Delta + 1)(2\Delta + \delta + 3)} \right) n.$$

Moreover, they also showed that if G is a graph with n vertices and m edges with no isolated vertex, then we have

$$(1.2) \quad \gamma_{SR}(G) \geq \frac{3n - 4m}{2}.$$

They investigated the relation between γ_{SR} and some other graphical parameters, and the signed Roman domination number of some special bipartite graphs. It is proved in [AAHL⁺14] that $\gamma_{SR}(K_n) = 1$ for each $n \neq 3$, $\gamma_{SR}(K_3) = 2$, $\gamma_{SR}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$, $\gamma_{SR}(P_n) = \left\lfloor \frac{2n}{3} \right\rfloor$, and that the only n -vertex graph G with $\gamma_{SR}(G) = n$ is the empty graph K_n .

Note that each SRDF f of G is uniquely determined by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$, where $V_i = \{u \in V(G) : f(u) = i\}$ for each $i \in \{-1, +1, 2\}$. Specially, $w(f) = 2|V_2| + |V_1| - |V_{-1}|$. For simplicity, we usually write $f = (V_{-1}, V_1, V_2)$ and, when $U \subseteq V(G)$ we denote the sum $\sum_{u \in U} f(u)$ by $f(U)$. If $w(f) = \gamma_{SR}(G)$, then f is called a $\gamma_{SR}(G)$ -*function* (also known as an *optimal SRDF*) on G . There have been several follow up work on SRDFs after the work of Ahangar et al. [AAHL⁺14]. For instance, Behtoei, Vatandoost and Azizi [BVA16] studied the signed Roman domination number of the join of graphs, and determined its value for the join of cycles, wheels, fans, and friendship graphs; while, Hong et. al. [HYZZ20] determined its value spider graphs and double star graphs.

In this paper, we investigate the signed Roman domination number of the Ladder graph LG_n , Circular Ladder graph LC_n and their complement graphs LG_n^C and LC_n^C . The paper is arranged as follows: in Section 2 we evaluate the signed Roman domination number for LG_n and its complement, in Section 3 we evaluate the signed Roman domination number for LC_n and its complement, we end the paper with some concluding remarks in Section 4.

2. SIGNED ROMAN DOMINATION NUMBER FOR LADDER GRAPHS LG_n AND ITS COMPLEMENT

A *Cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ with the vertex set $V(G_1) \times V(G_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if and only if either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$. We are interested in the following cartesian product in this section.

Definition 2. The *Ladder graph* of order $2n$, denoted by LG_n , is the Cartesian product of two path graphs, one of which has only one edge. In other words $LG_n := P_2 \times P_n$, where P_n is the path graph on n vertices.

Note, by the definition of LG_n , one can write

$$V(LG_n) = \{(1, i), (2, i) \mid i \in [n]\}, \text{ and}$$

$$E(LG_n) = \{(1, i), (2, i)\} \mid 1 \leq i \leq n\} \cup \{(1, i), (1, i+1)\}, \{(2, i), (2, i+1)\} \mid 1 \leq i \leq n-1\}.$$

For simplicity, we encode the vertices $(1, i)$ and $(2, i)$ with $1i$ and $2i$ for $i = 1, 2, \dots, n$.

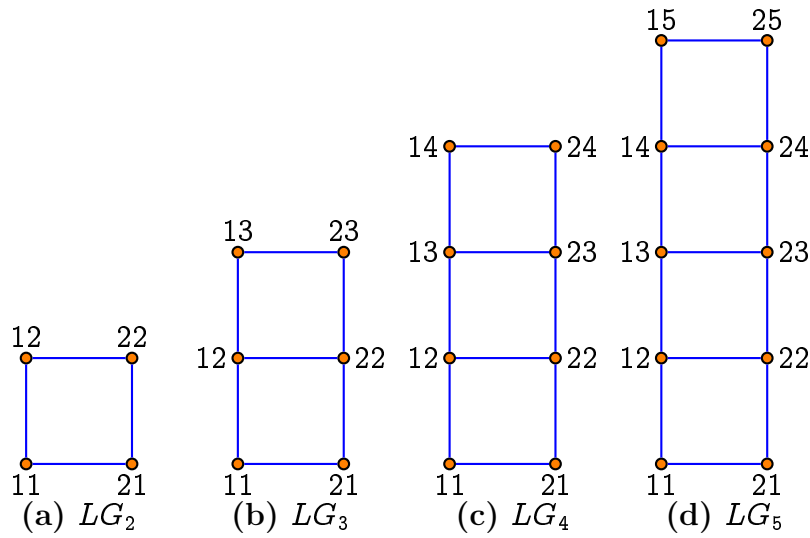


FIGURE 1. Ladder graphs of orders 2, 3, 4 and 5.

It is easy to observe that, for LG_n we have

(i) $|E(LG_n)| = 3n - 2$, and

(ii) LG_n has four vertices of degree 2 and $2n - 4$ vertices of degree 3.

We now proceed to find the signed Roman domination number for Ladder graphs.

Theorem 1. Let LG_n be the Ladder graph of order $2n$ with $n \geq 2$. Then

$$\gamma_{SR}(LG_n) = \left\lfloor \frac{n+2}{2} \right\rfloor + 1.$$

From (1.1) we get a bound of

$$\gamma_{SR}(LG_n) \geq \frac{2n}{11}.$$

Proof of Theorem 1. For the graph LG_n consider the function

$$f : V(LG_n) \rightarrow \{-1, 1, 2\}$$

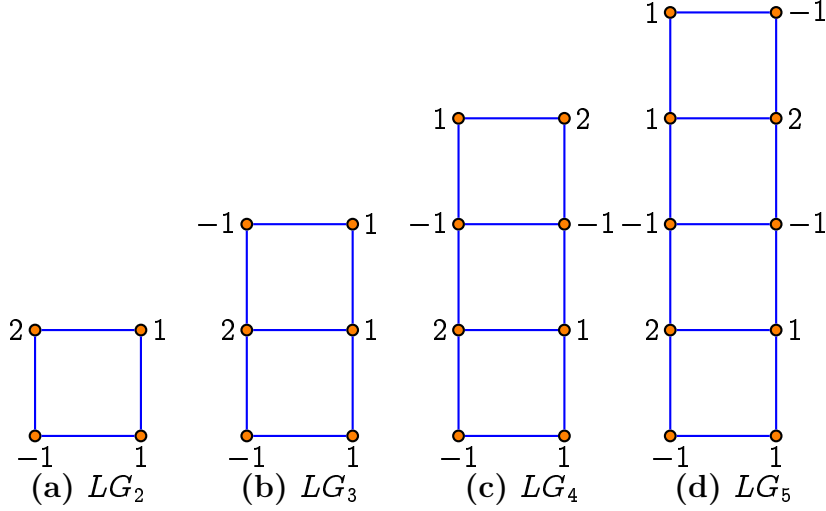


FIGURE 2. The SRDF for the Ladder graphs of order 2, 3, 4 and 5.

given by

$$f(v) = \begin{cases} -1 & \text{if } v = (1, i), i \geq 1 \text{ and } i \text{ is odd, except when } i = n \equiv 1 \pmod{4}; \\ & v = (2, i), i \geq 3 \text{ and } i \text{ is odd, except when } i = n \equiv 3 \pmod{4}; \\ 1 & \text{if } v = (2, 1), v = (1, i) \text{ and } i \equiv 0 \pmod{4}, v = (1, n) \text{ for } n \equiv 1 \pmod{4}; \\ & v = (2, i) \text{ and } i \equiv 2 \pmod{4}, v = (2, n) \text{ and } n \equiv 2, 3 \pmod{4}; \\ 2 & \text{if } v = (1, i) \text{ and } i \equiv 2 \pmod{4}, v = (2, i) \text{ and } i \equiv 0 \pmod{4}. \end{cases}$$

By our construction, it is easy to see that f is a SRDF. Except for the $(1, 1)$, $(2, 1)$, $(1, n)$ and $(2, n)$ vertices, every vertex with label 2 is adjacent to three vertices, two of which are labelled -1 and one of them is labelled 1; every vertex with label -1 is adjacent to three vertices, labelled 1, 2 and -1 ; and, every vertex with label 1 is adjacent to three vertices, two of which are labelled -1 and one of them is labelled 2. This establishes the conditions for the function to be a SRDF¹. By the construction of our labels, it is easy to verify the following properties:

$$\begin{aligned} |V_{-1}| &= n - 1, \\ |V_1| &= \begin{cases} \lfloor n/2 \rfloor + 1, & \text{if } n \text{ is even,} \\ \lfloor n/2 \rfloor + 2, & \text{if } n \text{ is odd,} \end{cases} \\ |V_2| &= \lfloor \frac{n}{2} \rfloor. \end{aligned}$$

From now on, we will use the following fact without commentary: for all $n \geq 0$, we have

$$\lfloor \frac{n}{2} \rfloor = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

With this in hand, it is easy to see that the weight of the function f defined above is given by

$$2|V_2| + |V_1| - |V_{-1}| = \lfloor \frac{n+2}{2} \rfloor + 1.$$

Hence, we conclude that $\gamma_{SR}(LG_n) \leq \lfloor \frac{n+2}{2} \rfloor + 1$, for all $n \geq 2$.

¹We write this explanation explicitly here, but from now on we will omit such an explanation as the next cases are similar.

We now want to show that this is the minimum weight. Assume that

$$g : V(LG_n) \rightarrow \{-1, 1, 2\}$$

is another arbitrary SRDF on LG_n . By the construction of the Ladder graph there are 4 vertices of degree 2, namely, the vertices $x = (1, 1), y = (2, 1), z = (1, n)$, and $w = (2, n)$. Clearly it is not possible to have $g(x) = g(y) = -1$, as this would violate the first condition in the definition of a SRDF. A similar argument also holds for the vertices z and w . So we must have one of the following cases:

- $g(x) = -1$ and $g(y) = 1$ or $g(x) = 1$ and $g(y) = -1$,
- $g(x) = g(y) = 1$,
- $g(x) = 1$ and $g(y) = 2$ or $g(x) = 2$ and $g(y) = 1$,
- $g(x) = -1$ and $g(y) = 2$ or $g(x) = 2$ and $g(y) = -1$, and
- $g(x) = g(y) = 2$.

It is easy to check that following the greedy algorithm, each of the above cases leads to the following inequalities:

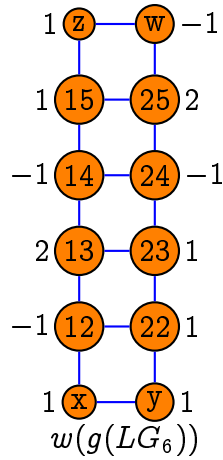
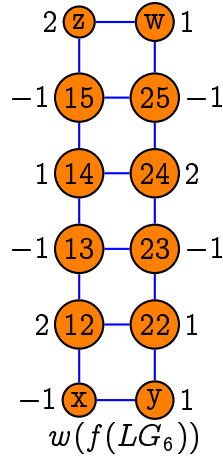
$$\begin{aligned} |V_{-1}| &\leq (n - 1), \\ |V_1| &\geq \begin{cases} \lfloor n/2 \rfloor + 1, & \text{if } n \text{ is even,} \\ \lfloor n/2 \rfloor + 2, & \text{if } n \text{ is odd,} \end{cases} \\ |V_2| &\geq \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

Here by greedy algorithm, we mean a constructive step by step approach where we assign our new label to a vertex by considering two key points: Firstly, the new label should comply with the SRDF conditions, and secondly, we must select the minimum possible value from the set $\{-1, 1, 2\}$ with respect to the previous labeled vertices. This approach will be clearer by doing an example:

Let's consider the ladder graph LG_6 . Following our recipe we have the following SRDF labelling:

$$f(v) = \begin{cases} -1 & \text{if } v = (1, i), i \geq 1 \text{ and } i \text{ is odd, except when } i = n \equiv 1 \pmod{4}; \\ & v = (2, i), i \geq 3 \text{ and } i \text{ is odd, except when } i = n \equiv 3 \pmod{4}; \\ 1 & \text{if } v = (2, 1), v = (1, i) \text{ and } i \equiv 0 \pmod{4}, v = (1, n) \text{ for } n \equiv 1 \pmod{4}; \\ & v = (2, i) \text{ and } i \equiv 2 \pmod{4}, v = (2, n) \text{ and } n \equiv 2, 3 \pmod{4}; \\ 2 & \text{if } v = (1, i) \text{ and } i \equiv 2 \pmod{4}, v = (2, i) \text{ and } i \equiv 0 \pmod{4}. \end{cases}$$

$$\gamma_{SR}(LG_6) = \left\lfloor \frac{6+2}{2} \right\rfloor + 1 = \left\lfloor \frac{6+2}{2} \right\rfloor + 1 = 5.$$



Now, let's assume another $g : V(LG_6) \rightarrow \{-1, 1, 2\}$ where $g(x) = g(y) = 1$ we follow the greedy algorithm for this case as an illustration.

Just keep in mind that we construct our labeling under greedy algorithm with respect to g in such a way that in each step we keep the two SRDF conditions satisfied and that we choose the minimum value for labeling our next vertices. Now we can assign $g(12) = -1$ or $g(22) = -1$ but not both since it violates the first condition of an SRDF. So we can assign $g(12) = -1$ and $g(22) = 1$ to keep it minimum.

Now the only option for vertex 13 is $g(13) = 2$ since we already assigned $g(12) = -1$ and hence the second condition of SRDF must be satisfied. To keep our new labeling minimum, we have to assign $g(23) = 1$. By following the same reasoning we can assign $g(14) = g(24) = -1$. For the next step, we can not assign $g(15) = -1$ since it violate the first condition of SRDF for vertex 14 with label $g(14) = -1$, so, to keep our g -labeling minimum we can put $g(15) = 1$ and for vertex 25, we must assign $g(25) = 2$ since $g(24) = -1$. Finally, for the vertices w and z we can assign $g(w) = -1$ and $g(z) = 2$ to keep our labeling both valid and minimum with respect to the previous labelling.

Now, we can see that

$$w(g) = 6 > 5 = w(f).$$

This illustrate a greedy algorithm approach for the case where $g(x) = g(y) = 1$. The arguments for the other cases are very similar.

Consequently, we have

$$w(g) = 2|V_2| + |V_1| - |V_{-1}| \geq \left\lfloor \frac{n+2}{2} \right\rfloor + 1$$

This completes the proof. \square

Now, we turn our attention to the complement of the Ladder graph LG_n^C and determine their signed Roman domination number. As an illustration, the complement of LG_n^C when $n = 2$ and 3 is shown in Figure 3 with $\gamma_{SR}(LG_2^C) = 2$ and $\gamma_{SR}(LG_3^C) = 3$.

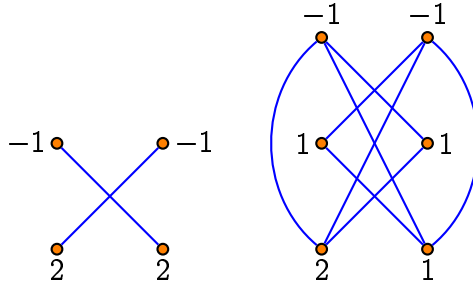


FIGURE 3. SRDF for LG_2^C and LG_3^C .

Theorem 2. Let LG_n^C be complement of the Ladder graph. Then, for all $n \geq 4$, we have

$$\gamma_{SR}(LG_n^C) = 2.$$

From (1.1), we get the bound of $\gamma_{SR}(LG_n^C) > 1$ for $n \geq 2$.

Proof of Theorem 2. For the graph LG_n^C , consider the function

$$f : V(LG_n^C) \rightarrow \{-1, 1, 2\}$$

given by

$$f(v) = \begin{cases} -1 & \text{if } v = (1, i), 2 \leq i \leq n-1, v = (2, 2) \text{ and } v = (2, n-1); \\ 1 & \text{if } v = (2, i) \text{ with } i \neq 2, n-1; \\ 2 & \text{if } v = (1, 1) \text{ and } v = (1, n). \end{cases}$$

Clearly, f is a signed Roman domination function. By the construction of our labels, it is easy to verify the following

$$|V_{-1}| = n, \quad |V_1| = n-2, \quad \text{and} \quad |V_2| = 2.$$

Then, we have

$$\gamma_{SR}(LG_n^C) \leq 2|V_2| + |V_1| - |V_{-1}| = 2.$$

We now show that this is the minimum weight function. Let us assume that

$$g : V(LG_n^C) \rightarrow \{-1, 1, 2\}$$

is another arbitrary SRDF on LG_n^C , where $n \geq 4$. By the construction of LG_n^C , there are four vertices of degree $2n-3$ and the other $2n-4$ internal vertices are of degree $2n-4$. Let the four vertices that have degree $2n-3$ be $x = (1, 1)$, $y = (2, 1)$, $z = (1, n)$, and $w = (2, n)$.

Clearly $g(x) = g(y) = 1$ is not possible as this would violate the second condition of being a SRDF, as we know $g(1, i) = -1$ for $2 \leq i \leq n-1$ in this case, otherwise we will get $w(g) > w(f)$. Similarly we can conclude that $g(w) = g(z) = 1$ is not possible.

If $g(y) = -1$, since y is adjacent to all vertices $v = (1, i)$ where $2 \leq i \leq n$ and $v = (2, i)$ where $3 \leq i \leq n$, in LG_n^C , we have

$$\begin{aligned} \sum_{v \in N_{LG_n^C}[y]} g(v) &= \sum_{i=2}^n g(1, i) + \sum_{i=1, i \neq 2}^n g(2, i) \\ &= (-1) \cdot (n-2) + 2 + 1 \cdot (n-3) - 2 < 0. \end{aligned}$$

This clearly violates the second condition for g being a SRDF. Similarly if $g(2, i) = -1$ for $3 \leq i \leq n$ and $i \neq n-1$, we will get $\sum_{v \in N_{LG_n^C}[y]} g(v) < 1$, which also violates the second condition for g being a SRDF. This completes the proof. \square

3. SIGNED ROMAN DOMINATION NUMBER FOR LADDER GRAPHS CIRCULAR LADDER GRAPH AND ITS COMPLEMENT

In this section we look at a different cartesian product of graphs, which we define below.

Definition 3. The *circular Ladder graph* of order $2n$, denoted by LC_n , is the Cartesian product of a cycle graph C_n , $n \geq 3$ and an edge P_2 . In other words:

$$LC_n := C_n \times P_2.$$

It is easy to check that LC_n has $2n$ vertices and $3n$ edges. As an illustration, the circular Ladder graphs of orders 3, 4 and 5 are shown in Figure 4. For simplicity, we encode the vertices $(1, i)$ and $(2, i)$ with $1i$ and $2i$ for $i = 1, 2, \dots, n$.

Example 1. For LC_5 , the signed Roman domination number is equal to 4 (see Figure 4 (c)).

Theorem 3. Let LC_n be the circular Ladder graph of the order $2n$. Then, for all $n \geq 3$, $n \neq 5$, we have

$$\gamma_{SR}(LC_n) = \begin{cases} \left\lfloor \frac{n+2}{2} \right\rfloor + 1, & \text{if } n \equiv 0, 2, 3 \pmod{4}, \\ \left\lfloor \frac{n+2}{2} \right\rfloor + 2, & \text{if } n \equiv 1 \pmod{4}. \end{cases}$$

From (1.1) we get a bound of

$$\gamma_{SR}(LG_n) \geq \frac{n}{4}.$$

Proof. For the circular Ladder graph LC_n , $n \geq 3$ consider the function

$$f : V(LC_n) \rightarrow \{-1, 1, 2\}$$

given by

$$f(v) = \begin{cases} -1, & \text{if } v = (1, i) \text{ for } i \geq 1 \text{ and } i \text{ is odd; } v = (2, i) \text{ for } i \geq 3 \text{ and } i \text{ is odd } (i \neq n). \\ 1, & \text{if } v = (2, 1); v = (1, i) \text{ and } i \equiv 0 \pmod{4}; v = (2, i) \text{ and } i \equiv 2 \pmod{4}; \\ & v = (2, n) \text{ and } n \equiv 3 \pmod{4} \\ 2, & \text{if } v = (1, i) \text{ and } i \equiv 2 \pmod{4}; v = (2, i) \text{ and } i \equiv 0 \pmod{4}; \\ & v = (2, n) \text{ and } n \equiv 0, 1 \pmod{4}. \end{cases}$$

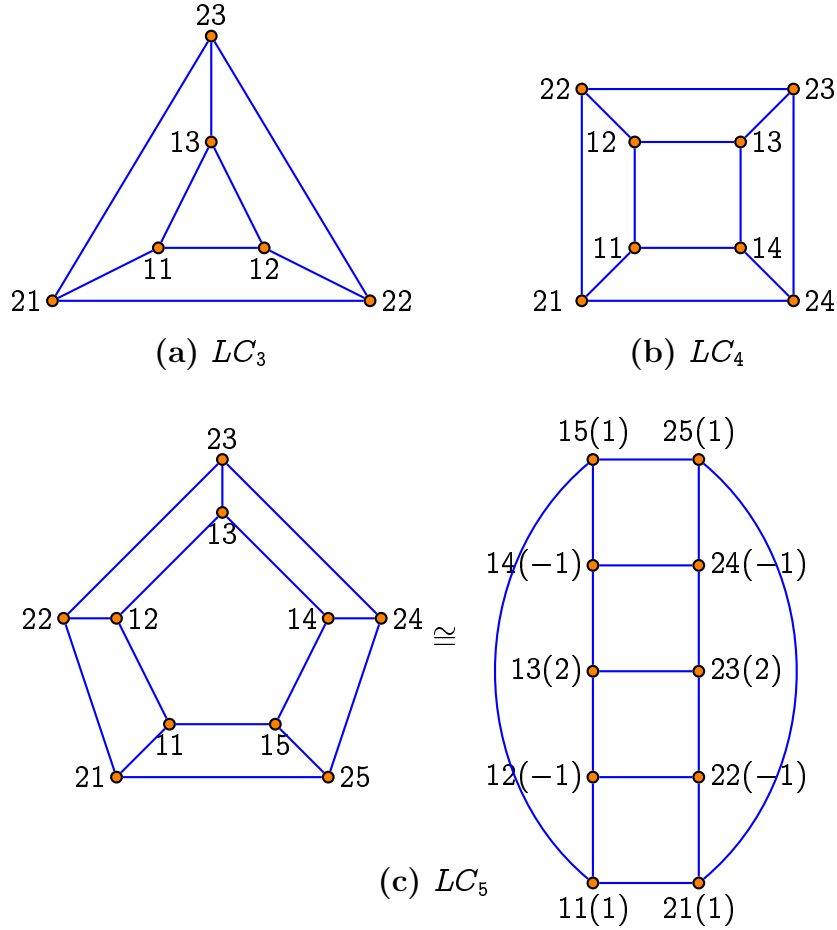


FIGURE 4. The circular Ladder graphs of orders 3, 4 and 5.

It is clear that f is a SRDF. By our labeling, it is easy to show that

$$\begin{aligned}
 |V_{-1}| &= n - 1, \\
 |V_1| &= \begin{cases} \left\lfloor \frac{n+2}{2} \right\rfloor, & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\ \left\lceil \frac{n+2}{2} \right\rceil, & \text{if } n \equiv 3 \pmod{4}. \end{cases} \\
 |V_2| &= \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } n \equiv 0, 2, 3 \pmod{4}, \\ \left\lceil \frac{n+2}{2} \right\rceil, & \text{if } n \equiv 1 \pmod{4}. \end{cases}
 \end{aligned}$$

It is easy to verify the following inequality

$$\gamma_{SR}(LC_n) \leq 2|V_2| + |V_1| - |V_{-1}| = \begin{cases} \left\lfloor \frac{n+2}{2} \right\rfloor + 1, & \text{if } n \equiv 0, 2, 3 \pmod{4}, \\ \left\lceil \frac{n+2}{2} \right\rceil + 2, & \text{if } n \equiv 1 \pmod{4}. \end{cases}$$

We now show that our function is the minimum SRDF. Let us assume that

$$g : V(LC_n) \rightarrow \{-1, 1, 2\}$$

is another arbitrary SRDF on LC_n . By the construction of the circular Ladder graph all vertices have the same degree (that is, LC_n is a 3-regular graph). So, whether we select the vertices from the interior or exterior, it makes no difference. Therefore, we will choose the four exterior vertices, namely $x = (1, 1)$, $y = (2, 1)$, $w = (1, n)$ and $z = (2, n)$.

Clearly, it is not possible to have the following cases

- $g(x) = g(y) = g(w) = -1$,
- $g(x) = g(y) = g(z) = -1$,
- $g(x) = g(w) = g(z) = -1$,
- $g(y) = g(z) = g(w) = -1$, and
- $g(x) = g(y) = g(w) = g(z) = -1$.

This is because if any of the above occurs, then we will have the following cases (in order)

- $\sum_{v \in N_{LC_n}[x]} g(v) < 0$,
- $\sum_{v \in N_{LC_n}[y]} g(v) < 0$,
- $\sum_{v \in N_{LC_n}[w]} g(v) < 0$,
- $\sum_{v \in N_{LC_n}[z]} g(v) < 0$, and
- $\sum_{v \in N_{LC_n}[u]} g(v) < 0$ for $u \in \{x, y, w, z\}$.

So, we must have one of the following cases (some cases will be excluded by the symmetric role the vertices play):

- $g(x) = g(y) = -1$ and $g(w) = g(z) = 1$ (or, $g(x) = g(y) = 1$ and $g(w) = g(z) = -1$),
- $g(x) = g(y) = g(z) = g(w) = 1$,
- $g(x) = g(y) = -1$, $g(w) = 1$ and $g(z) = 2$ (or, $g(x) = g(y) = -1$, $g(w) = 2$ and $g(z) = 1$),
- $g(x) = g(y) = g(w) = g(z) = 2$,
- $g(x) = g(y) = 2$ and $g(w) = g(z) = 1$,
- $g(x) = g(y) = g(w) = 1$ and $g(z) = 2$,
- $g(x) = g(y) = g(w) = 2$ and $g(z) = 1$,
- $g(x) = g(y) = 2$ and $g(w) = g(z) = -1$; and
- $g(x) = g(y) = g(w) = 2$ and $g(z) = -1$.

It is easy to check that following the greedy algorithm, each of the above cases give us

$$\begin{aligned} |V_{-1}| &\leq n - 1, \\ |V_1| &\geq \begin{cases} \left\lfloor \frac{n+2}{2} \right\rfloor, & \text{if } n \equiv 0, 1, 2 \pmod{4}; \\ \left\lceil \frac{n+2}{2} \right\rceil, & \text{if } n \equiv 3 \pmod{4}, \end{cases} \\ |V_2| &\geq \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } n \equiv 0, 2, 3 \pmod{4}; \\ \left\lceil \frac{n+2}{2} \right\rceil, & \text{if } n \equiv 1 \pmod{4}. \end{cases} \end{aligned}$$

Consequently, summing up these inequalities we have

$$w(g) = 2|V_2| + |V_1| - |V_{-1}| \geq \begin{cases} \left\lfloor \frac{n+2}{2} \right\rfloor + 1, & \text{if } n \equiv 0, 2, 3 \pmod{4}, \\ \left\lceil \frac{n+2}{2} \right\rceil + 2, & \text{if } n \equiv 1 \pmod{4}. \end{cases}$$

This completes the proof. \square

Now, we turn our attention to the complement of the circular Ladder graph LC_n^C .

Definition 4. The complement of the circular Ladder graph (denoted by LC_n^C) is the complement of the cartesian product of path graph P_2 and cycle graph C_n , that is

$$LC_n^C = (P_2 \times C_n)^C.$$

It is easy to see that for the complement of the circular Ladder graph of order $2n$ (that is LC_n^C), we have

- (i) $|V(LC_n^C)| = 2n$,
- (ii) $|E(LC_n^C)| = \frac{2n(2n-1)}{2} - |E(LC_n)|$,
- (iii) LC_n^C is a regular graph of degree $(2n-4)$.

Example 2. The complement of the circular Ladder graph of order 3 and 4 has signed Roman domination number 4 (see Figure 5).

Theorem 4. Let LC_n^C be the complement of the circular Ladder graph of order n . Then for all $n \geq 5$, we have

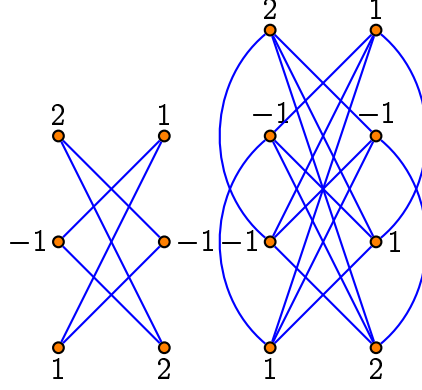
$$\gamma_{SR}(LC_n^C) = 3.$$

Proof. For the complement of circular Ladder graph LC_n^C when $n \geq 5$, consider the function

$$f : V(LC_n^C) \rightarrow \{-1, 1, 2\}$$

given by:

$$f(v) = \begin{cases} -1, & \text{if } v = (1, 1), v = (1, 4), v = (2, i) \text{ for } 1 \leq i \leq n, i \neq 2, 5, \\ 1, & \text{if } v = (1, i) \text{ for } 5 \leq i \leq n, v = (2, 2), \\ 2, & \text{if } v = (1, 2), v = (1, 3) \text{ and } v = (2, 5). \end{cases}$$

FIGURE 5. SRDF for LC_3^C and LC_4^C .

Clearly, f is a SRDF. By the construction of our labelling, it easy to show that

$$\begin{aligned} |V_{-1}| &= 2 + (n - 2) = n, \\ |V_1| &= (n - 4) + 1 = n - 3, \\ |V_2| &= 3. \end{aligned}$$

Hence, we have

$$\gamma_{SR}(LC_n^C) \leq 2|V_2| + |V_1| - |V_{-1}| = 3.$$

Now, we prove our function is the minimum SRDF. Let us assume that

$$g : V(LC_n^C) \rightarrow \{-1, 1, 2\}$$

be another arbitrary SRDF on LC_n^C , $n \geq 5$. Since the complement of the circular graph LC_n^C is a regular graph of degree $2n - 4$, so all vertices in LC_n^C is adjacent to $2n - 4$ vertices. To show our function is the minimum then we have the following cases for $g(v)$, depending on $f(v) = 2$ or $f(v) = 1$:

- (1) If $g(V(LC_n^C)) = f(V(LC_n^C))$ except $g(1, 3) \neq f(1, 3)$, since $f(1, 3) = 2$ but if $g(1, 3) = 1 < f(1, 3)$. Clearly it is not possible to have $g(1, 3) = 1$, Otherwise we will have $\sum_{v \in N_{LC_n^C}[x]} g(v) = 0$ where $x = (1, 1)$ be the exterior vertex of LC_n^C , so

$$\begin{aligned} \sum_{v \in N_{LC_n^C}[x]} g(v) &= \sum_{i=1, i \neq 2}^{n-1} g((1, i)) + \sum_{i=2}^n g((2, i)) \\ &= 0. \end{aligned}$$

- (2) If $g(V(LC_n^C)) = f(V(LC_n^C))$ except $g(1, 2) \neq f(1, 2)$, since $f(1, 2) = 2$, but $g(1, 2) = 1 < f(1, 2)$. Clearly, it is not possible to have $g(1, 2) = 1$ since otherwise we will have $\sum_{v \in N_{LC_n^C}[x]} g((1, i)) = 0$ for $i = 2, 3, 4$ or $v = (2, 3)$.

- (3) $g(V(LC_n^C)) = f(V(LC_n^C))$ except $g(2, 5) \neq f(2, 5)$ since $f(2, 5) = 2$, let $g(2, 5) = 1$. Clearly it is not possible to have $g(2, 5) = 1$ otherwise we will have $\sum_{v \in N_{LC_n^C}[x]} g(v) = 0$ similar to the first case.

- (4) If $g(V(LC_n^C)) = f(V(LC_n^C))$ except $g(2, 2) = -1$, because $f(2, 2) = 1$. Clearly, it is not possible to have $g(2, 2) = -1$, otherwise we will have

$$\begin{aligned} \sum_{v \in N_{LC_n^C}[x]} g(v) &= \sum_{i=1, i \neq 2}^{n-1} g((1, i)) + \sum_{i=2}^n g((2, i)) \\ &= -2. \end{aligned}$$

This completes the proof. □

4. CONCLUDING REMARKS

- (1) It is a natural next step to find the values of $\gamma_{SR}(G)$ when G is either a grid graph or its complement. We leave that problem open.
- (2) Some other closely related types of domination numbers have been studied in the literature. We point out two of these: the concept of signed Roman k -Domination in graphs by Henning and Volkmann [HV16] ($k = 1$ corresponds to a SRDF), and the concept of weak signed Roman Domination in graphs by Volkmann [Vol20]. It would be interesting to study the graphs we study in this paper for these domination numbers.

REFERENCES

- [AAHL⁺14] H. Abdollahzadeh Ahangar, Michael A. Henning, Christian Löwenstein, Yancai Zhao, and Vladimir Samodivkin. Signed Roman domination in graphs. *J. Comb. Optim.*, 27(2):241–255, 2014.
- [Ber73] Claude Berge. *Graphs and hypergraphs*. Translated by Edward Minieka, volume 6 of *North-Holland Math. Libr.* Elsevier (North-Holland), Amsterdam, 1973.
- [BVA16] A. Behtoei, E. Vatandoost, and Rajol Abad F. Azizi. Signed Roman domination number and join of graphs. *J. Algebr. Syst.*, 4(1):65–77, 2016.
- [GJ79] Michael R. Garey and David S. Johnson. *Computers and intractability. A guide to the theory of NP-completeness*. A Series of Books in the mathematical Sciences. San Francisco: W. H. Freeman and Company. X, 338 p. (1979)., 1979.
- [HH03] Michael A. Henning and Stephen T. Hedetniemi. Defending the Roman Empire—a new strategy. *Discrete Math.*, 266(1-3):239–251, 2003.
- [HHS98] Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater, editors. *Domination in graphs. Advanced topics*, volume 209 of *Pure Appl. Math.*, Marcel Dekker. New York, NY: Marcel Dekker, 1998.
- [HV16] Michael A. Henning and Lutz Volkmann. Signed Roman k -domination in graphs. *Graphs Comb.*, 32(1):175–190, 2016.
- [HYZZ20] Xia Hong, Tianhu Yu, Zhengbang Zha, and Huihui Zhang. The signed Roman domination number of two classes graphs. *Discrete Math. Algorithms Appl.*, 12(2):18, 2020. Id/No 2050024.
- [Ore62] Øystein Ore. *Theory of graphs*, volume 38 of *Colloq. Publ., Am. Math. Soc.* American Mathematical Society (AMS), Providence, RI, 1962.
- [Ste99] Ian Stewart. Defend the roman empire! *Sci. Amer.*, 281(6):136–138, 1999.
- [Vol20] Lutz Volkmann. Weak signed Roman domination in graphs. *Commun. Comb. Optim.*, 5(2):111–123, 2020.

UNIVERSITY OF DUHOK, UNIVERSITY CAMPUS, ZAKHO STREET, DUHOK, KURDISTAN REGION, IRAQ

Email address: dilpakhaje@uod.ac, delbrin.ahmed@uod.ac

SCHOOL OF MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS, LS2 9JT, UK

Email address: h.izanloo@leeds.ac.uk

MATHEMATICAL AND PHYSICAL SCIENCES DIVISION, SCHOOL OF ARTS AND SCIENCES, AHMEDABAD UNIVERSITY, NAVRANGPURA, AHMEDABAD - 380009, GUJARAT, INDIA

Email address: manjil@saikia.in