BIASES IN NON-UNITARY PARTITIONS

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Abstract. Recently, the concept of parity bias in integer partitions has been studied by several authors. We continue this study here, but for non-unitary partitions (namely, partitions with parts greater than 1). We prove analogous results for these restricted partitions as those that have been obtained by Kim, Kim and Lovejoy (2020) and Kim and Kim (2021). We also look at inequalities between two classes of partitions studied by Andrews (2019) where the parts are separated by parity (either all odd parts are smaller than all even parts or vice versa).

1. Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of n is a non-increasing sequence of natural numbers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Here, each λ_i is called a part of the partition λ of n (written as $\lambda \vdash n$) and the length of the partition, denoted by $\ell(\lambda)$ is k. Partitions have been studied since the time of Euler, and continues to be a serious topic for ongoing research in several directions. A good introduction to the subject is given in the masterly treatment of Andrews [And98].

In the theory of partitions, inequalities arising between two classes of partitions have a long tradition of study, see for instance work in this direction by Alder [Ald48], Andrews [And13], McLaughlin [ML16], Chern, Fu, and Tang [CFT18] and Berkovich and Uncu [BU19], among others. In 2020, Kim, Kim and Lovejoy [KKL20] introduced a phenomenon in integer partitions called *parity bias*, wherein the number of partitions of n with more odd parts (denoted by $p_o(n)$) are more in number than the number of partitions of n with more even parts (denoted by $p_e(n)$). That is, they proved for $n \neq 2$, $p_o(n) > p_e(n)$. They also conjectured a similar inequality for partitions with only distinct parts. For $n > 19$, they conjectured that $d_o(n) > d_e(n)$, where $d_o(n)$ (resp. $d_e(n)$) denotes the number of partitions of n with distinct parts with more odd parts (resp. even parts) than even parts (resp. odd parts). Further generalizations of the results of Kim, Kim and Lovejoy [KKL20] have been found by Kim and Kim [KK21] and Chern [Che22]. Most of the proofs of the results in these papers use techniques arising from q-series methods.

The first two authors, in collaboration with Banerjee, Bhattacharjee and Dastidar [BBD+22] proved both the above quoted result and conjecture of Kim, Kim and Lovejoy [KK21] using combinatorial means. In addition, they proved several more results on parity biases of partitions with restrictions on the set of parts. For a nonempty set $S \subsetneq \mathbb{Z}_{\geq 0}$, define

$$\begin{aligned} P_e^S(n) &:= \{\lambda \in P_e(n) : \lambda_i \notin S\}, \\ \text{and} \quad P_o^S(n) &:= \{\lambda \in P_o(n) : \lambda_i \notin S\}, \end{aligned}$$

where the set $P_e(n)$ (resp. $P_o(n)$) consists of all partitions of n with more even parts (resp. odd parts) than odd parts (resp. even parts). Let us denote the number of partitions of $P_e^S(n)$ (resp. $P_o^S(n)$) by $p_e^S(n)$ (resp. $p_o^S(n)$). Banerjee *et al.* [BBD+22] proved the following result.

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Theorem 1.1 (Banerjee *et al.*, [BBD+22]). *For positive integers n, the following inequalities are true (the range is given in the brackets),*

$$p_o^{\{1\}}(n) < p_e^{\{1\}}(n), \quad (n > 7), \tag{1}$$

$$p_o^{\{2\}}(n) > p_e^{\{2\}}(n), \quad (n \geq 1), \tag{2}$$

and

$$p_o^{\{1,2\}}(n) > p_e^{\{1,2\}}(n), \quad (n > 8). \tag{3}$$

All of the proofs of the above inequalities were by using combinatorial techniques. Although they do not use this term, but partitions where the part 1 does not appear are called *non-unitary partitions* and we will use this terminology in this paper.

The primary goal of this paper is to use analytical techniques and prove results of the type proved by Banerjee *et al.*, that is about parity biases in partitions with certain restrictions on its allowed parts. We reprove the inequality (1) using analytical techniques, as well as prove results in a similar setup for the biases discussed in the work of Kim and Kim [KK21]. We further look at some simply derived results on biases in partitions with a restriction on the size of the minimum part as well as on parity of the number of parts of a given parity. Our techniques can also be used to prove partition inequalities of the type where the number of partitions of a certain class of partitions are more than another class. This is explored for two classes of partitions studied by Andrews [And19] where the parts are separated by parity, where either all odd parts are smaller than all even parts or vice versa.

The paper is structured as follows: in Section 2 we state our main results, namely on biases in ordinary non-unitary partitions, in Section 3 we prove results on biases in partitions with restrictions on the smallest part, in Section 4 we look at inequalities on partitions with parts separated by parity. Finally we close the paper with some concluding remarks in Section 5.

2. Biases in Ordinary Non-Unitary Partitions

Using analytical techniques we will give a proof of following result which was proved by Banerjee *et. al.* [BBD+22] combinatorially. We modify the notation a bit and let $q_e(n)$ (resp. $q_o(n)$) be the number of non-unitary partitions of n where the number of even (resp. odd) parts are more than the number of odd (resp. even) parts.

Theorem 2.1 (Theorem 1.5, [BBD+22]). *For all positive integers $n \geq 8$, we have*

$$q_o(n) < q_e(n).$$

Let $p_{j,k,m}(n)$ be the number of partitions of n such that there are more parts congruent to j modulo m than parts congruent to k modulo m, for $m \geq 2$. Then, Kim and Kim [KK21] proved that for all positive integers $n \geq m^2 - m + 1$, we have

$$p_{1,0,m}(n) > p_{0,1,m}(n).$$

Let us now denote by $q_{j,k,m}(n)$ the number of non-unitary partitions of n such that there are more parts congruent to j modulo m than parts congruent to k modulo m, for $m > 2$. Then, we have the following result.

Theorem 2.2. *For $n \geq 4m + 3$, we have*

$$q_{0,1,m}(n) > q_{1,0,m}(n)$$

We need some preliminaries before we can prove the above results. We use the standard q-series notation

$$(a)_n = (a;q)_n = \prod_{k=1}^{n}(1 - aq^{k-1}), \quad |q| < 1,$$

and

$$(a_1, a_2, \dots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n.$$

Also, recall Heine's transformation [GR04, Appendix III.1], which says for $|z|, |q|, |b| \leq 1$, we have

$$\sum_{n\geq 0} \frac{(a)_n (b)_n}{(q)_n (c)_n} z^n = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \sum_{n\geq 0} \frac{(z)_n (c/b)_n}{(q)_n (az)_n} b^n. \tag{4}$$

By appropriately iterating Heine's transformation, we obtain [GR04, Appendix III.3] what is sometimes called the q-analogue of Euler's transformation, which says that for $|z|$, $|\frac{abz}{c}| \leq 1$, we have

$$\sum_{n\geq 0} \frac{(a)_n (b)_n}{(q)_n (c)_n} z^n = \frac{(abz/c)_\infty}{(z)_\infty} \sum_{n\geq 0} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} (abz/c)^n. \tag{5}$$

We also recall an identity of Sylvester [SF82, p. 281]: for $|q| \leq 1$, we have

$$(-xq)_\infty = \sum_{n\geq 0} \frac{(-xq)_n}{(q)_n} (1 + xq^{2n+1}) x^n q^{n(3n+1)/2}. \tag{6}$$

By standard combinatorial arguments, we have that $\dfrac{q^{bn}}{(q^2;q^2)_n}$ is the generating function for partitions with exactly n odd parts with the minimum odd part being at least b, as well as it is the generating function for partitions with exactly n even parts with the minimum even part being at least b. We will use this in the proof below without commentary.

Proof of Theorem 2.1. Let $P_o(q)$ (resp. $P_e(q)$) be the generating functions of $q_o(n)$ (resp. $q_e(n)$. Then, we have

$$P_o(q) = \sum_{n\geq 0} \frac{q^{3n}}{(q^2;q^2)_n^2} - \sum_{n\geq 0} \frac{q^{5n}}{(q^2;q^2)_n^2} = q^3 + q^5 + q^6 + q^7 + 2q^8 \cdots,$$

and,

$$P_e(q) = \frac{1}{(q^2;q)_\infty} - \sum_{n\geq 0} \frac{q^{3n}}{(q^2;q^2)_n^2} = q^2 + 2q^4 + 3q^6 + q^7 + 5q^8 \cdots.$$

Substituting $c = q^4, a, b \to 0, z = q^3, q \to q^2$ in equation (5) we get

$$\begin{aligned}
P_o(q) &= \sum_{n\geq 1} \frac{q^{3n}}{(q^2;q^2)_n^2} (1 - q^{2n}) \\
&= \frac{1}{(1-q^2)} \sum_{n\geq 1} \frac{q^{3n}}{(q^4;q^2)_{n-1}(q^2;q^2)_{n-1}} = \frac{q^3}{(1-q^2)} \sum_{n\geq 0} \frac{q^{3n}}{(q^4;q^2)_n (q^2;q^2)_n} \\
&= \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 0} \frac{q^{2n^2+5n+3}}{(q^2;q^2)_{n+1}(q^2;q^2)_n} = \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2+n}}{(q^2;q^2)_n (q^2;q^2)_{n-1}} \\
&= \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2+n}}{(q^2;q^2)_n^2} (1 - q^{2n}).
\end{aligned}$$

Substituting $c = q^2, a, b \to 0, z = q^3, q \to q^2$ in equation (5) we get

$$\begin{aligned}
P_e(q) &= \frac{1}{(q^3;q^2)_\infty}\frac{1}{(q^2;q^2)_\infty} - \sum_{n\geq 0}\frac{q^{3n}}{(q^2;q^2)_n^2} \\
&= \frac{1}{(q^3;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n^2}}{(q^2;q^2)_n^2} - \frac{1}{(q^3;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n^2+3n}}{(q^2;q^2)_n^2} \\
&= \frac{1}{(q^3;q^2)_\infty}\sum_{n\geq 1}\frac{q^{2n^2}}{(q^2;q^2)_n^2}(1-q^{3n}).
\end{aligned}$$

Now,

$$P_e(q) - P_o(q) = \frac{1}{(q^3;q^2)_\infty}\sum_{n\geq 1}\frac{q^{2n^2}}{(q^2;q^2)_n^2}(1-q^n).$$

Clearly, for the summands from $n = 2$ onward the coefficients are positive, because if n is even, then $1-q^n$ will be cancelled by a factor of $(q^2;q^2)_n$ and if n is odd, then it will be cancelled by a factor of $(q^3;q^2)_\infty$.

For, $n = 1$, we do the following. Put $x = 1$ in equation (6) to get the following

$$\begin{aligned}
q^3 + q^5 + \frac{1}{(q^3;q^2)_\infty}\frac{q^2(1-q)}{(1-q^2)^2} &= q^3(1+q^2) + \frac{q^2(1-q)^2}{(1-q^2)^2}(-q)_\infty \\
&= q^3(1+q^2) + \frac{q^2}{(1+q)^2}\sum_{n\geq 0}\frac{(-q)_n}{(q)_n}(1+q^{2n+1})q^{\frac{3n^2+n}{2}} \\
&= q^3(1+q^2) + \frac{q^2}{(1+q)} + \frac{q^4(1+q^3)}{(1-q^2)} \\
&\quad + \frac{q^2}{(1+q)^2}\sum_{n\geq 2}\frac{(-q)_n}{(q)_n}(1+q^{2n+1})q^{\frac{3n^2+n}{2}} \\
&= \frac{q^2(1+q^2)}{(1-q^2)} + \frac{q^2}{(1+q)^2}\sum_{n\geq 2}\frac{(-q)_n}{(q)_n}(1+q^{2n+1})q^{\frac{3n^2+n}{2}},
\end{aligned}$$

which gives us

$$\frac{1}{(q^3;q^2)_\infty}\frac{q^2(1-q)}{(1-q^2)^2} = -q^3 - q^5 + \frac{q^2(1+q^2)}{(1-q^2)} + \frac{q^2}{1-q^2}\sum_{n\geq 2}\frac{(-q^2)_{n-1}}{(q^2)_{n-1}}(1+q^{2n+1})q^{\frac{3n^2+n}{2}}.$$

We see that the coefficients for all terms are nonnegative except for q^3 and q^5. The terms of the expansion of the third summand of the RHS consists of terms of the form q^{2i} for all $i \in N$. For $n = 2$ the fourth summand of the RHS gives a series where the terms are of the form q^{2i+1} for all $i \in N$ and $i \geq 4$. For all $n > 2$ the minimum power of q in the expansion of the fourth term of RHS is greater than 9. Also, for all $n > 1$ the minimum power of q in the expansion of $P_e(q) - P_o(q)$ is greater than or equal to 8. So, in each case the coefficient of q^7 is 0. This completes the proof. □

Proof of Theorem 2.2. We start by acknowledging the fact that $\dfrac{q^{bn}}{(q^m;q^m)_n}$ is the generating function with partitions into n parts congruent to $b \pmod{m}$. Let $P_{1,0,m}(q)$ (resp. $P_{0,1m}(q)$) be the generating functions of $q_{1,0,m}(n)$ (resp. $q_{0,1,m}(n)$). Then, we have

$$P_{1,0,m}(q) = \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{q^{(m+1)n}}{(q^m;q^m)_n^2} - \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{q^{(m+1)n+mn}}{(q^m;q^m)_n^2},$$

and

$$P_{0,1,m}(q) = \frac{1}{(q^2;q)_\infty} - \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 0} \frac{q^{(m+1)n}}{(q^m;q^m)_n^2}.$$

Now,

$$\begin{aligned} P_{1,0,m}(q) &= \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 0} \frac{q^{(m+1)n}}{(q^m;q^m)_n^2}(1-q^{mn}) \\ &= \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 1} \frac{q^{(m+1)n}}{(q^m;q^m)_n(q^m;q^m)_{n-1}} \\ &= \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty} \frac{q^{m+1}}{(1-q^m)} \sum_{n\geq 0} \frac{q^{(m+1)n}}{(q^m,q^{2m};q^m)_n} \end{aligned}$$

(by substituting, $q \to q^m, a, b \to 0, c \to q^{2m}$ and $z \to q^{m+1}$ in equation (5), we get)

$$\begin{aligned} &= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty} \frac{q^{m+1}}{(1-q^m)} \sum_{n\geq 0} \frac{q^{mn^2+2mn+n}}{(q^m,q^{2m};q^m)_n} \\ &= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 1} \frac{q^{mn^2+n}(1-q^{mn})}{(q^m;q^m)_n^2}. \end{aligned} \tag{7}$$

Similarly, we have

$$\begin{aligned} P_{0,1,m}(q) &= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 0} \frac{q^{mn^2}}{(q^m;q^m)_n^2} - \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 0} \frac{q^{mn^2+(m+1)n}}{(q^m;q^m)_n^2} \\ &= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 0} \frac{q^{mn^2}}{(q^m;q^m)_n^2}(1-q^{(m+1)n}) \\ &= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 1} \frac{q^{mn^2}}{(q^m;q^m)_n^2}(1-q^{(m+1)n}). \end{aligned} \tag{8}$$

From equations (7) and (8), we get

$$P_{0,1,m}(q) - P_{1,0,m}(q) = \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 1} \frac{q^{mn^2}}{(q^m;q^m)_n^2}(1-q^n).$$

From Kim and Kim [KK21, Lemma 2.1], we see that the above difference has nonnegative coefficients for all q^k with $k > 2m+1$. The summand $n = 2$ is $\dfrac{(q^m;q^m)_\infty q^{4m}}{(q^3;q)_\infty (q^m;q^m)_2^2}$. This shows that coefficients of q^k are positive for $k \geq 4m+3$. In fact, the coefficient of q^{4m} is also positive. So, we have our result. $\square$

3. Further Biases in Restricted Partitions

The following results can also be proved using similar analytical techniques as in the proofs in the previous section.

Theorem 3.1. *Let the minimum part for each partition of n be m and et $n \geq 2m$ be an even number. Denote by $E_{me}(n)$ (resp. $O_{me}(n)$) the number of partitions of n with an even number of odd and even parts, where the number of even parts (resp. odd parts) is more than the number of odd parts (resp. even parts). Then, for all n, we have*

$$O_{me}(n) < E_{me}(n), \quad \text{if } m \text{ is even},$$

and,

$$O_{me}(n) > E_{me}(n), \quad \textit{if } m \textit{ is odd.}$$

Proof. Let the generating function of $E_{me}(n)$ (resp. $O_{me}(n)$) be $E_{me}(q)$ (resp. $O_{me}(q)$).

Case I: If m is odd, say $m = 2b-1$ for some $b \in \mathbb{Z}$. We have,

$$E_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} E_{me}(n)q^n = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{2bn}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{(2b-1)k}}{(q^2;q^2)_k}\right),$$

and,

$$O_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} O_{me}(n)q^n = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{(2b-1)n}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{2bk}}{(q^2;q^2)_k}\right).$$

Now,

$$O_{me}(q) - E_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{(2b-1)n}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{2bk}}{(q^2;q^2)_k}(1-q^{n-k})\right).$$

which clearly has non negative coefficients. This completes the proof of Case I.

Case II: If m is even, say $m = 2c$ for some $c \in \mathbb{Z}$. Then, we have

$$E_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{2cn}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{(2c-1)k}}{(q^2;q^2)_k}\right),$$

and,

$$O_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{(2c+1)n}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{2ck}}{(q^2;q^2)_k}\right).$$

Now,

$$E_{me}(q) - O_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{(2c)n}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{(2c+1)k}}{(q^2;q^2)_k}(1-q^{n-k})\right),$$

which clearly has positive coefficients and it can be seen that the minimum power of q is $2m$. □

Theorem 3.2. *Let the minimum part for each partition of n be m and let $n \geq m$ be an odd number. Denote by $E_{mo}(n)$ (resp. $O_{mo}(n)$) the number of partitions of n with odd number of even and odd parts, where the number of even parts (resp. odd parts) is more than the number of odd parts (resp. even parts). Then, we have*

$$O_{mo}(n) > E_{mo}(n), \quad \textit{if } m \textit{ is odd,}$$

and,

$$O_{mo}(n) < E_{mo}(n), \quad \textit{if } m \textit{ is even.}$$

The proof is similar to the proof of Theorem 3.1, so we leave it to the reader.

4. Inequalities between Partitions with Parts Separated by Parity

Andrews [And18, And19] studied partitions in which parts of a given parity are all smaller than those of the other parity, and proved several interesting results, which have been studied by other authors as well. In this short section we look at some inequalities between two of these classes of partitions that were studied by Andrews. We define,

$$P_{eu}^{ou}(n) := \text{Set of partitions of } n \text{ in which each even part is less than each odd part.}$$
$$P_{ou}^{eu}(n) := \text{Set of partitions of } n \text{ in which each odd part is less than each even part.}$$

Again, let us denote the cardinalities of these two sets by $p_{eu}^{ou}(n)$ and $p_{ou}^{eu}(n)$ respectively. We get the following two generating functions from Andrews [And19].

$$P_{eu}^{ou}(q) := \sum_{n\geq 0} p_{eu}^{ou}(n)q^n = \frac{1}{(1-q)(q^2;q^2)_\infty},$$
$$\text{and} \quad P_{ou}^{eu}(q) := \sum_{n\geq 0} p_{ou}^{eu}(n)q^n = \frac{1}{1-q}\left(\frac{1}{(q;q^2)_\infty} - \frac{1}{(q^2;q^2)_\infty}\right).$$

Note that the set $P_{eu}^{ou}(n)$ includes the partitions with all parts even or odd. But $P_{ou}^{eu}(n)$ does not include the partitions with all parts even.

Banerjee, Bhattacharjee and Dastidar (via a private communication to the second author) proved the following result combinatorially.

Theorem 4.1. *For all $n > 6$, we have*

$$p_{ou}^{eu}(n) > p_{eu}^{ou}(n).$$

For the sake of completeness, we give a proof of this result (which is different from that of Banerjee, Bhattacharjee and Dastidar) a little later.

Again, we look at non-unitary versions of these types of partitions. Let us denote by $Q_{eu}^{ou}(n)$ and $Q_{ou}^{eu}(n)$ the set of non-unitary partitions which are in the sets $P_{eu}^{ou}(n)$ and $P_{ou}^{eu}(n)$ respectively. Let us denote the cardinalities of these two sets by $q_{eu}^{ou}(n)$ and $q_{ou}^{eu}(n)$ respectively. If 1 is a part in any partition in any partition inside $P_{eu}^{ou}(n)$, then no even part is there in that partition. So, we get the following generating function.

$$Q_{eu}^{ou}(q) := \sum_{n\geq 0} q_{eu}^{ou}(n)q^n = \frac{1}{(1-q)(q^2;q^2)_\infty} - \frac{q}{(q;q^2)_\infty}.$$

If 1 is not a part in any partition inside $P_{ou}^{eu}(n)$, then the least odd part of that partition is greater than or equal to 3. So, in any case the partition can not contain 2 as a part. Therefore, we get the following generating function (for details see Andrews [And19].)

$$\begin{aligned}
Q_{ou}^{eu}(q) := \sum_{n\geq 0} q_{ou}^{eu}(n)q^n &= \sum_{n\geq 0} \frac{q^{2n+3}}{(q^3;q^2)_{n+1}(q^{2n+4};q^2)_\infty} \\
&= \frac{q}{(q^2;q^2)_\infty}\left(\sum_{n\geq 0}\frac{q^{2n}(q^2;q^2)_n}{(q^3;q^2)_n} - 1\right) \\
&= \frac{1}{(q;q^2)_\infty} - \frac{q+1}{(q^2;q^2)_\infty}.
\end{aligned}$$

We now have the following result.

Theorem 4.2. *For all $n > 3$, we have*

$$q_{ou}^{eu}(n) < q_{eu}^{ou}(n).$$

Before proving this result, we need some auxillary identities. Due to Euler, we know [And98, p. 19]

$$\frac{1}{(a;q)_\infty} = \sum_{n\geq 0} \frac{a^n}{(q;q)_n}.$$

Therefore,

$$\frac{1}{(q;q^2)_\infty} = \sum_{n\geq 0} \frac{q^n}{(q^2;q^2)_n} = \sum_{n\geq 0} \frac{q^n}{(-q)_n(q)_n},$$

and

$$\frac{1}{(q^2;q^2)_\infty} = \sum_{n\geq 0} \frac{q^{2n}}{(q^2;q^2)_n} = \sum_{n\geq 0} \frac{q^{2n}}{(-q)_n(q)_n}.$$

Here (and afterwards) we use the shorthand notation $(a)_n := (a;q)_n$ and $(a)_\infty = (a;q)_\infty$.

Now, Substituting $c = -q, a, b \to 0, z = q$ in equation (4) we get

$$\sum_{n\geq 0} \frac{q^n}{(-q)_n(q)_n} = \frac{1}{(-q)_\infty(q)_\infty} \sum_{n\geq 0} q^{\frac{n^2+n}{2}}.$$

Again, substituting $c = -q, a, b \to 0, z = q^2$ in equation (4) we get

$$\sum_{n\geq 0} \frac{q^{2n}}{(-q)_n(q)_n} = \frac{1}{(-q)_\infty(q)_\infty} \sum_{n\geq 0} (1-q^{n+1}) q^{\frac{n^2+n}{2}}.$$

We use these identities in the remainder of this section without commentary.

Proof of Theorem 4.2. We have,

$$\begin{aligned}
Q^{ou}_{eu}(q) - Q^{eu}_{ou}(q) &= \frac{2-q^2}{1-q} \cdot \frac{1}{(q^2;q^2)_\infty} - \frac{1+q}{(q;q^2)_\infty} = \sum_{n\geq 0} \frac{2-q^2}{1-q} \cdot \frac{q^{2n}}{(q^2;q^2)_n} - \sum_{n\geq 0} \frac{(1+q)q^n}{(q^2;q^2)_n} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} \left(\frac{(2-q^2)(1-q^{n+1})}{1-q} - (1+q) \right) q^{\frac{n^2+n}{2}} \\
&= \frac{1}{(q^2;q^2)_\infty} \left(\sum_{n\geq 0} (1+q+q^2+\cdots+q^n) q^{\frac{n(n+1)}{2}} - \sum_{n\geq 0} (1+q) q^{\frac{(n+1)(n+2)}{2}} \right) \\
&= \frac{1}{(q^2;q^2)_\infty} \left(1 + \sum_{n\geq 0} (1+q+q^2+\cdots+q^{n+1}) q^{\frac{(n+1)(n+2)}{2}} - \sum_{n\geq 0} (1+q) q^{\frac{(n+1)(n+2)}{2}} \right) \\
&= \frac{1}{(q^2;q^2)_\infty} \left(1 + \sum_{n\geq 1} (q^2+\cdots+q^{n+1}) q^{\frac{(n+1)(n+2)}{2}} \right).
\end{aligned}$$

Hence, coefficients of q^n in $Q^{ou}_{eu}(q) - Q^{eu}_{ou}(q)$ are positive for all $n > 3$. □

We end this section by giving a proof of Theorem 4.1.

Proof of Theorem 4.1. We have,

$$\begin{aligned}
P^{eu}_{ou}(q) - P^{ou}_{eu}(q) &= \frac{1}{1-q} \left(\frac{1}{(q;q^2)_\infty} - \frac{2}{(q^2;q^2)_\infty} \right) = \frac{1}{(1-q)(q^2;q^2)_\infty} \left(\frac{(q^2;q^2)^2_\infty}{(q;q)_\infty} - 2 \right) \\
&= \frac{1}{(1-q)(q^2;q^2)_\infty} \left(\sum_{n\geq 0} q^{\frac{n^2+n}{2}} - 2 \right),
\end{aligned}$$

where the last equality follows from [And98, p. 23].

We now note that the products on the RHS can be rewritten as

$$(1+q+q^2+q^3+\cdots)\prod_{i=1}^{\infty}(1+q^{2i}+q^{4i}+q^{6i}+\cdots)(-1+q+q^3+q^6+q^{10}+q^{15}+\cdots).$$

Let $(1+q+q^2+q^3+\cdots)\prod_{i=1}^{\infty}(1+q^{2i}+q^{4i}+q^{6i}+\cdots)=\sum\limits_{n\geq 0}a_nq^n$. Then we can prove that

$$a_{2n}=a_{2n+1},\quad \text{for all } n\geq 0,$$

and the series begins as

$$1+q+2q^2+2q^3+4q^4+4q^5+7q^6+7q^7+\cdots,$$

where the coefficients of q^n are clearly monotonically non-decreasing. Multiplying this with $(-1+q+q^3+q^6+q^{10}+q^{15}+\cdots)$ now shows that indeed the coefficients of q^{2n+1} in $P_{ou}^{eu}(q)-P_{eu}^{ou}(q)$ are nonnegative for $n\geq 1$ (since each instance of $a_{2n+1}q^{2n+1}$ multiplied with -1 will be cancelled out by $a_{2n}q^{2n}$ multiplied with q).

Let $\prod\limits_{i=1}^{\infty}(1+q^{2i}+q^{4i}+q^{6i}+\cdots)=\sum\limits_{n\geq 0}b_{2n}q^{2n}$, where b_{2n} is the number of partitions of $2n$ with all parts even. To prove that the coefficients of q^{2n} in $P_{ou}^{eu}(q)-P_{eu}^{ou}(q)$ are nonnegative for $n\geq 4$, we have to prove that

$$a_{2n-1}+a_{2n-3}>a_{2n},$$

which means

$$a_{2n-2}+a_{2n-3}>a_{2n}.$$

It is easy to see that

$$a_{2n}=\sum_{i=0}^{n}b_{2i},\quad \text{and}\quad a_{2n-3}=\sum_{i=0}^{n-2}b_{2i}.$$

This implies,

$$a_{2n-2}+a_{2n-3}-a_{2n}=\sum_{i=0}^{n-2}b_{2i}-b_{2n}.$$

So, to complete the proof, it is enough to show that

$$\sum_{i=0}^{n-2}b_{2i}-b_{2n}>0. \tag{9}$$

This is not difficult to see combinatorially. We define the set $\tilde{P}(2n)$ to be the set of partitions of $2n$ into even parts. Let $\tilde{A}(2n)=\tilde{P}(2n)\setminus\{(2n),(2+2+\cdots+2)\}$ Then we define an injection $\varphi:\tilde{A}(2n)\to\bigcup\limits_{i=1}^{n-2}\tilde{P}(2i)$ by mapping any partition λ in $\tilde{A}(2n)$ to a partition in $\tilde{P}(2i)$ for $n-2\leq i\geq 1$ by removing the largest part of λ. And we map $(2n)$ to $(2n-4)$, and $(2+2+\cdots+2)$ to $(2n-6)$, which is possible for all $n\geq 7$. This proves the inequality (9) for $n\geq 7$. So, coefficients of even powers of q in $P_{ou}^{eu}(q)-P_{eu}^{ou}(q)$ are positive for all $n\geq 14$. Verifying for the smaller even powers of q, we get the theorem. □

Remark 4.2.1. *In fact, it is possible to prove combinatorially that, for all $n\geq 7$, we have*

$$b_{2n-4}+b_{2n-6}+b_{2n-8}+b_{2n-10}>b_{2n}.$$

This will give an alternate justification of the previous proof without invoking the map φ.

5. Concluding Remarks

There are several natural questions that arise from our study, including several avenues for further research. We list below a selection of such questions and comments.

(1) Experiments suggest that the inequality in Theorem 2.1 can be strengthened. We conjecture that, for all $n > 9$ we have
$$3q_o(n) < 2q_e(n).$$
In fact, it is easy to see that this is true for all even n, since we get
$$2P_e(q) - 3P_o(q) = \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2}}{(q^2;q^2)_n^2}(1-q^n)^2(2+q^n),$$
and when n is even then $(1-q^n)^2$ is cancelled by a factor of $(q^2;q^2)_n^2$.

(2) Chern [Che22, Theorem 1.3] has recently proved for $m \geq 2$ and for integers a and b such that $1 \leq a < b \leq m$, we have
$$p_{a,b,m}(n) \geq p_{b,a,m}(n),$$
thus generalizing the results of Kim and Kim [KK21]. Limited data suggests that this inequality is reversed if we consider $q_{j,k,m}(n)$ instead of $p_{j,k,m}(n)$. It would be interesting to get a unified proof of this observation.

(3) Kim, Kim and Lovejoy [KK21] and Kim and Kim [KK21] also study asymptotics of some of their parity biases. It would be interesting to study such asymptotics for our cases as well.

(4) All the proofs in this paper are analytical. It would be interesting to get combinatorial proofs of some of these results.

(5) Analytical proofs of the inequalities (2) and (3) would also be of interest to see if we can get more generalized results of a similar flavour.

(6) Alanazi and Nyirenda [AN21] and Chern [Che21] study some more classes of partitions where the parts are separated by parity, following the work of Andrews [And19]. It would be interesting to see if inequalities of the type proved in Theorems 4.1 and 4.2 can be proved for these cases as well as for other classes studied by Andrews [And19].

(7) It appears that there is a lot of interesting (parity) biases to be unearthed for different types of partition functions, a systematic study of such (parity) biases would also be of interest.

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References

[Ald48] Henry L. Alder. The nonexistence of certain identities in the theory of partitions and compositions. *Bull. Amer. Math. Soc.*, 54:712–722, 1948.

[AN21] Abdulaziz M. Alanazi and Darlison Nyirenda. On Andrews' partitions with parts separated by parity. *Mathematics*, 9(21), 2021.

[And98] George E. Andrews. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[And13] George E. Andrews. Difference of partition functions: the anti-telescoping method. In *From Fourier analysis and number theory to Radon transforms and geometry*, volume 28 of *Dev. Math.*, pages 1–20. Springer, New York, 2013.

[And18] George E. Andrews. Integer partitions with even parts below odd parts and the mock theta functions. *Ann. Comb.*, 22(3):433–445, 2018.

[And19] George E. Andrews. Partitions with parts separated by parity. *Ann. Comb.*, 23(2):241–248, 2019.

[BBD+22] Koustav Banerjee, Sreerupa Bhattacharjee, Manosij Ghosh Dastidar, Pankaj Jyoti Mahanta, and Manjil P. Saikia. Parity biases in partitions and restricted partitions. *Eur. J. Comb.*, 103:19, 2022. Id/No 103522.

[BU19] Alexander Berkovich and Ali Kemal Uncu. Some elementary partition inequalities and their implications. *Ann. Comb.*, 23(2):263–284, 2019.

[CFT18] Shane Chern, Shishuo Fu, and Dazhao Tang. Some inequalities for k-colored partition functions. *Ramanujan J.*, 46(3):713–725, 2018.

[Che21] S. Chern. Note on partitions with even parts below odd parts. *Math. Notes*, 110(3-4):454–457, 2021.

[Che22] Shane Chern. Further results on biases in integer partitions. *Bull. Korean Math. Soc.*, 59(1):111–117, 2022.

[GR04] George Gasper and Mizan Rahman. *Basic hypergeometric series*, volume 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Richard Askey.

[KK21] Byungchan Kim and Eunmi Kim. Biases in integer partitions. *Bull. Aust. Math. Soc.*, 104(2):177–186, 2021.

[KKL20] Byungchan Kim, Eunmi Kim, and Jeremy Lovejoy. Parity bias in partitions. *European J. Combin.*, 89:103159, 19, 2020.

[ML16] James Mc Laughlin. Refinements of some partition inequalities. *Integers*, 16:Paper No. A66, 11, 2016.

[SF82] J. J. Sylvester and F. Franklin. A Constructive Theory of Partitions, Arranged in Three Acts, an Interact and an Exodion. *Amer. J. Math.*, 5(1-4):251–330, 1882.

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