Research Paper

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On near-perfect numbers with five prime factors

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Abstract: Let n be a positive integer and $\sigma(n)$ the sum of all the positive divisors of n. We call n a near-perfect number with redundant divisor d if $\sigma(n) = 2n + d$. Let n be an odd near-perfect number of the form $n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot p_4^{a_4} \cdot p_5^{a_5}$ where p_i's are odd primes and a_i's $(1 \leq i \leq 5)$ are positive integers. In this article, we prove that $3 \mid n$ and one of $5, 7, 11 \mid n$. We also show that there exists no odd near-perfect number when $n = 3^{a_1} \cdot 7^{a_2} \cdot p_3^{a_3} \cdot p_4^{a_4} \cdot p_5^{a_5}$ with $p_3 \in \{17, 19\}$ and when $n = 3^{a_1} \cdot 11^{a_2} \cdot p_3^{a_3} \cdot p_4^{a_4} \cdot p_5^{a_5}$.

Keywords: near-perfect numbers, perfect numbers, divisor sum function.

MSC: 11A25, 11A67

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1 Introduction

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_i^{a_i}$ be the canonical representation of a positive integer where p_i's are prime numbers and a_i's are natural numbers. If $\sigma(n)$ denote the sum of the positive divisors of n, then

$$\sigma(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \frac{p_2^{a_2+1} - 1}{p_2 - 1} \frac{p_3^{a_3+1} - 1}{p_3 - 1} \frac{p_4^{a_4+1} - 1}{p_4 - 1} \cdots \frac{p_i^{a_i+1} - 1}{p_i - 1}. \tag{1.1}$$

If $\sigma(n) < 2n$, then n is a deficient number, and n is an abundant number if $\sigma(n) > 2n$. n is a perfect number if $\sigma(n) = 2n$.

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Pollock and Shevelev introduced near-perfect numbers in 2012 [2]. A positive integer n is near-perfect if $\sigma(n) = 2n + d$, where d is a proper divisor of n, known as redundant divisor of n. They also gave a method for the construction of near-perfect numbers with two prime divisors, and they introduced the concept of k-near-perfect numbers. A positive integer n is a k-near-perfect number when at most, k proper divisors are removed from the summation to make the sum equal to $2n$. When $\sigma(n) = 2n - d$, n is called a deficient perfect number with deficient divisor d; also, when $d = 1$, n is called an almost perfect number. Ren and Chen [3] demonstrated that all near-perfect numbers with two prime factors besides 40 fall within one of the three constructions provided by Pollock and Shevelev [2]. In the same year, Tang, Ren and Li [6] proved that there are no odd near-perfect numbers with three prime factors and determined all deficient-perfect numbers with at most two distinct prime factors. Tang and Feng [4] proved that there are no odd deficient perfect numbers with three distinct prime factors. Later in 2016, Tang, Ma and Feng [5] demonstrated that the only odd near-perfect number with four distinct prime factors is $173369889 = 3^4 \times 7^2 \times 11^2 \times 19^2$ whose redundant divisor is $d = 3^2 \times 7 \times 11^2 \times 19^2$. Subsequently, Dutta and Saikia [1] proved a few properties of odd deficient perfect numbers with four different prime factors. Very recently, Yang and Togbe [7] showed that the only deficient perfect number with four distinct prime factors is $3^2 \times 7^2 \times 11^2 \times 13^2$ whose deficient divisor is $d = 3^2 \times 7 \times 13$.

Inspired by the findings of Tang, Ma and Feng, we investigate near-perfect numbers with five prime factors. Throughout this paper, we assume that

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot p_4^{a_4} \cdot p_5^{a_5}, \quad 3 \le p_1 < p_2 < p_3 < p_4 < p_5 \tag{1.2}$$

is an odd near-perfect number whose redundant divisor is

$$d = p_1^{b_1} \cdot p_2^{b_2} \cdot p_3^{b_3} \cdot p_4^{b_4} \cdot p_5^{b_5}, \quad \text{with} \quad b_1 + b_2 + b_3 + b_4 + b_5 < a_1 + a_2 + a_3 + a_4 + a_5,$$

where $p_i (1 \le i \le 5)$ are distinct primes, $a_i \in \mathbb{N}$, $\quad b_i \in \{0\} \bigcup \mathbb{N}, 1 \le i \le 5$. Since n is a near-perfect number, we have,

$$\sigma(n) = 2n + d \tag{1.3}$$

$$\Rightarrow 2n < \sigma(n)$$

$$\Rightarrow 2 < \frac{\sigma(n)}{n} \tag{1.4}$$

But

$$\sigma(n) = \prod_{i=1}^{5} \frac{p_i^{a_i+1} - 1}{p_i - 1} < \prod_{i=1}^{5} \frac{p_i^{a_i+1}}{p_i - 1} = \prod_{i=1}^{5} \frac{p_i^{a_i} p_i}{p_i - 1} = n \times \prod_{i=1}^{5} \frac{p_i^{a_i}}{p_i - 1}$$

$$\Rightarrow \frac{\sigma(n)}{n} < \prod_{i=1}^{5} \frac{p_i^{a_i}}{p_i - 1} \tag{1.5}$$

Thus combining equations (1.4) and (1.5) we get,

$$2 < \frac{\sigma(n)}{n} < \prod_{i=1}^{5} \frac{p_i}{p_i - 1}. \tag{1.6}$$

Additionally, since n is an odd near-perfect number, $\sigma(n) \equiv 1 \pmod 2$, therefore

$$\sigma(n) = \prod_{i=1}^{5} (1 + p_i + p_i^2 + p_i^3 + \cdots + p_i^{a_i})$$

is odd. This implies $1 + p_i + p_i^2 + p_i^3 + \cdots + p_i^{a_i}$ being odd, $p_i + p_i^2 + p_i^3 + \cdots + p_i^{a_i}$ must be even for each $1 \le i \le 5$, consequently $a_i \equiv 0 \pmod 2$ for each $1 \le i \le 5$.

We define a function

$$f(a_1, a_2, a_3, a_4, a_5) = \left(1 - \frac{1}{p_1^{a_1+1}}\right)\left(1 - \frac{1}{p_2^{a_2+1}}\right)\left(1 - \frac{1}{p_3^{a_3+1}}\right)\left(1 - \frac{1}{p_4^{a_4+1}}\right)\left(1 - \frac{1}{p_5^{a_5+1}}\right). \tag{1.7}$$

This function (1.7) is strictly less than 1. We will be using (1.7) often in the paper.

Theorem 1. *For any odd near-perfect number of the form* (1.2) *we have the following bounds:*

1. $p_1 = 3$,
2. $5 \le p_2 \le 11$,
3. *If* $p_2 = 7$, *then* $11 \le p_3 \le 19$,
4. *If* $p_2 = 7$ *and* $p_3 = 17$, *then* $19 \le p_4 \le 23$,
5. *If* $p_2 = 7$ *and* $p_3 = 19$, *then* $p_4 = 23$, *and*
6. *If* $p_2 = 11$, *then* $p_3 = 13$,
7. *If* $p_2 = 11$ *and* $p_3 = 13$, *then* $p_4 = 17$.

Proof. Assuming $p_1 \ge 5$, we get from (1.6),

$$2 < \frac{\sigma(n)}{n} < \frac{5.7.11.13.17}{4.6.10.12.16} = 1.84646 < 2.$$

This is not possible; therefore $p_1 = 3$. Similarly, by using (1.6), it is easy to find out that the only possible p_2's in n are $5, 7$ and 11. From (1.6), by setting $p_1 = 3, p_2 = 7$ and $p_3 \ge 23$ we get,

$$2 < \frac{\sigma(n)}{n} < \frac{3.7.23.29.31}{2.6.22.28.30} = 1.95805 < 2.$$

This is not possible, therefore $p_3 \in \{11, 13, 17, 19\}$. By setting $p_1 = 3, p_2 = 7, p_3 = 17$, and $p_4 \geq 29$ from (1.6) we get,

$$2 < \frac{\sigma(n)}{n} < \frac{3.7.17.29.31}{2.6.16.28.30} = 1.98997 < 2.$$

This is not possible, therefore $p_4 \leq 23$. Similarly, we obtain the values of p_4 for the other cases of p_3 when $p_2 = 7$ and $p_2 = 11$. They are listed in Table 1.

p_1	p_2	p_3	p_4
3	7	17	≤ 23
3	7	19	23
3	11	13	17

Tab. 1: Possible values of p_4's.

□

Theorem 1 shows that to study near-perfect numbers of the form (1.2) it is enough to consider $p_1 = 3$ and $p_2 = 5, 7$ or 11. The main results of this paper are the following:

Theorem 2. *There is no odd near-perfect number* $n = 3^{a_1} \cdot 7^{a_2} \cdot p_3^{a_3} \cdot p_4^{a_4} \cdot p_5^{a_3}$ *when* $p_3 \in \{17, 19\}$.

Theorem 3. *There is no odd near-perfect number* $n = 3^{a_1} \cdot 11^{a_2} \cdot p_3^{a_3} \cdot p_4^{a_4} \cdot p_5^{a_3}$.

This paper is organised as follows, by fixing $p_1 = 3, p_2 = 7$ and $p_3 = 17$,
- Section 2 discusses the various possibilities of p_5 when p_4 is fixed to 19.
- Section 3 discusses the various possibilities of p_5 when p_4 is fixed to 23.

By fixing $p_1 = 3, p_2 = 7$ and $p_3 = 19$,
- Section 4 discusses the various possibilities of p_5 when p_4 is fixed to 23.

By fixing $p_1 = 3, p_2 = 11$ and $p_3 = 13$,
- Section 5 discusses the various possibilities of p_5 when p_4 is fixed to 17.

2 Odd near-perfect number of the form $3^{a_1} 7^{a_2} 17^{a_3} 19^{a_4} p_5^{a_5}$

In this section, we prove the following result.

Proposition 1. *There exists no odd near-perfect number of the form* $n = 3^{a_1}7^{a_2}17^{a_3}19^{a_4}p_5^{a_5}$, *where* a_i*'s are natural numbers.*

Proof. Let $n = 3^{a_1}7^{a_2}17^{a_3}19^{a_4}p_5^{a_5}$ be an odd near-perfect number with redundant divisor $d = 3^{b_1}7^{b_2}17^{b_3}19^{b_4}p_5^{b_5}$, where $b_1+b_2+b_3+b_4+b_5 < a_1+a_2+a_3+a_4+a_5$ and $b_i \le a_i$, $i = 1,2,3,4,5$. Then by (1.1) and (1.3), we have

$$\sigma(n) = \frac{3^{a_1+1}-1}{2}\frac{7^{a_2+1}-1}{6}\frac{17^{a_3+1}-1}{16}\frac{19^{a_4+1}-1}{18}\frac{p_5^{a_5+1}-1}{p_5-1} = 2.3^{a_1}7^{a_2}17^{a_3}19^{a_4}p_5^{a_5} + 3^{b_1}7^{b_2}17^{b_3}19^{b_4}p_5^{b_5}. \quad (2.1)$$

And (1.7) becomes

$$f(a_1,a_2,a_3,a_4,a_5) = \left(1-\frac{1}{3^{a_1+1}}\right)\left(1-\frac{1}{7^{a_2+1}}\right)\left(1-\frac{1}{17^{a_3+1}}\right)\left(1-\frac{1}{19^{a_4+1}}\right)\left(1-\frac{1}{p_5^{a_5+1}}\right). \quad (2.2)$$

Using (1.6), we find $p_5 \le 53$, and from (2.1), we get the following bounds as laid down in Table 2 for the powers of the primes in n.

p_5	a_i
$23 \le p_5 \le 37$	$a_1 \ge 4$
$41 \le p_5 \le 43$	$a_1 \ge 4, a_2 \ge 4$
$p_5 = 47$	$a_1 \ge 6, a_2 \ge 4$
$p_5 = 53$	$a_1 \ge 8, a_2 \ge 6, a_3 \ge 4$

Tab. 2: The lower bounds of a_i's for $3^{a_1}7^{a_2}17^{a_3}19^{a_4}p_5^{a_5}$.

Now we will discuss the cases for $p_5 \in \{23,29,31,37,41,43,47,53\}$.

We define the function,

$$g(a_1,a_2,a_3,a_4,a_5) = \frac{2^8 \times 3^2 \times (p_5-1)}{7 \times 17 \times 19 \times p_5} + \frac{2^7 \times 3^2 \times (p_5-1)}{D}, \quad (2.3)$$

where

$$D = 3^{a_1-b_1} \times 7^{a_2-b_2+1} \times 17^{a_3-b_3+1} \times 19^{a_4-b_4+1} \times p_5^{a_5-b_5+1}. \quad (2.4)$$

From (2.1) and (2.2) it is clear that

$$g(a_1,a_2,a_3,a_4,a_5) = f(a_1,a_2,a_3,a_4,a_5) < 1.$$

Since, $17 \nmid (3^{a_1+1}-1)\times(7^{a_2+1}-1)\times(17^{a_3+1}-1)\times(19^{a_4+1}-1)\times(p_5^{a_5+1}-1)$ for $p_5 = 23, 29, 31, 37$ and 43, by (2.1), we get $b_3 = 0$.

We now consider the following six cases:

Case 1: $p_5 \in \{23, 29, 31\}$.

Now by Table 2 and equations (2.3), (2.2), we have for $p_5 = 23$,

$$g(a_1,a_2,a_3,a_4,a_5) \leq \frac{2^8\times 3^2\times 22}{7\times 17\times 19\times 23} + \frac{2^7\times 3^2\times 22}{7\times 17^3\times 19\times 23} = 0.976399$$

which contradicts $f(a_1,a_2,a_3,a_4,a_5) \geq 0.992553$.

For $p_5 = 29$,

$$g(a_1,a_2,a_3,a_4,a_5) \leq \frac{2^8\times 3^2\times 28}{7\times 17\times 19\times 29} + \frac{2^7\times 3^2\times 28}{7\times 17^3\times 19\times 29} = 0.985582$$

and $f(a_1,a_2,a_3,a_4,a_5) \geq 0.992594$. This is a contradiction.

For $p_5 = 31$,

$$g(a_1,a_2,a_3,a_4,a_5) \leq \frac{2^8\times 3^2\times 30}{7\times 17\times 19\times 31} + \frac{2^7\times 3^2\times 30}{7\times 17^3\times 19\times 31} = 0.987853$$

and $f(a_1,a_2,a_3,a_4,a_5) \geq 0.992601$, which is a contradiction.

Case 2: $p_5 = 37$.

By Table 2 and equation (2.2), we find that

$$f(a_1,a_2,a_3,a_4,a_5) \geq 0.992615. \tag{2.5}$$

It is easy to see that $37 \mid \sigma(7^{a_2})$ whenever $a_2 + 1 \equiv 9 \pmod{18}$. Therefore we have the following two sub-cases depending on a_2.

Sub-case 2.1: $a_2 + 1 \not\equiv 9 \pmod{18}$.

Since $a_2 + 1 \not\equiv 9 \pmod{18}$, therefore $37 \nmid \sigma(7^{a_2})$ and thus by (2.1) we get, $b_5 = 0$. Therefore by (2.3),

$$g(a_1,a_2,a_3,a_4,a_5) \leq \frac{2^8\times 3^2\times 36}{7\times 17\times 19\times 37} + \frac{2^7\times 3^2\times 36}{7\times 17^3\times 19\times 37^3} \leq 0.991478.$$

Since the functions f and g are equal, the above bound contradicts (2.5).

Sub-case 2.2: $a_2 + 1 \equiv 9 \pmod{18}$.

If $D \leq 3\times 7\times 17^2\times 19\times 37$, by (2.3), we get $g(a_1,a_2,a_3,a_4,a_5) = 1.0012 > 1$, which is not possible. Again if $D \geq 3\times 7^4\times 17\times 19\times 37$, from (2.3), we get $g(a_1,a_2,a_3,a_4,a_5) = 0.991959$, which contradicts (2.5). Since $b_3 = 0$, the only D's that satisfies (2.5) and the inequality

$$3\times 7\times 17^2\times 19\times 37 < D < 3\times 7^4\times 17\times 19\times 37 \tag{2.6}$$

are given below in Table 3 along with the corresponding results for $\sigma(n)$. In Table 3, l_i's, indicate the indices of prime factors $3, 7, 17, 19, 37$ of n in D respectively and t_i's, refer to the powers of primes $3, 7, 17, 19, 37$ respectively in $\sigma(n)$ where $1 \leq i \leq 5$, M denotes the co-efficient in $\sigma(n)$, obtained in (2.1). Here on wards, l_i's and t_i's denote the indices of each prime factor of D and $\sigma(n)$, respectively.

D $[3^{l_1} \times 7^{l_2} \times 17^{l_3} \times 19^{l_4} \times 37^{l_5}]$					$\sigma(n)$ $[M \times 3^{t_1} \times 7^{t_2} \times 17^{t_3} \times 19^{t_4} \times 37^{t_5}]$					
l_1	l_2	l_3	l_4	l_5	M	t_1	t_2	t_3	t_4	t_5
0	1	3	1	1	193	a_1+1	a_2	a_3-2	a_4	a_5

Tab. 3: Calculations of $\sigma(3^{a_1}7^{a_2}17^{a_3}19^{a_4}37^{a_5})$.

Since $37 \mid \sigma(7^{a_2})$ whenever $a_2+1 \equiv 9 \pmod{18}$, $1063 \mid \sigma(n)$. But none of the $\sigma(n)$ values found in Table 3 is divisible by 1063. This is a contradiction.

Case 3: $p_5 = 41$.

Using Table 2 and equation (2.2), we find that $f(a_1, a_2, a_3, a_4, a_5) \geq 0.995463$. But, $(7 \times 17) \nmid (3^{a_1+1}-1) \times (7^{a_2+1}-1) \times (17^{a_3+1}-1) \times (19^{a_4+1}-1) \times (41^{a_5+1}-1)$. Then by (2.1) we get, $b_2 = b_3 = 0$ and thus by (2.3),

$$g(a_1, a_2, a_3, a_4, a_5) \leq \frac{2^8 \times 3^2 \times 40}{7 \times 17 \times 19 \times 41} + \frac{2^7 \times 3^2 \times 40}{7^5 \times 17^3 \times 19 \times 41} \leq 0.994165.$$

This is not possible.

Case 4: $p_5 = 43$.

By Table 2 and equation (2.2), we get

$$f(a_1, a_2, a_3, a_4, a_5) \geq 0.995465. \tag{2.7}$$

It is easy to see that $43 \mid \sigma(17^{a_3})$ whenever $a_3+1 \equiv 21 \pmod{42}$. Therefore we have the following two sub-cases depending on a_3.

Sub-case 4.1: $a_3+1 \not\equiv 21 \pmod{42}$.

In this case, $43 \nmid \sigma(17^{a_3})$ and thus by (2.1) we get, $b_5 = 0$. Therefore, by (2.3) we get,

$$g(a_1, a_2, a_3, a_4, a_5) \leq \frac{2^8 \times 3^2 \times 42}{7 \times 17 \times 19 \times 43} + \frac{2^7 \times 3^2 \times 42}{7 \times 17^3 \times 19 \times 43^3} \leq 0.995321.$$

Since the functions f and g are equal, the above bound contradicts(2.7).

Sub-case 4.2: $a_3+1 \equiv 21 \pmod{42}$.

If $D \leq 3 \times 7 \times 17^2 \times 19 \times 43$, by (2.3) we get, $g(a_1, a_2, a_3, a_4, a_5) = 1.00508 > 1$, which is not possible. If $D \geq 3 \times 7 \times 17^4 \times 19 \times 43$, by (2.3), we get $g(a_1, a_2, a_3, a_4, a_5) = 0.995354 < f(a_1, a_2, a_3, a_4, a_5)$, Since the functions f and g are equal, the above bound contradicts (2.7). Since $b_3 = 0$, the D's that satisfy (2.5) and the inequality

$$3 \times 7 \times 17^2 \times 19 \times 43 < D < 3 \times 7 \times 17^4 \times 19 \times 43.$$

are given below in Table 4 along with the corresponding results for $\sigma(n)$.

D $[3^{l_1} \times 7^{l_2} \times 17^{l_3} \times 19^{l_4} \times 43^{l_5}]$					$\sigma(n)$ $[M \times 3^{t_1} \times 7^{t_2} \times 17^{t_3} \times 19^{t_4} \times 43^{t_5}]$					
l_1	l_2	l_3	l_4	l_5	M	t_1	t_2	t_3	t_4	t_5
0	1	3	1	1	193	a_1+1	a_2+1	a_3-2	a_4	a_5
0	2	3	1	1	229	a_1+1	a_2-1	a_3-2	a_4	a_5
1	1	3	1	1	1735	a_1-1	a_2	a_3-2	a_4	a_5
2	1	3	1	1	121	a_1-2	a_2	a_3-2	a_4	a_5+1

Tab. 4: Calculations of $\sigma(3^{a_1}7^{a_2}17^{a_3}19^{a_4}43^{a_5})$.

Since $43 \mid \sigma(17^{a_3})$ whenever $a_3+1 \equiv 21 \pmod{42}$, $940143709 \mid \sigma(n)$. But none of the $\sigma(n)$ values, given in Table 4, is divisible by 940143709. This leads to a contradiction.

Case 5: $p_5 = 47$.

We observe that, $(7 \times 17) \nmid (3^{a_1+1}-1) \times (7^{a_2+1}-1) \times (17^{a_3+1}-1) \times (19^{a_4+1}-1) \times (47^{a_5+1}-1)$, therefore by (2.1), we get $b_2 = b_3 = 0$. Then using Table 2 and by (2.2) and (2.3), we get

$$g(a_1,a_2,a_3,a_4,a_5) \leq \frac{2^8 \times 3^2 \times 46}{7 \times 17 \times 19 \times 47} + \frac{2^7 \times 3^2 \times 46}{7^5 \times 17^3 \times 19 \times 47} \leq 0.997338.$$

and $f(a_1,a_2,a_3,a_4,a_5) \geq 0.999125$. This is not possible.

Case 6: $p_5 = 53$.

Here, $(17 \times 53) \mid (3^{a_1+1}-1) \times (7^{a_2+1}-1) \times (17^{a_3+1}-1) \times (19^{a_4+1}-1) \times (53^{a_5+1}-1)$ and thus by (2.1), $b_3 = b_5 = 0$. Then by using Table 2, $f(a_1,a_2,a_3,a_4,a_5) \geq 0.999795$ and

$$g(a_1,a_2,a_3,a_4,a_5) \leq \frac{2^8 \times 3^2 \times 52}{7 \times 17 \times 19 \times 53} + \frac{2^7 \times 3^2 \times 52}{7 \times 17^5 \times 19 \times 53^3} \leq 0.999791$$

This is a contradiction.

This proves that there is no odd near-perfect number of the form $n = 3^{a_1}7^{a_2}17^{a_3}19^{a_4}p_5^{a_5}$. □

3 Odd near-perfect number of the form $3^{a_1}7^{a_2}17^{a_3}23^{a_4}p_5^{a_5}$

In this section, we prove the following result.

Proposition 2. *There exists no odd near-perfect number of the form* $n = 3^{a_1}7^{a_2}17^{a_3}23^{a_4}p_5^{a_5}$*, where* a_i*'s are natural numbers.*

Proof. Let $n = 3^{a_1}7^{a_2}17^{a_3}23^{a_4}p_5^{a_5}$ be an odd near-perfect number with redundant divisor $d = 3^{b_1}7^{b_2}17^{b_3}23^{b_4}p_5^{b_5}$, where $b_1+b_2+b_3+b_4+b_5 < a_1+a_2+a_3+a_4+a_5$ and $b_i \le a_i$, $i = 1,2,3,4,5$. Then by (1.1) and (1.3), we have

$$\sigma(n) = \frac{3^{a_1+1}-1}{2}\frac{7^{a_2+1}-1}{6}\frac{17^{a_3+1}-1}{16}\frac{23^{a_4+1}-1}{22}\frac{p_5^{a_5+1}-1}{p_5-1} = 2\cdot 3^{a_1}7^{a_2}17^{a_3}23^{a_4}p_5^{a_5} + 3^{b_1}7^{b_2}17^{b_3}23^{b_4}p_5^{b_5}. \quad (3.1)$$

and (1.7) becomes

$$f(a_1,a_2,a_3,a_4,a_5) = \left(1-\frac{1}{3^{a_1+1}}\right)\left(1-\frac{1}{7^{a_2+1}}\right)\left(1-\frac{1}{17^{a_3+1}}\right)\left(1-\frac{1}{23^{a_4+1}}\right)\left(1-\frac{1}{p_5^{a_5+1}}\right). \quad (3.2)$$

Using (1.6), we find that $p_5 \le 31$, and from (3.1), we get the following bounds as laid done in Table 5 for the powers of the primes in n. Now we will

p_i	a_i
$p_5 = 29$	$a_1, a_2 \ge 4$
$p_5 = 31$	$a_1, a_2, a_3 \ge 4$

Tab. 5: The lower bounds of a_i's for $3^{a_1}7^{a_2}17^{a_3}23^{a_4}p_5^{a_5}$.

discuss the cases for $p_5 = \{29, 31\}$. We define the function,

$$g(a_1.a_2,a_3,a_4,a_5) = \frac{2^8\times 11\times(p_5-1)}{7\times 17\times 23\times p_5} + \frac{2^7\times 11\times(p_5-1)}{D}, \quad (3.3)$$

where

$$D = 3^{a_1-b_1}\times 7^{a_2-b_2+1}\times 17^{a_3-b_3+1}\times 23^{a_4-b_4+1}\times p_5^{a_5-b_5+1}. \quad (3.4)$$

Then from (3.1) and (3.2) it is clear that,

$$g(a_1,a_2,a_3,a_4,a_5) = f(a_1,a_2,a_3,a_4,a_5) < 1.$$

Since, $17 \nmid (3^{a_1+1}-1)\times(7^{a_2+1}-1)\times(17^{a_3+1}-1)\times(23^{a_4+1}-1)\times(p_5^{a_5+1}-1)$ for $p_5 = 29$ and $p_5 = 31$, by (3.1) we get $b_3 = 0$.

By using (3.3) and Table 5 for $p_5 = 29$ we have,

$$g(a_1, a_2, a_3, a_4, a_5) \leq \frac{2^8 \times 11 \times 28}{7 \times 17 \times 23 \times 29} + \frac{2^7 \times 11 \times 28}{7 \times 17^3 \times 23 \times 29} \leq 0.995104,$$

which contradicts $f(a_1, a_2, a_3, a_4, a_5) \geq 0.9955$. Therefore $p_5 = 29$ is not possible. Similarly for $p_5 = 31$ we have,

$$g(a_1, a_2, a_3, a_4, a_5) \leq \frac{2^8 \times 11 \times 30}{7 \times 17 \times 23 \times 31} + \frac{2^7 \times 11 \times 30}{7 \times 17^5 \times 23 \times 31} \leq 0.995681,$$

which contradicts $f(a_1, a_2, a_3, a_4, a_5) \geq 0.99571$. Therefore $p_5 = 31$ is not possible.

This proves that there is no odd near-perfect number of the form $n = 3^{a_1} 7^{a_2} 17^{a_3} 23^{a_4} p_5^{a_5}$. □

4 Odd near-perfect number of the form $3^{a_1} 7^{a_2} 19^{a_3} 23^{a_4} p^{a_5}$

In this section, we prove the following result.

Proposition 3. *There exists no odd near-perfect number of the form* $n = 3^{a_1} 7^{a_2} 19^{a_3} 23^{a_4} p^{a_5}$, *where* a_i*'s are natural numbers.*

Proof. Let $n = 3^{a_1} 7^{a_2} 19^{a_3} 23^{a_4} p_5^{a_5}$ be an odd near-perfect number with redundant divisor $d = 3^{b_1} 7^{b_2} 19^{b_3} 23^{b_4} p_5^{b_5}$, where $b_1 + b_2 + b_3 + b_4 + b_5 < a_1 + a_2 + a_3 + a_4 + a_5$ and $b_i \leq a_i$, $i = 1, 2, 3, 4, 5$. Then by (1.1) and (1.3) , we have

$$\sigma(n) = \frac{3^{a_1+1} - 1}{2} \frac{7^{a_2+1} - 1}{6} \frac{19^{a_3+1} - 1}{18} \frac{23^{a_4+1} - 1}{22} \frac{p_5^{a_5+1} - 1}{p_5 - 1}$$
$$= 2 \times 3^{a_1} 7^{a_2} 19^{a_3} 23^{a_4} p_5^{a_5} + 3^{b_1} 7^{b_2} 19^{b_3} 23^{b_4} p_5^{b_5}. \quad (4.1)$$

and (1.7) becomes

$$f(a_1, a_2, a_3, a_4, a_5) = \left(1 - \frac{1}{3^{a_1+1}}\right) \left(1 - \frac{1}{7^{a_2+1}}\right) \left(1 - \frac{1}{19^{a_3+1}}\right) \left(1 - \frac{1}{23^{a_4+1}}\right) \left(1 - \frac{1}{p_5^{a_5+1}}\right). \quad (4.2)$$

Using (1.6), we find $p_5 = 29$, and from (4.1), we get the following bounds as laid down in Table 6 for the powers of the primes in n.

p_i	a_i
$p_5 = 29$	$a_1 \geq 8, a_2 \geq 6, a_i \geq 4$ where $i = 3, 4, 5.$

Tab. 6: The lower bounds of a_i's for $3^{a_1}7^{a_2}19^{a_3}23^{a_4}p_5^{a_5}$.

We define,

$$g(a_1.a_2, a_3, a_4, a_5) = \frac{2^7 \times 3^2 \times 11}{19 \times 23 \times 29} + \frac{2^6 \times 3^2 \times 11}{D} \tag{4.3}$$

where,

$$D = 3^{a_1-b_1} \times 7^{a_2-b_2} \times 19^{a_3-b_3+1} \times 23^{a_4-b_4+1} \times 29^{a_5-b_5+1}. \tag{4.4}$$

From (4.1) and (4.2) it is clear that,

$$g(a_1, a_2, a_3, a_4, a_5) = f(a_1, a_2, a_3, a_4, a_5) < 1.$$

By (4.2) we find that

$$f(a_1, a_2, a_3, a_4, a_5) \geq 0.999947. \tag{4.5}$$

If $D \leq 3 \times 7^3 \times 19 \times 23 \times 29$, by (4.3), $g(a_1, a_2, a_3, a_4, a_5) = 1.00041 > 1$, this is not possible. If $D \geq 3 \times 7^5 \times 19 \times 23 \times 29$, by (4.3) we get, $g(a_1, a_2, a_3, a_4, a_5) = 0.999931$ which is a contradiction by (4.5). Therefore, the only D's that satisfy the (4.5) and inequality

$$3 \times 7^3 \times 19 \times 23 \times 29 < D < 3 \times 7^5 \times 19 \times 23 \times 29.$$

are given in Table 7.

D $[3^{l_1} \times 7^{l_2} \times 19^{l_3} \times 23^{l_4} \times 29^{l_5}]$					$\sigma(n)$ $[M \times 3^{t_1} \times 7^{t_2} \times 19^{t_3} \times 23^{t_4} \times 29^{t_5}]$					
l_1	l_2	l_3	l_4	l_5	M	t_1	t_2	t_3	t_4	t_5
0	0	2	1	3	3551	a_1+2	a_2	a_3-1	a_4	a_5-2
0	0	2	2	2	1207	a_1+1	a_2+1	a_3-1	a_4-1	$a_{-}1$
0	0	2	2	1	125	a_1	a_2+1	a_3-1	a_4-1	a_5
0	0	3	1	2	20939	a_1	a_2	a_3-2	a_4	a_5-1
0	0	3	2	1	16607	a_1	a_2	a_3-2	a_4-1	a_5
0	0	4	1	1	4573	a_1+1	a_2	a_3-3	a_4	a_5
0	2	3	1	1	3931	a_1+2	a_2-2	a_3-2	a_4	a_5
0	3	1	1	2	865	a_1	a_2-3	a_3	a_4+1	a_5-1

l_1	l_2	l_3	l_4	l_5	M	t_1	t_2	t_3	t_4	t_5
0	3	1	2	1	15779	a_1	a_2-3	a_3	a_4-1	a_5
0	3	2	1	1	4345	a_1	a_2-3	a_3-1	a_4	a_5
0	5	1	1	1	514	a_1+4	a_2-5	a_3	a_4	a_5
2	0	1	1	3	15139	a_1-2	a_2	a_3	a_4	a_5-2
3	0	1	2	2	36019	a_1-3	a_2	a_3	a_4-1	a_5-1
3	0	1	3	1	583	a_1-3	a_2+2	a_3	a_4-2	a_5
3	0	2	1	2	29755	a_1-3	a_2	a_3-1	a_4	a_5-1
3	0	2	2	1	23599	a_1-3	a_2	a_3-1	a_4-1	a_5
3	0	3	1	1	2785	a_1-3	a_2	a_3-2	a_4	a_5
5	0	1	1	2	14095	a_1-5	a_2	a_3	a_4	a_5-1
6	0	1	2	1	1765	a_1-6	a_2	a_3	a_4-1	a_5
6	0	2	1	1	277035	a_1-6	a_2	a_3-1	a_4	a_5
8	0	1	1	1	13123	a_1-8	a_2	a_3	a_4	a_5
1	1	1	1	3	35323	a_1-1	a_2-1	a_3	a_4+1	a_5-2
1	1	1	2	2	28015	a_1-1	a_2-1	a_3	a_4-1	a_5-1
1	1	1	3	1	22219	a_1-1	a_2-1	a_3	a_4-2	a_5
1	1	2	1	2	23143	a_1-1	a_2-1	a_3-1	a_4	a_5-1
1	1	2	2	1	18355	a_1-1	a_2-1	a_3-1	a_4-1	a_5
1	1	3	1	1	15163	a_1-1	a_2-1	a_3-2	a_4	a_5
1	4	1	1	1	14407	a_1-1	a_2-4	a_3	a_4	a_5
2	2	1	1	2	25579	a_1-2	a_2-2	a_3	a_4	a_5-1
2	2	1	2	1	20287	a_1-2	a_2-2	a_3	a_4-1	a_5
2	2	2	1	1	16759	a_1-2	a_2-2	a_3-1	a_4	a_5
3	3	1	1	1	18523	a_1-3	a_2-3	a_3	a_4	a_5
4	1	1	1	2	32887	a_1-4	a_2-1	a_3	a_4	a_5-1
4	1	1	2	1	26083	a_1-4	a_2-1	a_3	a_4-1	a_5
4	1	2	1	1	743	a_1-4	a_2-1	a_3-1	a_4	a_5+1
5	2	2	1	1	23815	a_1-5	a_2-2	a_3-1	a_4	a_5
7	1	1	1	1	30619	a_1-7	a_2-1	a_3	a_4	a_5

Tab. 7: Calculation of $\sigma(3^{a_1}7^{a_2}19^{a_3}23^{a_4}29^{a_5})$.

Now $23 \mid \sigma(3^{a_1})$ and $23 \mid \sigma(29^{a_5})$, when $a_1+1 \equiv 11 \pmod{22}$ and $a_5+1 \equiv 11 \pmod{22}$ respectively. Therefore, $18944890940537 \mid \sigma(29^{a_5})$ and $3851 \mid \sigma(3^{a_1})$ which implies $18944890940537 \mid \sigma(n)$ and $3851 \mid \sigma(n)$. But $\sigma(n)$ is not divisible by 3851 or 18944890940537 in Table 7. This is a contradiction.

This proves that there is no odd near-perfect number of the form $n = 3^{a_1}7^{a_2}19^{a_3}23^{a_4}29^{a_5}$. □

5 Odd near-perfect number of the form $3^{a_1}11^{a_2}13^{a_3}17^{a_4}p_5^{a_5}$

In this section, we prove the following result.

Proposition 4. *There exists no odd near-perfect number of the form* $n = 3^{a_1}11^{a_2}13^{a_3}17^{a_4}p_5^{a_5}$*, where* a_i*'s are natural numbers.*

Proof. Let $n = 3^{a_1}11^{a_2}13^{a_3}17^{a_4}p_5^{a_5}$ be an odd near- perfect number with redundant divisor $d = 3^{b_1}11^{b_2}13^{b_3}17^{b_4}p_5^{b_5}$, where $b_1 + b_2 + b_3 + b_4 + b_5 < a_1 + a_2 + a_3 + a_4 + a_5$ and $b_i \le a_i$, $i = 1, 2, 3, 4, 5$. Then by (1.1) and (1.3), we have

$$\sigma(n) = \frac{3^{a_1+1}-1}{2}\frac{11^{a_2+1}-1}{10}\frac{13^{a_3+1}-1}{12}\frac{17^{a_4+1}-1}{16}\frac{p_5^{a_5+1}-1}{p_5-1}$$
$$= 2 \times 3^{a_1}11^{a_2}13^{a_3}17^{a_4}p_5^{a_5} + 3^{b_1}11^{b_2}13^{b_3}17^{b_4}p_5^{b_5}. \quad (5.1)$$

and (1.7) becomes

$$f(a_1, a_2, a_3, a_4, a_5) = \left(1 - \frac{1}{3^{a_1+1}}\right)\left(1 - \frac{1}{11^{a_2+1}}\right)\left(1 - \frac{1}{13^{a_3+1}}\right)\left(1 - \frac{1}{17^{a_4+1}}\right)\left(1 - \frac{1}{p_5^{a_5+1}}\right). \quad (5.2)$$

By using (1.6), we find that $p_5 = 19$ and from (5.1) and (1.6), we get the following bounds as laid down in Table 8 for the powers of the primes in n.

p_i	a_i
$p_5 = 19$	$a_1 \ge 6$

Tab. 8: The lower bounds of a_i's for $3^{a_1}11^{a_2}13^{a_3}17^{a_4}p_5^{a_5}$.

We define the function,

$$g(a_1.a_2, a_3, a_4, a_5) = \frac{2^{10} \times 3^2 \times 5}{11 \times 13 \times 17 \times 19} + \frac{2^9 \times 3^2 \times 5}{D} \quad (5.3)$$

where,

$$D = 3^{a_1-b_1} \times 11^{a_2-b_2+1} \times 13^{a_3-b_3+1} \times 17^{a_4-b_4+1} \times 19^{a_5-b_5+1}. \quad (5.4)$$

By using (5.1) and (5.2) it is clear that,

$$g(a_1, a_2, a_3, a_4, a_5) = f(a_1, a_2, a_3, a_4, a_5) < 1.$$

By (5.1) and Table 8, we find that

$$f(a_1, a_2, a_3, a_4, a_5) \geq 0.997988. \tag{5.5}$$

If $D \leq 3 \times 11^2 \times 13 \times 17 \times 19$, by (5.3) we get $g(a_1, a_2, a_3, a_4, a_5) = 1.01276 > 1$, this is not possible. If $D \geq 3 \times 11^4 \times 13 \times 17 \times 19$, from (5.3) we get $g(a_1, a_2, a_3, a_4, a_5) = 0.997765$, which contradicts (5.5). Therefore,

$$3 \times 11^2 \times 13 \times 17 \times 19 < D < 3 \times 11^4 \times 13 \times 17 \times 19. \tag{5.6}$$

The D's that satisfy (5.6) and (5.3) are given in Table 9.

D $[3^{l_1} \times 11^{l_2} \times 13^{l_3} \times 17^{l_4} \times 19^{l_5}]$					$\sigma(n)$ $[M \times 3^{t_1} \times 11^{t_2} \times 13^{t_3} \times 17^{t_4} \times 19^{t_5}]$					
l_1	l_2	l_3	l_4	l_5	M	t_1	t_2	t_3	t_4	t_5
0	1	1	1	3	241	a_1+1	a_2	a_3	a_4	a_5-2
0	1	1	2	2	647	a_1+1	a_2	a_3	a_4-1	a_5-1
0	1	2	1	2	5	a_1+2	a_2+1	a_3-1	a_4	a_5-1
0	1	2	2	1	443	a_1	a_2	a_3-1	a_4-1	a_5
0	4	1	1	1	2663	a_1	a_2-3	a_3	a_4	a_5
1	1	1	1	3	197	a_1-1	a_2+1	a_3	a_4	a_5-2
1	1	1	2	2	1939	a_1-1	a_2	a_3	a_4-1	a_5-1
1	1	1	3	1	1735	a_1-1	a_2	a_3	a_4-2	a_5
1	1	2	1	2	1483	a_1-1	a_2	a_3-1	a_4	a_5-1
1	1	2	2	1	1327	a_1-1	a_2	a_3-1	a_4-1	a_5
1	1	3	1	1	1015	a_1-1	a_2	a_3-2	a_4	a_5
1	2	1	1	2	1255	a_1-1	a_2-1	a_3	a_4	a_5-1
1	2	1	2	1	1123	a_1-1	a_2-1	a_3	a_4-1	a_5
1	2	2	1	1	859	a_1-1	a_2-1	a_3-1	a_4	a_5
1	3	1	1	1	727	a_1-1	a_2-2	a_3	a_4	a_5
2	2	2	1	1	2575	a_1-2	a_2-1	a_3-1	a_4	a_5
2	3	1	1	1	2179	a_1-2	a_2-2	a_3	a_4	a_5
3	1	1	1	2	79	a_1-3	a_2	a_3+1	a_4	a_5-1
3	1	1	2	1	919	a_1-3	a_2	a_3	a_4-1	a_5
3	1	2	1	1	37	a_1-3	a_2	a_3-1	a_4	a_5+1
3	2	1	1	1	35	a_1-3	a_2-1	a_3	a_4+1	a_5
4	1	1	2	1	145	a_1-4	a_2	a_3	a_4-1	a_5+1

l_1	l_2	l_3	l_4	l_5	M	t_1	t_2	t_3	t_4	t_5
4	1	2	1	1	2107	a_1-4	a_2	a_3-1	a_4	a_5
4	2	1	1	1	1783	a_1-4	a_2-1	a_3	a_4	a_5
5	1	1	1	1	487	a_1-5	a_2	a_3	a_4	a_5
6	1	1	1	1	1459	a_1-6	a_2	a_3	a_4	a_5

Tab. 9: Calculation of $\sigma(3^{a_1}11^{a_2}13^{a_3}17^{a_4}19^{a_5})$.

Since $3 \mid \sigma(13^{a_3})$ and $3 \mid \sigma(19^{a_5})$, when $a_3+1 \equiv 3 \pmod 6$ and $a_5+1 \equiv 3 \pmod 6$ respectively, therefore $61 \mid \sigma(13^{a_3})$ and $127 \mid \sigma(19^{a_5})$ which implies $61 \mid \sigma(n)$ and $127 \mid \sigma(n)$, but the above relations in Table 9 is neither divisible by 127 nor by 61. This is not possible. □

6 Proof of the main Theorems 2 and 3

From Proposition 1,2 and 3 it is clear that there exists no odd near-perfect number of the form $n = 3^{a_1} \cdot 7^{a_2} \cdot p_3^{a_3} \cdot p_4^{a_4} \cdot p_5^{a_5}$ when $p_3 \in \{17, 19\}$ and from Proposition 4, it is clear that there exists no odd near-perfect number of the form $n = 3^{a_1} \cdot 11^{a_2} \cdot p_3^{a_3} \cdot p_4^{a_4} \cdot p_5^{a_5}$.

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