On the equation $\sigma(n) = \left(\frac{p}{p-1}\right) n$

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Abstract. We show that the equation $\sigma(n) = \left(\frac{p}{p-1}\right) n$, where n is a positive integer and p is a prime has no solutions for $p \geq 5$ and only one solution for $p = 3$.

It is well-known that a number n such that $\sigma(n) = 2n$ is called a *perfect number*, where $\sigma(n)$ denotes the sum of all positive divisors of n. Even perfect numbers have been completely characterized by the work of Euclid and Euler, who showed that such an even perfect number is always of the form $2^{p-1}(2^p - 1)$, where both p and $2^p - 1$ are prime numbers. There is no known odd perfect number and it is an open problem to determine such numbers.

Motivated by the inherent beauty and simplicity of these numbers, mathematicians have played with various generalizations and extensions of perfect numbers. One such generalization is the concept of a k-perfect number: a number n is called a k-perfect number if $\sigma(n) = kn$. Clearly, the perfect numbers are 2-perfect. (See for instance, the work of the second author with Laugier and Sarmah [1] for some of these generalizations.) Sándor (in an unpublished preprint titled 'An extension of k-perfect numbers', available online at https://rgmia.org/papers/v9n4/art37.pdf) further generalized this concept to what he called k_p-perfect numbers which we define below.

Definition 1. A natural number n is called a k_p-perfect number if the following holds

$$\sigma(n) = \left(\frac{p}{p-1}\right) n.$$

Clearly, all k_2-perfect numbers are perfect. One of the main results of Sándor was the following.

Theorem 1 (Sándor). *All k_p-perfect numbers divisible by p have the form $n = p^q(p^{a+1} - 1)$, where $a \geq 1$ is an integer and $p^{a+1} - 1$ is a prime.*

In this note, we show that the only k_p-perfect number for $p \geq 3$ is 2. This is proved in the following result.

Theorem 2. *There are no k_p-perfect numbers for $p \geq 5$. The only k_3 perfect number is 2.*

Proof. Let $n = p^\ell x$ be a k_p-perfect number, where p is a prime, not dividing x and $\ell \geq 0$. By the multiplicity of σ we have

$$\sigma(n) = \frac{p^{\ell+1} - 1}{p - 1}\sigma(x) = \frac{pn}{p-1}.$$

This implies

$$p^{\ell+1}x = (p^{\ell+1} - 1)\sigma(x), \tag{1}$$

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which gives us $x \neq 1$ and hence $p^{\ell+1}-1$ divides x. Let $x = y(p^{\ell+1}-1)$, for some integer y. Then, $\sigma(x) = x + y + \sigma'(x)$, where $\sigma'(x)$ is now the sum of the divisors of x, apart from x and y.

Now, from equation (1) and $x = y(p^{\ell+1}-1)$ we have $p^{\ell+1}x = (p^{\ell+1}-1)(x+y+\sigma'(x))$, this gives us $\sigma'(x)(p^{\ell+1}-1) = 0$. Since $\ell \geq 0$, so we have $\sigma'(x) = 0$. This is only possible when $y = 1$ and x is a prime. But for all $p \geq 5$, $x = p^{\ell+1}-1$ cannot be a prime, so no k_p-perfect numbers exist for $p \geq 5$.

For the second part, when $p = 3$ we have from above $x = 3^{\ell+1}-1$ is a prime. When $\ell = 0$ we get $x = 2$ which gives $n = 2$ is a k_3-perfect number. For $\ell \geq 1$, x is always even and cannot be a prime, so no such k_3-perfect numbers can exist. □

References

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