ARITHMETIC PROPERTIES MODULO POWERS OF 2 AND 3 FOR OVERPARTITION k-TUPLES WITH ODD PARTS

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ABSTRACT. Recently, Drema and N. Saikia (2023) and M. P. Saikia, Sarma, and Sellers (2023) proved several congruences modulo powers of 2 for overpartition triples with odd parts. In this paper we study further divisibility properties of overpartition k-tuples with odd parts using elementary means as well as properties of modular forms. In particular, we prove several congruences modulo multiples of 3, and an infinite family of congruences modulo powers of 3; we also prove some cases of a conjecture of Saikia, Sarma and Sellers.

1. INTRODUCTION

A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that the parts λ_i's sum up to n. For instance, $4, 3+1, 2+2, 2+1+1$ and $1+1+1+1$ are the five partitions of 4. The number of partitions of n is denoted by $p(n)$, and its generating function found by Euler is given by

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_\infty := \prod_{i\geq 0}(1-aq^i), \quad |q|<1.$$

Throughout the paper, we will use the notation $f_k := (q^k;q^k)_\infty$.

The arithmetic properties of partitions have been studied for a long time and several beautiful congruences satisfied by the partition function have been found. This avenue of study has also trickled down to other classes of partitions. For instance, a widely studied class of partitions are the overpartitions. An overpartition of a nonnegative integer n is a non-increasing sequence of natural numbers whose sum is n, and where the first occurrence (or equivalently, the last occurrence) of a number may be overlined. The eight overpartitions of 3 are

$$3, \bar{3}, 2+1, \bar{2}+1, 2+\bar{1}, \bar{2}+\bar{1}, 1+1+1, \text{ and } \bar{1}+1+1.$$

The number of overpartitions of n is denoted by $\overline{p}(n)$ and its generating function is given by

$$\sum_{n\geq 0} \overline{p}(n)q^n = \frac{f_2}{f_1^2}.$$

Generalizing the idea of overpartitions, we can define an overpartition k-tuple, as was done by Keister, Sellers and Vary [KSV09]. An overpartition k-tuple of n is a k-tuple of overpartitions $(\pi_1, \pi_2, \ldots, \pi_k)$ such that the sum of the parts of π_i's equal n. The generating function for the number of overpartition k-tuples of n, denoted by $\overline{p}_k(n)$ is given by

$$\sum_{n\geq 0} \overline{p}_k(n)q^n = \frac{f_2^k}{f_1^{2k}}.$$

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If we restrict all our parts in such an overpartition k-tuple to be even, then we have an overpartition k-tuple $(\xi_1, \xi_2, \dots, \xi_k)$ of n where all parts of ξ_i's are odd. The generating function for the number of overpartition k-tuples of n with odd parts, denoted by $\overline{OPT}_k(n)$ is given by

$$\sum_{n\geq 0} \overline{OPT}_k(n) q^n = \frac{f_2^{3k}}{f_1^{2k} f_4^k}. \tag{1.1}$$

The study of the arithmetic properties of the case $k = 1$ was initiated by Hirschhorn and Sellers [HS06], which led to a lot of follow-up work by other mathematicians (See, for example, [Che14], [Mer22]). The case $k = 2$ has also been studied, the interested reader can look at the work of Lin [Lin12] for some representative work. Kim [Kim11] further studied the case using the theory of modular forms. The case $k = 3$ here corresponds to overpartition tuples with odd parts, which were very recently studied by Drema and N. Saikia [DS23] and M. P. Saikia, Sarma and Sellers [SSS23].

The work by Drema and Saikia [DS23] focused mostly on finding congruences modulo small powers of 2 for $\overline{OPT}_3(n)$. The work of Saikia, Sarma and Sellers [SSS23] focused mostly on finding arithmetic properties of $\overline{OPT}_k(n)$ for $k = 3, 4$ and odd values modulo powers of 2. In the present paper, we extend the study of the arithmetic properties of these functions. In particular, we prove some congruences modulo multiples of 3 which have not appeared earlier. We also prove an infinite family of congruences modulo powers of 3. However first, we state an unexpected equality below.

Theorem 1.1. *For all $n \geq 1$, we have*

$$\overline{OPT}_4(2n) = 2 \cdot \overline{OPT}_8(n).$$

We give a simple proof of the equality in Section 3. Now, we move ahead to some congruences satisfied by $\overline{OPT}_3(n)$.

Theorem 1.2. *For all $n \geq 0$, we have*

$$\overline{OPT}_3(3n+1) \equiv 0 \pmod{6}, \tag{1.2}$$
$$\overline{OPT}_3(12n+7) \equiv 0 \pmod{12}, \tag{1.3}$$
$$\overline{OPT}_3(12n+10) \equiv 0 \pmod{12}, \tag{1.4}$$
$$\overline{OPT}_3(3n+2) \equiv 0 \pmod{18}, \tag{1.5}$$
$$\overline{OPT}_3(6n+5) \equiv 0 \pmod{36}, \tag{1.6}$$
$$\overline{OPT}_3(24n+23) \equiv 0 \pmod{144}. \tag{1.7}$$

The proof of Theorem 1.2 is via elementary techniques and is given in Section 4. The above list is far from exhaustive. In the next theorem, we state a family of congruences for $\overline{OPT}_3(n)$ modulo 4.

Theorem 1.3. *For all $n \geq 0$, primes $p \geq 5$, quadratic non-residues r modulo p, and $A \in \{0, 1, 2\}$ such that $Ap \equiv 2r + 1 \pmod{3}$, we have*

$$\overline{OPT}_3(3pn + R) \equiv 0 \pmod{4}, \tag{1.8}$$

where

$$R = \begin{cases} 2(Ap+r) & \text{if } 2(Ap+r) < 3p, \\ 2(Ap+r) - 3p & \text{if } 2(Ap+r) > 3p. \end{cases}$$

We give an elementary proof of this result in Section 5.

Saikia, Sarma and Sellers [SSS23] prove several results for $\overline{OPT}_k(n)$, depending on whether k is even or odd. For instance, one of their result is the following theorem.

Theorem 1.4. [SSS23, Theorem 6] *Let $k = (2^m)r$ with $m > 0$ and r odd. Then for all $n \geq 0$ we have*

$$\overline{OPT}_k(n) \equiv 0 \pmod{2^{m+1}}.$$

They [SSS23] conjecture at the end of their paper a more general result for even values of k, which is given below.

Conjecture 1.1. [SSS23, Conjecture 1] *For all $i \geq 1$, $n \geq 0$ and odd r, we have*

$$\begin{aligned}
\overline{OPT}_{2^i r}(8n+1) &\equiv 0 \pmod{2^{i+1}},\\
\overline{OPT}_{2^i r}(8n+2) &\equiv 0 \pmod{2^{2i+1}},\\
\overline{OPT}_{2^i r}(8n+3) &\equiv 0 \pmod{2^{i+3}},\\
\overline{OPT}_{2^i r}(8n+4) &\equiv 0 \pmod{2^{2i+4}},\\
\overline{OPT}_{2^i r}(8n+5) &\equiv 0 \pmod{2^{i+2}},\\
\overline{OPT}_{2^i r}(8n+6) &\equiv 0 \pmod{2^{2i+3}},\\
\overline{OPT}_{2^i r}(8n+7) &\equiv 0 \pmod{2^{i+4}}.
\end{aligned}$$

We prove three cases of this conjecture here. In fact, we are able to give a better result for one case.

Theorem 1.5. *For all $i \geq 1$, $n \geq 0$ and odd r, we have*

$$\overline{OPT}_{2^i r}(8n+1) \equiv 0 \pmod{2^{i+1}}, \tag{1.9}$$

$$\overline{OPT}_{2^i r}(4n+3) \equiv 0 \pmod{2^{i+3}}, \tag{1.10}$$

$$\overline{OPT}_{2^i r}(8n+5) \equiv 0 \pmod{2^{i+2}}. \tag{1.11}$$

The proof of Theorem 1.5 is via elementary means and is given in Section 6.

While we were unable to prove the remaining four cases of Conjecture 1.1, we were however able to prove several particular cases of some of them, using an algorithmic approach due to Radu [Rad09, Rad15]. Note here that the power of 2 in the fourth congruence of Conjecture 1.1 is corrected below.

Theorem 1.6. *For all $1 \leq i \leq 5$, $r \in \{1,3,5\}$ and $n \geq 0$, we have*

$$\begin{aligned}
\overline{OPT}_{2^i r}(8n+2) &\equiv 0 \pmod{2^{2i+1}},\\
\overline{OPT}_{2^i r}(8n+4) &\equiv 0 \pmod{2^{2i+3}},\\
\overline{OPT}_{2^i r}(8n+6) &\equiv 0 \pmod{2^{2i+3}}.
\end{aligned}$$

The proof of Theorem 1.6 is given in Section 7.

Remark 1.1. *We however, note that for $i = 2^k$ for some $k \geq 1$, odd r and for all $n \geq 0$, numerically it seems that the following congruence does hold*

$$\overline{OPT}_{2^i r}(8n+4) \equiv 0 \pmod{2^{2i+4}}.$$

In fact, the same proof as the proof of Theorem 1.6 will work for $i \in \{2,4\}$ and $r \in \{1,3,5,7\}$ in this case as well.

We now move towards proving arithmetic properties of $\overline{OPT}_k(n)$ modulo powers of 3. So far, no congruence for modulo arbitrary powers of 3 are known for general values of k. We fill in this gap via the next theorem.

Theorem 1.7. *For all $i \geq 1$ and $n \geq 0$, we have*

$$\overline{OPT}_{3^i}(3n+2) \equiv 0 \pmod{3^{i+1}}. \tag{1.12}$$

The proof of Theorem 1.7 is via elementary means and is given in Section 8.

Several authors (see [GJ22] and [Sin24] for two recent examples which have motivated our results below) have over the years also studied divisibility and density properties of various types of partition functions using the theory of modular forms. We close our results by giving some representative examples of such results in Theorems 1.8 and 1.9 below.

Theorem 1.8. *Let $k, n \geq 0$ be integers, for each i with $1 \leq i \leq k+1$, if $p_i \geq 3$ is a prime such that $p_i \not\equiv 1 \pmod 8$, then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$ we have*

$$\overline{OPT}_3(8p_1^2p_2^2 \dots p_{k+1}^2 n + p_1^2p_2^2 \dots p_k^2 p_{k+1}(8j + p_{k+1}) + 1) \equiv 0 \pmod 8.$$

There are other results of the same type as Theorem 1.8. Since the proof techniques are similar, we do not state them here.

For a fixed positive integer k, Gordon and Ono [GO97] proved that $p(n)$ is divisible by 2^k for almost all n. Several mathematicians have done similar work related to other classes of partitions, for instance by Ray and Barman [RB20], Veena and Fathima [VF21], etc. It can be shown easily [DS23, Eq. (43)] that

$$\overline{OPT}_3(2n+1) \equiv 0 \pmod 2,$$

for all $n \geq 0$. The authors in [SSS23] have found several new congruences satisfied by the $\overline{OPT}_{2k+1}(n)$ function modulo powers of 2. Motivated by this, we look at the case for $\overline{OPT}_3(n)$ and study its properties.

Theorem 1.9. *Let k be a fixed positive integer with $k \geq 4$ and p $(\neq 3)$ be a prime, then $\overline{OPT}_3(n)$ is almost always divisible by p^k, that is*

$$\lim_{X \to \infty} \frac{|\{n \leq X \, : \, \overline{OPT}_3(n) \equiv 0 \pmod{p^k}\}|}{X} = 1.$$

The proofs of Theorems 1.8 and 1.9 are given in Section 9.

The paper is structured as follows: in Section 2 we state some preliminary results which are using for our proofs, Sections 3 – 9 contains the proofs of our results, we end our paper with some concluding remarks (and conjectures) in Section 10.

2. Preliminaries

2.1. **Elementary Results.**

It can be shown easily [DS23, Eq. (43)] that

$$\overline{OPT}_3(2n+1) \equiv 0 \pmod 2,$$

for all $n \geq 0$. Drema and Saikia [DS23, Eq. (88)] have also shown that

$$\overline{OPT}_3(3n+1) \equiv \overline{OPT}_3(3n+2) \equiv 0 \pmod 3. \tag{2.1}$$

Now, working modulo 2, we have

$$\sum_{n \geq 0} \overline{OPT}_3(n) q^n \equiv 1 \pmod 2. \tag{2.2}$$

Hence, for $n \geq 1$, we have

$$\overline{OPT}_3(n) \equiv 0 \pmod 2. \tag{2.3}$$

The last congruence is reminiscent of the following congruence for overpartitions

$$\overline{p}(n) \equiv 0 \pmod 2, \quad \text{for all } n > 0.$$

Some known 2-, 3-dissections (see for example [BD22, Lemmas 2, 3 and 4]) are stated in the following lemma, which will be used subsequently.

Lemma 2.1. *We have*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8}, \tag{2.4}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.5}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2}, \tag{2.6}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.7}$$

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}, \tag{2.8}$$

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}, \tag{2.9}$$

$$= a_3 f_3 - 3q f_9^3, \tag{2.10}$$

$$\frac{1}{f_1^3} = a_3^2 \frac{f_9^3}{f_3^{10}} + 3a_3 q \frac{f_9^6}{f_3^{11}} + 9q^2 \frac{f_9^9}{f_3^{12}}, \tag{2.11}$$

where $a_n = a(q^n) := \sum_{j,k=-\infty}^{\infty} q^{n\cdot(j^2+jk+k^2)}$ *is one of Borweins' cubic theta functions.*

We need the following lemma, which is a refined form of a result of Keister, Sellers and Vary [KSV09, Lemma 7].

Lemma 2.2. *Let m be a non-negative integer, then for all $1 \le n \le 2^m$, we have*

$$\binom{2^m}{n} 2^n \equiv 0 \pmod{2^{m+\lfloor \frac{n-1}{2} \rfloor + 1}}.$$

In fact, we also need the following lemma, whose proof follows exactly the line of proof as Lemma 2.2, so we omit the proof here.

Lemma 2.3. *Let m be a nonnegative integer, then for all $1 \le n \le 3^m$, we have*

$$\binom{3^m}{n} 3^n \equiv 0 \pmod{3^{m+\lfloor \frac{n-1}{3} \rfloor + 1}}.$$

We also need the following results from Hirschhorn and Sellers [HS06, Theorems 1.1 and 1.2].

Lemma 2.4. *For all $n \ge 1$, we have*

$$\overline{OPT}_1(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n \text{ is a square or twice a square}, \\ 0 \pmod 4 & \text{otherwise}. \end{cases}$$

Lemma 2.5. *For all $n \ge 1$, we have*

$$\overline{OPT}_1(n) \equiv \begin{cases} 2 \ \text{or} \ 6 \pmod 8 & \text{if } n \text{ is a square or twice a square}, \\ 0 \ \text{or} \ 4 \pmod 8 & \text{otherwise}. \end{cases}$$

2.2. **A primer on modular forms.** We recall some aspects of modular forms that will be used in Sections 7 and 9. Let $\mathbb{H}$ be the complex upper half-plane. For a positive integer N, we define the following matrix groups:

$$\Gamma := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in \Gamma : n \in \mathbb{Z} \right\}.$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \equiv 0 \pmod N \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod N \right\}$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N, \text{ and } b \equiv c \equiv 0 \pmod N \right\}.$$

A congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup satisfying $\Gamma(N) \subseteq \Gamma$ for some N and the smallest such N s called the level of Γ. The group

$$\mathrm{GL}_2^+(\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \dfrac{az+b}{cz+d}$. We identify ∞ with $\dfrac{1}{0}$. We also define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \dfrac{r}{s} = \dfrac{ar+bs}{cr+ds}$, where $\dfrac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This will give an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then a cusp of Γ is an equivalence class in $\mathbb{Q} \cup \{\infty\}$.

We denote by $M_\ell(\Gamma_1(N))$ for a positive integer ℓ, the complex vector space of modular forms of weight ℓ with respect to $\Gamma_1(N)$. Also

$$[\Gamma : \Gamma_0(N)] := N \prod_{\ell | N} \left(1 + \frac{1}{\ell}\right),$$

where ℓ is a prime number.

Definition 2.1. [Ono04, Definition 1.15] *If χ is a Dirichlet character modulo N, then a form $f(x) \in M_\ell(\Gamma_1(N))$ has Nebentypus character χ if*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$.

Recall that the Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer. We now recall two theorems from [Ono04, p. 18] which will be used to prove our result.

Theorem 2.1. [Ono04, Theorem 1.64 and Theorem 1.65] *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient with $\ell = \dfrac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z}$, with*

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. *Here the character* χ *is defined by* $\chi(d) := \left(\frac{(-1)^\ell \prod_{\delta|N} \delta^{r_\delta}}{d}\right)$. *In addition, if* c, d, *and* N *are positive integers with* $d|N$ *and* $\gcd(c,d) = 1$, *then the order of vanishing of* $f(z)$ *at the cusp* $\frac{c}{d}$ *is* $\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}$.

Suppose that ℓ is a positive integer and that $f(z)$ is an eta-quotient satisfying the conditions of the above theorem. If $f(z)$ is holomorphic at all of the cups of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$.

Theorem 2.2. [Ono04, due to Serre, p. 43] *If* $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_\ell(\Gamma_0(N), \chi)$ *has Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} c(n) q^n \in \mathbb{Z}[[q]],$$

then for each positive integer m *there exists a constant* $\alpha > 0$ *such that*

$$|\{n \le X \;:\; c(n) \not\equiv 0 \pmod{m}\}| = \mathcal{O}\left(\frac{X}{(\log X)^\alpha}\right).$$

We finally recall the definition of Hecke operators. Let m be a positive integer and

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_\ell(\Gamma_0(N), \chi).$$

Then the Hecke operator T_m on $f(z)$ is defined as

$$f(Z)|T_m := \sum_{n=0}^{\infty} \left(\sum_{d|\gcd(n,m)} \chi(d) d^{\ell-1} a\left(\frac{nm}{d^2}\right) \right) q^n.$$

In particular, if $m = p$ is a prime, then we have

$$f(Z)|T_m := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p) p^{\ell-1} a\left(\frac{n}{p}\right) \right) q^n. \tag{2.12}$$

Note that $a(n) = 0$ unless $n \ge 0$.

2.3. **Radu's Algorithm.** We need some preliminary results, which describe an algorithmic approach to proving partition concurrences, developed by Radu [Rad15, Rad09]. For integers $M \ge 1$, suppose that $R(M)$ is the set of all the integer sequences

$$(r_\delta) := (r_{\delta_1}, r_{\delta_2}, r_{\delta_3}, \ldots, r_{\delta_k})$$

indexed by all the positive divisors δ of M, where $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$. For integers $m \ge 1$, $(r_\delta) \in R(M)$, and $t \in \{0, 1, 2, \ldots, m-1\}$, we define the set $P(t)$ as

$$P(t) := \Bigg\{ t' \in \{0, 1, 2, \ldots, m-1\} : t' \equiv ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod{m} \\ \text{for some } [s]_{24m} \in \mathbb{S}_{24m} \Bigg\}. \tag{2.13}$$

For integers $N \ge 1$, $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $(r_\delta) \in R(M)$, and $(r'_\delta) \in R(N)$, we also define

$$p(\gamma) := \min_{\lambda \in \{0,1,\ldots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd(\delta(a + k\lambda c), mc)^2}{\delta m},$$

$$p'(\gamma) := \frac{1}{24} \sum_{\delta|N} r'_\delta \frac{\gcd(\delta, c)^2}{\delta}.$$

For integers $m \geq 1$; $M \geq 1$, $N \geq 1$, $t \in \{0, 1, 2, \dots, m-1\}$, $k := \gcd(m^2-1, 24)$, and $(r_\delta) \in R(M)$, define Δ^* to be the set of all tuples $(m, M, N, t, (r_\delta))$ such that all of the following conditions are satisfied

1. Prime divisors of m are also prime divisors of N;
2. If $\delta \mid M$, then $\delta \mid mN$ for all $\delta \geq 1$ with $r_\delta \neq 0$;
3. $24 \mid kN \sum_{\delta|M} \frac{r_\delta mN}{\delta}$;
4. $8 \mid kN \sum_{\delta|M} r_\delta$;
5. $\frac{24m}{\left(-24kt - k\sum_{\delta|M}\delta r_\delta, 24m\right)} \mid N$;
6. If $2|m$ then either $4|kN$ and $8|\delta N$ or $2|s$ and $8|(1-j)N$, where $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s \cdot j$.

We now state a result of Radu [Rad09], which we use in completing the proofs of Theorems 1.3 and 1.6.

Lemma 2.6. [Rad09, Lemma 4.5] *Suppose that $(m, M, N, t, (r_\delta)) \in \Delta^*$, $(r'_\delta) := (r'_\delta)_{\delta|N} \in R(N)$, $\{\gamma_1, \gamma_2, \dots, \gamma_n\} \subseteq \Gamma$ is a complete set of representatives of the double cosets of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$, and $t_{\min} := \min_{t' \in P(t)} t'$,*

$$\nu := \frac{1}{24}\left(\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta r'_\delta - \frac{1}{m}\sum_{\delta|M}\delta r_\delta\right) - \frac{t_{min}}{m}, \tag{2.14}$$

$p(\gamma_j) + p'(\gamma_j) \geq 0$ for all $1 \leq j \leq n$, and $\sum_{n=0}^{\infty} A(n)q^n := \prod_{\delta|M} f_\delta^{r_\delta}$. If for some integers $u \geq 1$, all $t' \in P(t)$, and $0 \leq n \leq \lfloor \nu \rfloor$, $A(mn + t') \equiv 0 \pmod{u}$ is true, then for integers $n \geq 0$ and all $t' \in P(t)$, we have $A(mn + t') \equiv 0 \pmod{u}$.

The following lemma supports Lemma 2.6 in the proofs of Theorems 1.3 and 1.6.

Lemma 2.7. [RS11, Lemma 2.6] *Let N or $\frac{N}{2}$ be a square-free integer, then we have*

$$\bigcup_{\delta|N} \Gamma_0(N)\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\Gamma_\infty = \Gamma.$$

3. Proof of Theorem 1.1

Using (2.7), we have

$$\sum_{n=0}^{\infty} \overline{OPT}_4(n)q^n = \frac{f_2^{12}}{f_1^8 f_4^4} = \frac{f_4^{24}}{f_2^{16} f_8^8} + 8q\frac{f_4^{12}}{f_2^{12}} + 16q^2\frac{f_8^8}{f_2^8},$$

which gives

$$\sum_{n=0}^{\infty} \overline{OPT}_4(2n)q^n = \frac{f_2^{24}}{f_1^{16} f_4^8} + 16q\frac{f_4^8}{f_1^8}. \tag{3.1}$$

Again squaring (2.7) and then replacing q by $-q$, we have

$$\frac{1}{f_1^8} = \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q\frac{f_4^{16}}{f_2^{24}} + 16q^2\frac{f_8^8 f_4^4}{f_2^{20}},$$
$$\frac{f_1^8 f_4^8}{f_2^{24}} = \frac{f_4^{28}}{f_2^{28} f_8^8} - 8q\frac{f_4^{16}}{f_2^{24}} + 16q^2\frac{f_8^8 f_4^4}{f_2^{20}}.$$

The two identities above imply

$$\frac{f_2^{24}}{f_1^{16}f_4^8}=1+16q\frac{f_4^8}{f_1^8},$$

which together with (3.1), gives

$$\sum_{n=0}^{\infty}\overline{OPT}_4(2n)q^n=2\frac{f_2^{24}}{f_1^{16}f_4^8}-1=2\sum_{n=0}^{\infty}\overline{OPT}_8(n)q^n-1.$$

This equality proves Theorem 1.1. □

4. Elementary proof of Theorem 1.2

From (1.1), we recall

$$\sum_{n\geq 0}\overline{OPT}_3(n)q^n=\frac{f_2^9}{f_1^6f_4^3}=\frac{f_2^9}{f_4^3}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right),\tag{4.1}$$

where we have employed (2.5) and (2.7). Now, working modulo 4, we have,

$$\sum_{n\geq 0}\overline{OPT}_3(n)q^n\equiv\frac{f_2^7}{f_4^3}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\pmod 4.\tag{4.2}$$

Extracting the terms involving q^{2n+1}, dividing both sides by q and replacing q^2 by q, we have

$$\sum_{n\geq 0}\overline{OPT}_3(2n+1)q^n\equiv 2\frac{f_1^2f_8^2}{f_2f_4}\equiv 2f_2^6\pmod 4.\tag{4.3}$$

Now, extracting the terms involving q^{2n} from (4.2) and replacing q^2 by q, we have

$$\sum_{n\geq 0}\overline{OPT}_3(2n)q^n\equiv\frac{f_1^2f_4^5}{f_2^3f_8^2}\equiv\frac{f_4^5}{f_2^3f_8^2}\left(\frac{f_2f_8^5}{f_4^2f_{16}^2}-2q\frac{f_2f_{16}^2}{f_8}\right)\pmod 4,\tag{4.4}$$

where in the last step, we have employed (2.4). Equations (4.3) and (4.4) will be recalled in a while. We now move on to prove the congruences.

Proof of (1.2). From equations (2.1) and (2.3) for $n\geq 0$, we have

$$\overline{OPT}_3(3n+1)\equiv\overline{OPT}_3(3n+2)\equiv 0\pmod 6.\tag{4.5}$$

This proves (1.2). In fact, $\overline{OPT}_3(3n+2)\equiv 0\pmod{18}$. We prove this next.

Proof of (1.5). Working modulo 9, we obtain

$$\begin{aligned}\sum_{n\geq 0}\overline{OPT}_3(n)q^n&\equiv\frac{f_6^3}{f_3^3f_{12}^3}f_1^3\left(f_4^3\right)^2\\&\equiv\frac{f_6^3}{f_3^3f_{12}^3}\left(\frac{f_6f_9^6}{f_3f_{18}^3}-3qf_9^3+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}\right)\\&\quad\times\left(\frac{f_{24}f_{36}^6}{f_{12}f_{72}^3}-3q^4f_{36}^3+4q^{12}\frac{f_{12}^2f_{72}^6}{f_{24}^2f_{36}^3}\right)^2\pmod 9,\end{aligned}\tag{4.6}$$

where in the last step, we have employed (2.9).

Extracting the terms involving q^{3n+2}, dividing both sides by q^2 and replacing q^3 by q, we arrive at

$$\begin{aligned}\overline{OPT}_3(3n+2)&\equiv 18\frac{f_2^3f_3^3f_8f_{12}^9}{f_1^3f_4^4f_{24}^3}q+9\frac{f_2^4f_3^6f_{12}^6}{f_1^4f_4^3f_6^3}q^2+36\frac{f_2f_6^6f_{12}^6}{f_1f_3^3f_4^3}q^3+72\frac{f_2^3f_3^3f_{24}^6}{f_1^3f_4f_8^2}q^5\\&\equiv 0\pmod 9.\end{aligned}\tag{4.7}$$

Hence, from (2.3) and (4.7), we obtain

$$\overline{OPT}_3(3n+2) \equiv 0 \pmod{18}.$$

This proves (1.5).

In the remainder of this section, we will use $\overline{OPT}_3(3n+1) \equiv 0 \pmod 3$ and (4.7) above without commentary.

Proof of (1.3). Next, we have

$$\overline{OPT}_3(12n+7) = \overline{OPT}_3(4(3n+1)+3) \equiv 0 \pmod 4, \tag{4.8}$$

using [SSS23, Theorem 1]. Also,

$$\overline{OPT}_3(12n+7) = \overline{OPT}_3(3(4n+2)+1) \equiv 0 \pmod 3. \tag{4.9}$$

From (4.8) and (4.9), we conclude

$$\overline{OPT}_3(12n+7) \equiv 0 \pmod{12}.$$

This proves (1.3).

Proof of (1.4). Again, extracting the terms involving odd powers of q from (4.4), we have

$$\sum_{n\geq 0} \overline{OPT}_3(4n+2)q^n \equiv 2\frac{f_2^5 f_8^2}{f_1^2 f_4^3} \equiv 2f_4^3 \equiv 2\left(\frac{f_{24}f_{36}^6}{f_{12}f_{72}^3} - 3q^4 f_{36}^3 + 4q^{12}\frac{f_{12}^2 f_{72}^6}{f_{24}^2 f_{36}^3}\right) \pmod 4.$$

Extracting the terms involving q^{3n+2}, dividing both sides by q^2 and replacing q^3 by q, we obtain

$$\overline{OPT}_3(12n+10) \equiv 0 \pmod 4 \tag{4.10}$$

and

$$\overline{OPT}_3(12n+10) = \overline{OPT}_3(3(4n+3)+1) \equiv 0 \pmod 3. \tag{4.11}$$

Combining (4.11) and (4.10), we arrive at

$$\overline{OPT}_3(12n+10) \equiv 0 \pmod{12}.$$

This proves (1.4).

Proof of (1.6). Next, we have

$$\overline{OPT}_3(6n+5) = \overline{OPT}_3(3(2n+1)+2) \equiv 0 \pmod 9. \tag{4.12}$$

Also, from (4.3), we recall

$$\sum_{n\geq 0} \overline{OPT}_3(2n+1)q^n \equiv 2f_4^3 \pmod 4. \tag{4.13}$$

Employing (2.9) in (4.13) and extracting the odd powered terms of q, we obtain

$$\sum_{n\geq 0} \overline{OPT}_3(2n+1)q^n \equiv 2f_4^3 \equiv 2\left(\frac{f_{24}f_{36}^6}{f_{12}f_{72}^3} - 3q^4 f_{36}^3 + 4q^{12}\frac{f_{12}^2 f_{72}^6}{f_{24}^2 f_{36}^3}\right) \pmod 4.$$

Extracting the terms involving q^{3n+2}, we obtain

$$\overline{OPT}_3(6n+5) \equiv 0 \pmod 4. \tag{4.14}$$

Combining (4.14) and (4.12), we obtain

$$\overline{OPT}_3(6n+5) \equiv 0 \pmod{36}.$$

This proves (1.6).

Proof of (1.7). Using (1.5), we know that $\overline{OPT}_3(24n+23) \equiv 0 \pmod 9$. So, to prove (1.7) we only need to check whether the desired congruence is divisible by 16. Using (4.1), we have

$$\sum_{n=0}^{\infty} \overline{OPT}_3(n)q^n = \frac{f_2^9}{f_1^6 f_4^3} \equiv \frac{f_1^{16} f_2}{f_1^6 f_4^3} \equiv \frac{f_2}{f_4^3} \cdot f_1^2 \cdot \left(f_1^4\right)^2$$

$$\equiv \frac{f_2}{f_4^3}\left(\frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}\right)^2 \left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}\right) \pmod{16}.$$

Extracting the terms involving q^{2n+1}, dividing both sides by q and replacing q^2 by q , we have

$$\sum_{n=0}^{\infty} \overline{OPT}_3(2n+1)q^n \equiv 8\frac{f_1^2 f_2^3 f_4^5}{f_8^2} - 2\frac{f_2^{17} f_8^2}{f_1^2 f_4^9} \pmod{16}.$$

Employing (2.4) and (2.5) and extracting the odd powered terms of q, we obtain

$$\sum_{n=0}^{\infty} \overline{OPT}_3(4n+3)q^n \equiv -4\frac{f_1^{12} f_4 f_8^2}{f_2^7} \pmod{16}.$$

Using (2.6), we extract the terms involving q^{2n+1} to arrive at

$$\sum_{n=0}^{\infty} \overline{OPT}_3(8n+7)q^n \equiv 48\frac{f_2^{19}}{f_1^9 f_4^2} + 256q\frac{f_4^{14}}{f_1 f_2^5} \equiv 0 \pmod{16}.$$

This completes the proof of (1.7). □

5. Proof of Theorem 1.3

Combining the results [SSS23, Theorem 7] and [HS06, Theorem 1.1], we know that

$$\overline{OPT}_3(n) \equiv 0 \pmod 4$$

for all $n \geq 1$ if and only if n is neither a square nor twice a square. So, proving (1.8) is equivalent to proving that $3pn + R$ is neither a square nor twice a square for all $n \geq 0$, primes $p \geq 5$, and R defined in Theorem 1.3. First, note that since $Ap \equiv 2r + 1 \pmod 3$,

$$R \equiv \begin{cases} 2(3r+1) \pmod 3 & \text{if } 2(Ap+r) < 3p, \\ 2(3r+1) - 3p \pmod 3 & \text{if } 2(Ap+r) > 3p \end{cases}$$

$$\equiv \begin{cases} 2 \pmod 3 & \text{if } 2(Ap+r) < 3p, \\ 2 \pmod 3 & \text{if } 2(Ap+r) > 3p. \end{cases}$$

Therefore, $3pn + R \equiv 2 \pmod 3$ is never a square for all $n \geq 0$, primes $p \geq 5$, and R.

If $R = 2(Ap + r)$, then we have the following two cases depending on the parity of n.

1. When $n = 2m$ for some m, $3pn + R = 2(p(3m + A) + r)$ is not twice a square since $p(3m + A) + r$ can not be a square.
2. When n is odd, $3pn + R$ is odd. Therefore, $3pn + R$ is not twice a square.

If $R = 2(Ap + r) - 3p$, then again we have the following two cases depending on the parity of n.

1. When n is even, $3pn + R$ is odd. Therefore, $3pn + R$ is not twice a square.
2. When $n = 2m + 1$ for some m, $3pn + R = 2(p(3m + A) + r)$ is not twice a square since $p(3m + A) + r$ can not be a square.

Thus, $3pn+R$ is neither a square nor twice a square for any value of n, p, and R. This completes the proof of Theorem 1.3. □

6. Proof of Theorem 1.5

We note that the first congruence (1.9) follows immediately from Theorem 1.4. We prove the second congruence (1.10) next.

Using (2.5), we have

$$\sum_{n=0}^{\infty}\overline{OPT}_{2^i\cdot r}(n)q^n = \frac{f_2^{3\cdot 2^i\cdot r}}{f_4^{2^i\cdot r}}\left(\frac{1}{f_1^2}\right)^{2^i\cdot r} = \frac{f_2^{3\cdot 2^i\cdot r}}{f_4^{2^i\cdot r}}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)^{2^i\cdot r}$$
$$= \sum_{k=0}^{2^i\cdot r}2^k\binom{2^i\cdot r}{k}q^k\frac{f_8^{5\cdot 2^i\cdot r-6k}}{f_2^{2^{i+1}\cdot r}f_4^{2^i\cdot r-2k}f_{16}^{2^{i+1}\cdot r-4k}},$$

from which

$$\sum_{n=0}^{\infty}\overline{OPT}_{2^i\cdot r}(2n+1)q^n = \sum_{k=0}^{2^{i-1}\cdot r-1/2}2^{2k+1}\binom{2^i\cdot r}{2k+1}q^k\frac{f_4^{5\cdot 2^i\cdot r-12k-6}}{f_1^{2^{i+1}\cdot r}f_2^{2^i\cdot r-4k-2}f_8^{2^{i+1}\cdot r-8k-4}}$$
$$= \sum_{k=0}^{2^{i-1}\cdot r-1/2}2^{2k+1}\binom{2^i\cdot r}{2k+1}q^k\frac{f_4^{5\cdot 2^i\cdot r-12k-6}}{f_2^{2^i\cdot r-4k-2}f_8^{2^{i+1}\cdot r-8k-4}}\cdot\left(\frac{1}{f_1^2}\right)^{2^i\cdot r}$$
$$= \sum_{k=0}^{2^{i-1}\cdot r-1/2}\sum_{t=0}^{2^i\cdot r}2^{2k+t+1}\binom{2^i\cdot r}{2k+1}\binom{2^i\cdot r}{t}$$
$$\times\, q^{k+t}\frac{f_4^{5\cdot 2^i\cdot r-12k+2t-6}f_8^{3\cdot 2^i\cdot r+8k-6t+4}}{f_2^{3\cdot 2^{i+1}\cdot r-4k-2}f_{16}^{2^{i+1}\cdot r-4t}},$$

which is equivalent to

$$\sum_{n=0}^{\infty}\overline{OPT}_{2^i\cdot r}(2n+1)q^n = \sum_{\alpha=0,\beta=0}^{1}\sum_{k=0}^{2^{i-2}\cdot r-1/4-\alpha/2}\sum_{t=0}^{2^{i-1}\cdot r-\beta/2}2^{4k+2t+2\alpha+\beta+1}\binom{2^i\cdot r}{4k+2\alpha+1}$$
$$\times\binom{2^i\cdot r}{2t+\beta}q^{2k+2t+\alpha+\beta}\frac{f_4^{5\cdot 2^i\cdot r-24k+4t-12\alpha+2\beta-6}f_8^{3\cdot 2^i\cdot r+16k-12t+8\alpha-6\beta+4}}{f_2^{3\cdot 2^{i+1}\cdot r-8k-4\alpha-2}f_{16}^{2^{i+1}\cdot r-8t-4\beta}}.$$

From the above identity, we extract the terms that involve odd exponents of q,

$$\sum_{n=0}^{\infty}\overline{OPT}_{2^i\cdot r}(4n+3)q^n = \sum_{\substack{\alpha,\beta\\ \alpha+\beta=1}}^{1}\sum_{k=0}^{2^{i-2}\cdot r-1/4-\alpha/2}\sum_{t=0}^{2^{i-1}\cdot r-\beta/2}2^{4k+2t+2\alpha+\beta+1}\binom{2^i\cdot r}{4k+2\alpha+1}$$
$$\times\binom{2^i\cdot r}{2t+\beta}q^{k+t}\frac{f_2^{5\cdot 2^i\cdot r-24k+4t-12\alpha+2\beta-6}f_4^{3\cdot 2^i\cdot r+16k-12t+8\alpha-6\beta+4}}{f_1^{3\cdot 2^{i+1}\cdot r-8k-4\alpha-2}f_8^{2^{i+1}\cdot r-8t-4\beta}}. \tag{6.1}$$

By Lemma 2.2, we have

$$2^{4k+2t+2\alpha+\beta+1}\binom{2^i\cdot r}{4k+2\alpha+1}\binom{2^i\cdot r}{2t+\beta}\equiv 0 \pmod{2^{i+3}} \tag{6.2}$$

for the tuples $(\alpha,\beta)=(0,1)$ and $(1,0)$ except when $k=0$, $t=0$, and $\beta=0$. So, we consider

$$2^{4\cdot 0+2\cdot 0+2\cdot 1+0+1}\binom{2^i\cdot r}{4\cdot 0+2\cdot 1+1}\binom{2^i\cdot r}{2\cdot 0+0} = 2^3\cdot\frac{2^i\cdot r\,(2^i\cdot r-1)\,(2^{i-1}\cdot r-1)}{3}$$
$$\equiv 0 \pmod{2^{i+3}}.$$

Therefore, (6.1) and (6.2) give (1.10).

The third congruence (1.11) follows from a result analogous to a result of Keister, Vary and Sellers [KSV09, Theorem 9]. □

Lemma 6.1. *Let $k = 2^m r$, $m > 0$ and r be odd, then for all $n \geq 1$ we have*

$$\overline{OPT}_{2^m r}(n) \equiv \begin{cases} 2^{m+1} \pmod{2^{m+2}} & \text{if } n \text{ is a square, twice a square or four times a square,} \\ 0 \pmod{2^{m+2}} & \text{otherwise.} \end{cases}$$

Proof. We prove the result by induction on m. The base case $m = 1$ is given first, where we show

$$\overline{OPT}_{2r}(n) \equiv \begin{cases} 4 \pmod 8 & \text{if } n \text{ is a square, twice a square or four times a square,} \\ 0 \pmod 8 & \text{otherwise.} \end{cases}$$

We have

$$\sum_{n\geq 0} \overline{OPT}_{2r}(n)q^n = \left(\left(\sum_{n\geq 0} \overline{OPT}_1(n)q^n\right)^2\right)^r$$

$$= \left(\left(1 + \sum_{\substack{n>0 \\ n \text{ is a square}}} \overline{OPT}_1(n)q^n + \sum_{\substack{n>0 \\ n \text{ is not a square}}} \overline{OPT}_1(n)q^n\right)^2\right)^r.$$

We expand the above square and look at each term separately:

$$1 + \left(\sum_{\substack{n>0 \\ n \text{ is a square}}} \overline{OPT}_1(n)q^n\right)^2 + \underbrace{\left(\sum_{\substack{n>0 \\ n \text{ is not a square}}} \overline{OPT}_1(n)q^n\right)^2}_{A} + 2\left(\sum_{\substack{n>0 \\ n \text{ is a square}}} \overline{OPT}_1(n)q^n\right)$$

$$+ \underbrace{2\left(\sum_{\substack{n>0 \\ n \text{ is not a square}}} \overline{OPT}_1(n)q^n\right)}_{B} + \underbrace{2\left(\sum_{\substack{n>0 \\ n \text{ is a square}}} \overline{OPT}_1(n)q^n\right)\left(\sum_{\substack{n>0 \\ n \text{ is not a square}}} \overline{OPT}_1(n)q^n\right)}_{C}.$$

First, we look at the term labelled A:

$$A = \left(\sum_{\substack{n>0 \\ n=2\times \text{ a square}}} \overline{OPT}_1(n)q^n + \sum_{\substack{n>0 \\ n\neq 2\times \text{ a square or a square}}} \overline{OPT}_1(n)q^n\right)^2$$

$$\equiv \left(\sum_{\substack{n>0 \\ n=2\times \text{a square}}} \overline{OPT}_1(n)q^n\right)^2 \pmod 8,$$

where we have used Lemma 2.4 and 2.5 to obtain the last step. Similarly, another application of Lemma 2.4 would give us

$$B \equiv 2\left(\sum_{\substack{n>0 \\ n=2\times\text{a square}}} \overline{OPT}_1(n)q^n\right).$$

Finally, the term labelled C gives us

$$C = 2\left(\sum_{\substack{n>0 \\ n=2\times\text{a square}}} \overline{OPT}_1(n)q^n + \sum_{\substack{n>0 \\ n\neq 2\times\text{a square or a square}}} \overline{OPT}_1(n)q^n\right)\left(\sum_{\substack{n>0 \\ n=\text{a square}}} \overline{OPT}_1(n)q^n\right)$$

$$\equiv 2\left(\sum_{\substack{n>0\\ n=2\times\text{a square}}}\overline{OPT}_1(n)q^n\right)\left(\sum_{\substack{n>0\\ n=\text{a square}}}\overline{OPT}_1(n)q^n\right) \pmod 8$$
$$\equiv 0 \pmod 8.$$

Here we have applied Lemma 2.4 to obtain each of the two steps.

Collecting all of the above together, we obtain

$$\sum_{n\geq 0}\overline{OPT}_{2r}(n)q^n = \left(1+\left(\sum_{\substack{n>0\\ n=\text{a square}}}\overline{OPT}_1(n)q^n\right)^2+2\left(\sum_{\substack{n>0\\ n=\text{a square}}}\overline{OPT}_1(n)q^n\right)\right.$$
$$\left.+\left(\sum_{\substack{n>0\\ n=2\times\text{a square}}}\overline{OPT}_1(n)q^n\right)^2+2\left(\sum_{\substack{n>0\\ n=2\times\text{a square}}}\overline{OPT}_1(n)q^n\right)\right)^r \pmod 8$$
$$=\left(1+\left(\sum_{n\geq 1}\overline{OPT}_1(n^2)q^{n^2}\right)^2+\left(\sum_{n\geq 1}\overline{OPT}_1(2n^2)q^{2n^2}\right)^2\right.$$
$$\left.+2\left(\sum_{n\geq 1}\overline{OPT}_1(n^2)q^{n^2}\right)+2\left(\sum_{n\geq 1}\overline{OPT}_1(2n^2)q^{2n^2}\right)\right)^r \pmod 8.$$

Applying Lemma 2.4 in the above we obtain

$$\sum_{n\geq 0}\overline{OPT}_{2r}(n)q^n = \left(1+4\sum_{n\geq 1}q^{n^2}+4\sum_{n\geq 1}q^{2n^2}+4\left(\sum_{n\geq 1}q^{n^2}\right)^2+4\left(\sum_{n\geq 1}q^{2n^2}\right)^2\right)^r \pmod 8.$$

Using the fact that $(q^{n_1}+q^{n_2}+\cdots)^2=(q^{2n_1}+q^{2n_2}+\cdots)+2(q^{n_1+n_2}+\cdots)$ in the above, we obtain

$$\sum_{n\geq 0}\overline{OPT}_{2r}(n)q^n = \left(1+4\sum_{n\geq 1}q^{n^2}+4\sum_{n\geq 1}q^{2n^2}+4\left(\sum_{n\geq 1}q^{2n^2}+2\sum_{\substack{n_1,n_2>0\\ n_1\neq n_2}}q^{n_1^2+n_2^2}\right)\right.$$
$$\left.+4\left(\sum_{n\geq 1}q^{4n^2}+2\sum_{\substack{n_1,n_2>0\\ n_1\neq n_2}}q^{2n_1^2+2n_2^2}\right)\right)^r \pmod 8$$
$$=\left(1+4\left(\sum_{m\geq 1}q^{m^2}+\sum_{n\geq 1}q^{2n^2}+\sum_{k\geq 1}q^{4k^2}\right)\right)^r \pmod 8$$
$$=\sum_{j\geq 0}\binom{r}{j}4^j\left(\sum_{m\geq 1}q^{m^2}+\sum_{n\geq 1}q^{2n^2}+\sum_{k\geq 1}q^{4k^2}\right)^j$$
$$\equiv 1+4\left(\sum_{m\geq 1}q^{m^2}+\sum_{n\geq 1}q^{2n^2}+\sum_{k\geq 1}q^{4k^2}\right) \pmod 8.$$

The last step follows since r is odd. This proves the base case.

For the induction step, we assume

$$\overline{OPT}_{2^m r}(n)\equiv\begin{cases}2^{m+1} \pmod{2^{m+2}} & \text{if } n \text{ is a square, twice a square or four times a square,}\\ 0 \pmod{2^{m+2}} & \text{otherwise.}\end{cases}$$

and show that

$$\overline{OPT}_{2^{m+1}r}(n) \equiv \begin{cases} 2^{m+2} \pmod{2^{m+3}} & \text{if } n \text{ is a square, twice a square or four times a square,} \\ 0 \pmod{2^{m+3}} & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned}
&\sum_{n\geq 0}\overline{OPT}_{2^{m+1}r}(n)q^n \\
&= \left(\sum_{n\geq 0}\overline{OPT}_{2^m r}(n)q^n\right)^2 \\
&= \left(1+\sum_{\substack{n>0\\ n\neq 1\times\text{or } 2\times\text{or } 4\times\text{square}}}\overline{OPT}_{2^m r}(n)q^n+\sum_{\substack{n>0\\ n=1\times\text{or } 2\times\text{or } 4\times\text{square}}}\overline{OPT}_{2^m r}(n)q^n\right)^2 \\
&= 1+\left(\sum_{\substack{n>0\\ n\neq 1\times\text{or } 2\times\text{or } 4\times\text{square}}}\overline{OPT}_{2^m r}(n)q^n\right)^2+\left(\sum_{\substack{n>0\\ n=1\times\text{or } 2\times\text{or } 4\times\text{square}}}\overline{OPT}_{2^m r}(n)q^n\right)^2 \\
&\quad +2\left(\sum_{\substack{n>0\\ n\neq 1\times\text{or } 2\times\text{or } 4\times\text{square}}}\overline{OPT}_{2^m r}(n)q^n\right)+2\left(\sum_{\substack{n>0\\ n=1\times\text{or } 2\times\text{or } 4\times\text{square}}}\overline{OPT}_{2^m r}(n)q^n\right) \\
&\quad +2\left(\sum_{\substack{n>0\\ n\neq 1\times\text{or } 2\times\text{or } 4\times\text{square}}}\overline{OPT}_{2^m r}(n)q^n\right)\times\left(\sum_{\substack{n>0\\ n=1\times\text{or } 2\times\text{or } 4\times\text{square}}}\overline{OPT}_{2^m r}(n)q^n\right).
\end{aligned}$$

Using a similar technique like the base case and using the induction hypothesis we arrive at

$$\begin{aligned}
\sum_{n\geq 0}\overline{OPT}_{2^{m+1}r}(n)q^n &\equiv 1+2\sum_{n\geq 1}\overline{OPT}_{2^m r}(n^2)q^{n^2}+2\sum_{n\geq 1}\overline{OPT}_{2^m r}(2n^2)q^{2n^2} \\
&\quad +2\sum_{n\geq 1}\overline{OPT}_{2^m r}(4n^2)q^{4n^2}+\left(\sum_{n\geq 1}\overline{OPT}_{2^m r}(n^2)q^{n^2}\right. \\
&\quad \left.+\sum_{n\geq 1}\overline{OPT}_{2^m r}(2n^2)q^{2n^2}+\sum_{n\geq 1}\overline{OPT}_{2^m r}(4n^2)q^{4n^2}\right)^2 \pmod{2^{m+3}} \\
&\equiv 1+2\left(\sum_{n\geq 1}\overline{OPT}_{2^m r}(n^2)q^{n^2}+\sum_{n\geq 1}\overline{OPT}_{2^m r}(2n^2)q^{2n^2}+\right. \\
&\quad \left.+\sum_{n\geq 1}\overline{OPT}_{2^m r}(4n^2)q^{4n^2}\right) \pmod{2^{m+3}}.
\end{aligned}$$

From the induction hypothesis, the coefficients of the last term are congruent to 2^{m+1} or $2^{m+1}+2^{m+2}$ or $2^{m+1}+2^{m+2}+2^{m+3} \pmod{2^{m+3}}$, and with the factor of 2 in front of it, we finally arrive at

$$\sum_{n\geq 0}\overline{OPT}_{2^{m+1}r}(n)q^n \equiv 1+2^{m+2}\left(\sum_{n\geq 1}q^{n^2}+\sum_{m\geq 1}q^{2m^2}+\sum_{k\geq 1}q^{4k^2}\right) \pmod{2^{m+3}}. \tag{6.3}$$

This completes the induction. □

The third congruence (1.11) follows easily from Lemma 6.1. Clearly, $8n+5$ is not twice or four times a square. Also, $8n+5$ is not a square as it is odd and odd squares are $\equiv 1 \pmod 8$.

7. Proof of Theorem 1.6

We use the material in Subsection 2.3 without commentary.

For the purposes of Theorem 1.6, it is enough to take

$$(m, M, N, t, (r_\delta)) = (8, 4, 4, 2, (-2^{i+1}r, 3\cdot 2^i r, -2^i r)).$$

Later we will specialize $r \in \{1,3,5\}$. It is routine to check that this choice satisfies the Δ^* conditions. By equation (2.13) we see that $P(t) = \{\ell\}$ for $\ell \in \{2,4,6\}$. For the choice of $(r'_\delta) = (3\cdot 2^{i+1}r, 0, 0)$ we see that

$$p\left(\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\right) + p'\left(\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\right) \geq 0 \quad \text{for all } \delta \mid N,$$

and

$$\nu = 14\cdot 2^{i-2}r - \frac{\ell}{8}, \tag{7.1}$$

where $\ell = t$ for different choices of t. So we need to check the congruences for all $n \leq \lfloor \nu \rfloor$, $1 \leq i \leq 5$ and $r \in \{1,3,5\}$ and then by Lemmas 2.6 and 2.7 we would have proved our result. This can be checked using Mathematica and hence the result follows. □

8. Proof of Theorem 1.7

We have

$$\sum_{n=0}^{\infty} \overline{OPT}_{3^i}(n)q^n = \frac{f_2^{3\cdot 3^i}}{f_1^{2\cdot 3^i} f_4^{3^i}} = \frac{f_{-1}^{3^i}}{f_1^{3^i}},$$

where for positive odd integers k, we take $f_{-k} := \left(-q^k; -q^k\right)_\infty$ and it is known that $f_{-k} = f_{2k}^3/\left(f_k f_{4k}\right)$. In the above identity, we invoke the 3-dissections (2.10) and (2.11) and then use binomial expansions to arrive at

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{OPT}_{3^i}(n)q^n \\
&= \left(a_{-3}f_{-3} + 3qf_{-9}^3\right)^{3^{i-1}} \times \left(a_3^2 \frac{f_9^3}{f_3^{10}} + 3a_3 q \frac{f_9^6}{f_3^{11}} + 9q^2 \frac{f_9^9}{f_3^{12}}\right)^{3^{i-1}} \\
&= \left(\sum_{t=0}^{3^{i-1}} 3^t \binom{3^{i-1}}{t} q^t a_{-3}^{3^{i-1}} f_{-3}^{3^{i-1}-t} f_{-9}^{3t}\right) \\
&\quad \times \left(\sum_{\substack{\ell=0,m=0,r=0\\ \ell+m+r=3^{i-1}}}^{3^{i-1}} 3^{m+2r}\binom{3^{i-1}}{\ell, m, r} q^{m+2r} a_3^{3^{2\ell+m}} \frac{f_9^{3\ell+9m+6r}}{f_3^{10\ell+11m+12r}}\right) \\
&= \left(\sum_{\alpha=0}^{2}\sum_{t=0}^{3^{i-2}-\alpha/3} 3^{3t+\alpha}\binom{3^{i-1}}{3t+\alpha} q^{3t+\alpha} a_{-3}^{3^{i-1}} f_{-3}^{3^{i-1}-3t-\alpha} f_{-9}^{9t+3\alpha}\right) \\
&\quad \times \left(\sum_{\beta=0,\gamma=0}^{2} \sum_{\substack{\ell=0,m=0,r=0\\ \ell+3m+3r+\beta+\gamma=3^{i-1}}}^{3^{i-1},\ 3^{i-2}-\beta/3,\ 3^{i-2}-\gamma/3} 3^{3m+6r+\beta+2\gamma}\binom{3^{i-1}}{\ell, 3m+\beta, 3r+\gamma}\right.
\end{aligned}$$

$$\times\, q^{3m+6r+\beta+2\gamma} a_3^{3^{2\ell+3m+\beta}} \frac{f_9^{3\ell+27m+18r+9\beta+6\gamma}}{f_3^{10\ell+33m+36r+11\beta+12\gamma}}\Bigg).$$

From the above product of sums, extracting the terms that involve q^{3n+2}, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{OPT}_{3^i}(3n+2)q^n\\ &= \sum_{\substack{\alpha,\beta,\gamma\\ \alpha+\beta+2\gamma\equiv 2 \pmod 3}}^{2} \sum_{\substack{t,\ell,m,r\\ \ell+3m+3r+1=3^{i-1}}}^{3^{i-2}-\alpha/3,\ 3^{i-1},\ 3^{i-2}-\beta/3,\ 3^{i-2}-\gamma/3} 3^{3t+3m+6r+\alpha+\beta+2\gamma}\binom{3^{i-1}}{3t+\alpha}\\ &\quad\times\binom{3^{i-1}}{\ell,3m+\beta,3r+\gamma} q^{t+m+2r+\alpha/3+\beta/3+2\gamma/3-2/3} a_{-1}^{3^{i-1}} a_1^{3^{2\ell+3m+\beta}} f_{-1}^{3^{i-1}-3t-\alpha} f_{-3}^{9t+3\alpha}\\ &\quad\times\frac{f_3^{3\ell+27m+18r+9\beta+6\gamma}}{f_1^{10\ell+33m+36r+11\beta+12\gamma}}. \end{aligned}\tag{8.1}$$

Now, it remains to be examined if

$$3^{3t+3m+6r+\alpha+\beta+2\gamma}\binom{3^{i-1}}{3t+\alpha}\binom{3^{i-1}}{\ell,3m+\beta,3r+\gamma}\equiv 0 \pmod{3^{i+1}},\tag{8.2}$$

for the tuples

$$(\alpha,\beta,\gamma)\in\{(0,0,1),(0,1,2),(0,2,0),(1,0,2),(1,1,0),(1,2,1),(2,0,0),(2,1,1),(2,2,2)\}.$$

This follows immediately from Lemma 2.3 for $1\le t\le 3^{i-1}-1$. Next, for the case $t=0$ and $\alpha=0$, we need to check whether

$$3^{3m+6r+\beta+2\gamma}\binom{3^{i-1}}{\ell,3m+\beta,3r+\gamma}\equiv 0 \pmod{3^{i+1}}.\tag{8.3}$$

This can easily be seen as we have $\alpha=0$ and hence $\beta+2\gamma$ must be at least 2. Also, $\binom{3^{i-1}}{\ell,3m+\beta,3r+\gamma}$ is divisible by 3^{i-1}. Therefore, (8.1), (8.2) and (8.3) together give (1.12). □

9. Proofs of Theorems 1.8 and 1.9

Proof of Theorem 1.8. From [SSS23, Eq. (44)] we have

$$\sum_{n\ge 0}\overline{OPT}_3(8n+1)q^n\equiv 6f_1f_2 \pmod 8.$$

This implies

$$\sum_{n\ge 0}\overline{OPT}_3(8n+1)q^{8n+1}\equiv 6\eta(8z)\eta(16z) \pmod 8.\tag{9.1}$$

Let $\eta(8z)\eta(16z):=\sum_{n=1}^{\infty}a(n)q^n$, then $a(n)=0$ if $n\not\equiv 1 \pmod 8$ for all $n\ge 0$. This implies

$$\overline{OPT}_3(8n+1)\equiv 6a(8n+1) \pmod 8.\tag{9.2}$$

From Theorem 2.1 we have $\eta(8z)\eta(16z)\in S_1(\Gamma_0(128),\chi_1)$, where χ_1 is a Nebentypus character and is given by $\chi_1(\bullet)=\left(\frac{-1\cdot 2^7}{\bullet}\right)$.

Since $\eta(8z)\eta(16z)$ is a Hecke eigenform (see [Mar96, pp. 4854]), so equation (2.12) implies

$$\eta(8z)\eta(16z)|T_p=\sum_{n=1}^{\infty}\left(a(pn)+\chi_1(p)a\left(\frac{n}{p}\right)\right)q^n=\lambda(p)\sum_{n=1}^{\infty}a(n)q^n.$$

This implies

$$a(pn)+\chi_1(p)a\left(\frac{n}{p}\right)=\lambda(p)a(n).\tag{9.3}$$

Putting $n = 1$ we note that $a(1) = 1$, so $\lambda(p) = a(p)$ and since $a(p) = 0$ for all $p \not\equiv 1 \pmod 8$ we have $\lambda(p) = 0$. From (9.3) we obtain,

$$a(pn) + \chi_1(p)a\left(\frac{n}{p}\right) = 0. \tag{9.4}$$

In equation (9.4) setting $n = pn + r$ we obtain, for all $n \geq 0$ and $p \nmid r$,

$$a(p^2n + pr) = 0, \tag{9.5}$$

and replacing n by pn in (9.4) we obtain

$$a(p^2n) \equiv -\chi_1(p)a(n) \pmod 4. \tag{9.6}$$

Let $A(n) := a(8n+1)$, and let p be a prime such that $p \not\equiv 1 \pmod 8$. Replacing n by $8n - pr + 1$ in (9.5) we obtain

$$A\left(p^2n + \frac{p^2-1}{8} + pr\frac{1-p^2}{8}\right) = 0. \tag{9.7}$$

Setting $n = 8n + 1$ in (9.6), we have

$$A\left(p^2n + \frac{p^2-1}{8}\right) \equiv -\chi_1(p)A(n) \pmod 4. \tag{9.8}$$

Since $p \geq 3$ is a prime, so $8|(1-p^2)$ and $\gcd\left(\frac{1-p^2}{8}, p\right) = 1$. So, r runs over a residue system excluding the multiples of p, and so does $\frac{1-p^2}{8}r$. We can rewrite (9.7) as

$$A\left(p^2n + \frac{p^2-1}{8} + pj\right) \equiv 0 \pmod 4, \tag{9.9}$$

where $p \nmid j$.

For primes $p_i \geq 3$ such that $p_i \not\equiv 1 \pmod 8$ we can use (9.8) repeatedly to obtain

$$A\left(p_1^2p_2^2\dots p_k^2n + \frac{p_1^2p_2^2\dots p_k^2-1}{8}\right) \equiv (-1)^k\chi_1(p_1)\chi_1(p_2)\cdots\chi_1(p_k)A(n) \pmod 4. \tag{9.10}$$

This needs the observation

$$p_1^2p_2^2\dots p_k^2n + \frac{p_1^2p_2^2\dots p_k^2-1}{8} = p_1^2\left(p_2^2\dots p_k^2n + \frac{p_2^2\dots p_k^2-1}{8}\right) + \frac{p_1^2-1}{8}.$$

Let $j \not\equiv 0 \pmod{p_{k+1}}$, then (9.9) and (9.10) gives us

$$A\left(p_1^2p_2^2\dots p_{k+1}^2n + \frac{p_1^2p_2^2\dots p_{k+1}^2-1}{8} + p_1^2p_2^2\dots p_k^2p_{k+1}j\right) \equiv 0 \pmod 4. \tag{9.11}$$

Finally, equations (9.11) and (9.1) gives us the desired result. □

Proof of Theorem 1.9. **Case (i):** When $p = 2$.

Recall

$$\sum_{n\geq 0}\overline{OPT}_3(n)q^n = \frac{f_2^9}{f_1^6f_4^3}. \tag{9.12}$$

Rewriting this in terms of eta-quotients we have

$$\sum_{n\geq 0}\overline{OPT}_3(n)q^{24n} = \frac{\eta^9(48z)}{\eta^6(24z)\eta^3(96z)}. \tag{9.13}$$

Let

$$B(z) = \frac{\eta^2(24z)}{\eta(48z)}.$$

By the binomial theorem we have

$$B_{2^k}(z) \equiv 1 \pmod{2^{k+1}}.$$

Now, define

$$C_k(z) := \frac{\eta^9(48z)}{\eta^6(24z)\eta^3(96z)} B_{2^k}(z) = \frac{\eta^{2^{k+1}-6}(24z)}{\eta^{2^k-9}(48z)\eta^3(96z)}.$$

By equation (9.12) and (9.13) we have

$$C_k(z) \equiv \sum_{n\geq 0} \overline{OPT}_3(n)q^{24n} \pmod{2^{k+1}} \tag{9.14}$$

By Theorem 2.1 $C_k(z)$ is a form of weight 2^{k-1} on $\Gamma_0(768)$. The cusps of $\Gamma_0(768)$ are represented by fractions $\frac{c}{d}$, where $d|768$ and $\gcd(c,d)=1$. $C_k(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,24)^2}{24}(2^{k+1}-6) + \frac{\gcd(d,48)^2}{48}(9-2^k) - 3\frac{\gcd(d,96)^2}{96} \geq 0. \tag{9.15}$$

Using Mathematica, we verify the inequality (9.15) for all divisors of 768. So, by Theorem 2.1, $C_k(z) \in M_{2^{k-1}}\left(\Gamma_0(768), \left(\frac{2^{2^{k+1}+6}\cdot 3^{2^k}}{\bullet}\right)\right)$. By Theorem 2.2, the Fourier coefficients of $C_k(z)$ are almost always divisible by $m = 2^k$. By (9.14), the same is true for $\overline{OPT}_3(n)$ and hence the result follows.

Case (ii): When $p(\neq 3)$ is an odd prime.

The proof of the case when p is an odd prime is very similar, so we sketch it here.

For $a \geq 1$, let us define

$$A^{p^k}(z) = \frac{\eta^{p^{a+k}}(24z)}{\eta^{p^k}(24p^a z)} \equiv 1 \pmod{p^{k+1}},$$

and

$$B_{p,k}(z) = \frac{\eta^9(48z)}{\eta^6(24z)\eta^3(96z)} A^{p^k}(z).$$

Modulo p^{k+1}, we have

$$B_{p,k}(z) \equiv \frac{\eta^9(48z)}{\eta^6(24z)\eta^3(96z)} \equiv \sum_{n\geq 0} \overline{OPT}_3(n)q^{24n} \pmod{p^k}.$$

The weight of the eta-quotient $B_{p,k}(z)$ is $\frac{p^k}{2}(p^a-1)$. Let the level of the eta-quotient be $96u$, where u is the smallest integer satisfying

$$(p^{a+k}-6)\frac{96u}{24} + 9\frac{96u}{48} - 3\frac{96u}{96} - p^k\frac{96u}{24p^a} \equiv 0 \pmod{24}.$$

This is equivalent to

$$(4p^{k-a}(p^{2a}-1)-9)u \equiv 0 \pmod{24}. \tag{9.16}$$

For all primes $p \neq 3$ we have $3|p^{2a}-1$, so from (9.16) we can conclude that $u=8$.

By Theorem 2.1, the cusps of $\Gamma_0(768)$ are given by $\frac{c}{d}$ with $\gcd(c,d)=1$. So $B_{p,z}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$(p^{a+k}-6)\frac{\gcd(d,24)^2}{24} + 9\frac{\gcd(d,48)^2}{48} - 3\frac{\gcd(d,96)^2}{96} - p^k\frac{\gcd(d,24p^a)^2}{24p^a} \geq 0. \tag{9.17}$$

We need to check whether (9.17) is true for all $d|768$ and then by Theorems 2.1 and 2.2 we are done. We leave this to the reader to verify. □

10. Concluding Remarks

(1) Theorem 1.1 looks ripe for a combinatorial proof. We leave this as an open problem.
(2) Can we extend Theorem 1.3 to the $\overline{OPT}_{2k+1}(n)$ function?

(3) There are other candidates for a result like Theorem 1.8. For instance, using some of the identities in the proof of Theorem 10 in [SSS23] there might be some results for $\overline{OPT}_4(n)$.
(4) It is evident that changing the values of i and r in equation (7.1) we can also prove numerically several more cases of Conjecture 1.1, but we refrain from doing this, as it is just a matter of computing power.
(5) Based on strong numerical evidence, we pose the following conjectures.

Conjecture 10.1. *For all $i, j \geq 1$ and k not a multiple of 2 or 3, we have*

$$\overline{OPT}_{3^i \cdot 2^j \cdot k}(3n+2) \equiv 0 \pmod{3^{i+1} \cdot 2^{j+2}}.$$

Conjecture 10.2. *For all $i \geq 1$ and j not a power of 2, nor a multiple of 2 or divisible by 3, we have*

$$\overline{OPT}_{3^i \cdot j}(3n+2) \equiv 0 \pmod{3^{i+1} \cdot 2}.$$

Conjecture 10.3. *For all $i, j \geq 1$ and k not a multiple of 2 or 3, we have*

$$\overline{OPT}_{3^i \cdot 2^j \cdot k}(3n+1) \equiv 0 \pmod{3^{i} \cdot 2^{j+1}}.$$

Conjecture 10.4. *For all $i \geq 1$ and j not a power of 2, nor a multiple of 2 or divisible by 3, we have*

$$\overline{OPT}_{3^i \cdot j}(3n+1) \equiv 0 \pmod{3^{i} \cdot 2}.$$

We can prove several cases of the above conjectures using the technique used in the proof of Theorem 1.6.

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