BIASES IN PARTITIONS AND RESTRICTED PARTITIONS

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Abstract. Recently, the concept of parity bias in integer partitions has been studied by several authors. We continue this study here, but for restricted partitions (namely, partitions with parts greater than 1). We prove analogous results for these restricted partitions as those that have been obtained by Kim, Kim and Lovejoy (2020) and Kim and Kim (2021). We also discuss related results for overpartitions and other classes of restricted partitions.

1. Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of n is a non-increasing sequence of natural numbers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Here, each λ_i is called a part of the partition λ of n (written as $\lambda \vdash n$) and the length of the partition, denoted by $\ell(\lambda)$ is k. Partitions have been studied since the time of Euler, and continues to be a serious topic for ongoing research in several directions. A good introduction to the subject is given in the masterly treatment of Andrews [And98].

In the theory of partitions, inequalities arising between two classes of partitions have a long tradition of study, see for instance work in this direction by Alder [Ald48], Andrews [And13], McLaughlin [ML16], Chern, Fu, and Tang [CFT18], Berkovich and Uncu [BU19], among others. In 2020, Kim, Kim and Lovejoy [KKL20] introduced a phenomenon in integer partitions called *parity bias*, wherein the number of partitions of n with more odd parts (denoted by $p_o(n)$) are more in number than the number of partitions of n with more even parts (denoted by $p_e(n)$). That is, they proved for $n \neq 2$, $p_o(n) > p_e(n)$. They also conjectured a similar inequality for partitions with only distinct parts. For $n > 19$, they conjectured that $d_o(n) > d_e(n)$, where $d_o(n)$ (resp. $d_e(n)$) denotes the number of partitions of n with distinct parts with more odd parts (resp. even parts) than odd parts (resp. even parts). Further generalizations of the results of Kim, Kim and Lovejoy [KKL20] have been found by Kim and Kim [KK21] and Chern [Che21]. Most of the proofs of the results in these papers use techniques arising from q-series methods.

Banerjee *et al.* [BBD⁺21] proved both the above quoted result and conjecture of Kim, Kim and Lovejoy [KK21] using combinatorial means. In addition, they proved several more results on parity biases of partitions with restrictions on the set of parts. For a nonempty set $S \subsetneq \mathbb{Z}_{\geq 0}$, define

$$P_e^S(n) := \{\lambda \in P_e(n) : \lambda_i \notin S\}$$
$$\text{and} \quad P_o^S(n) := \{\lambda \in P_o(n) : \lambda_i \notin S\},$$

where the set $P_e(n)$ (resp. $P_o(n)$) consists of all partitions of n with more even parts (resp. odd parts) than odd parts (resp. even parts). Let us denote the number of partitions of $P_e^S(n)$ (resp. $P_o^S(n)$) by $p_e^S(n)$ (resp. $p_o^S(n)$). Banerjee *et al.* [BBD⁺21] proved the following result.

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Theorem 1.1 (Banerjee *et al.*, [BBD+21]). *For positive integers n, the following inequalities are true (the range is given in the brackets),*

$$p_o^{\{1\}}(n) < p_e^{\{1\}}(n), \quad (n > 7), \tag{1}$$

$$p_o^{\{2\}}(n) > p_e^{\{2\}}(n), \quad (n \geq 1), \tag{2}$$

and

$$p_o^{\{1,2\}}(n) > p_e^{\{1,2\}}(n), \quad (n > 8). \tag{3}$$

All of the proofs of the above inequalities by Banerjee *et al.* were by using combinatorial techniques.

The primary goal of this paper is to use analytical techniques and prove results of the type proved by Banerjee *et al.*, that is about parity biases in partitions with certain restrictions on its allowed parts. We reprove the inequality (1) using analytical techniques, as well as prove results in a similar setup for the biases discussed in the work of Kim and Kim [KK21]. Overpartitions (to be defined later) also appear in the work of Kim, Kim and Lovejoy [KKL20], where they study a different type of bias for overpartitions. In this paper we will also refine their result.

The paper is structured as follows: in Section 2 we state our main results, in Section 3 we prove the results stated in Section 2, in Section 4 we state and prove some other results using similar methods used in Section 3, and finally we close the paper with some concluding remarks in Section 5.

2. Main Results

Using analytical techniques we will give a proof of following result which was proved by Banerjee *et. al.* [BBD+21] combinatorially. We modify the notation a bit and let $q_e(n)$ (resp. $q_o(n)$) be the number of partitions of n with parts greater than 1 where the number of even (resp. odd) parts are more than the number of odd (resp. even) parts.

Theorem 2.1 (Theorem 1.5, [BBD+21]). *For all positive integers $n \geq 8$, we have*

$$q_o(n) < q_e(n).$$

Let $p_{j,k,m}(n)$ be the number of partitions of n such that there are more parts congruent to j modulo m than parts congruent to k modulo m, for $m \geq 2$. Then, Kim and Kim [KK21] proved that for all positive integers $n \geq m^2 - m + 1$, we have

$$p_{1,0,m}(n) > p_{0,1,m}(n).$$

Let us now denote by $q_{j,k,m}(n)$ the number of partitions of n into parts greater than 1 such that there are more parts congruent to j modulo m than parts congruent to k modulo m, for $m \geq 2$. Then, we have the following result.

Theorem 2.2. *For $n \geq 4m + 3$, we have*

$$q_{0,1,m} > q_{1,0,m}.$$

An overpartition of n is a non-increasing sequence of positive integers whose sum is n and in which the first occurence of a number may be overlined. These classes of partitions were introduced by Corteel and Lovejoy [CL04]. Kim, Kim and Lovejoy [KKL20] proved the following result about biases in overpartitions.

Theorem 2.3 (Theorem 8, [KKL20]). *If $\overline{p}_u(n)$ (resp. $\overline{p}_o(n)$) is the number of overpartitions of n with more unoverlined (resp. overlined) parts than overlined (resp. unoverlined) parts, then the difference $\overline{p}_u(n) - \overline{p}_o(n)$ is equal to the number of overpartitions of n where the number of unoverlined parts is at least two more than the number of overlined parts.*

Here, we refine this theorem to say that the result is true even if we choose the unoverlined parts to be the multiples of a particular natural number. Define $\overline{p}_{bu}(n)$ (resp. $\overline{p}_{bo}(n)$) to be the number of overpartitions of n with more unoverlined parts (resp. overlined parts) which are multiples of b than overlined parts (resp. unoverlined parts). Then the following result is true.

Theorem 2.4. *The difference $\overline{p}_{bu}(n)-\overline{p}_{bo}(n)$ is equal to the number of overpartitions of n where the number of unoverlined parts is at least two more than the number of overlined parts.*

Taking $b=1$ in Theorem 2.4 gives us Theorem 2.3.

Further, define $\overline{p}_{cu}(n)$ (resp. $\overline{p}_{co}(n)$) to be the number of overpartitions of n with parts that are multiples of c and more unoverlined (resp. overlined) parts than overlined (resp. unoverlined) parts.

Theorem 2.5. *The difference $\overline{p}_{cu}(n)-\overline{p}_{co}(n)$ is equal to the number of overpartitions of n with parts that are multiples of c where the number of unoverlined parts is at least two more than the number of overlined parts.*

Taking $c=1$ in Theorem 2.5 gives Theorem 2.3.

The following results can also be proved using similar analytical techniques as the above results.

Theorem 2.6. *Let n be an even number and let $E_e(n)$ (resp. $O_e(n)$) be the number of partitions of n with even number of even and odd parts, where the number of even parts (resp. odd parts) are more than odd parts (resp. even parts). Then, for all n, we have*

$$O_e(n) > E_e(n).$$

Theorem 2.7. *Let n be an odd number. Let $E_o(n)$ (resp. $O_o(n)$) be the number of partitions of n with odd number of odd and even parts, with more even parts (resp. odd parts) than even parts (resp. odd parts). Then, we have*

$$O_o(n) > E_o(n).$$

Similar results are true if we restrict the parts to a minimum value.

Theorem 2.8. *Let the minimum part for each partition of n be m. Let $n \geq 2m$ be an even number. Let, $E_{me}(n)$ (resp. $O_{me}(n)$) be the number of partitions of n with even number of odd and even parts, where the number of even parts (resp. odd parts) is more than the number of odd parts (resp. even parts). Then, for all n, we have*

$$O_{me}(n) < E_{me}(n), \quad \text{if } m \text{ is even},$$

and,

$$O_{me}(n) > E_{me}(n), \quad \text{if } m \text{ is odd}.$$

Theorem 2.9. *Let the minimum part for each partition of n be m. Let $n \geq m$ be an odd number. Let, $E_{mo}(n)$ (resp. $O_{mo}(n)$) be the number of partitions of n with odd number of even and odd parts, where the number of even parts (resp. odd parts) is more than the number of odd parts (resp. even parts). Then, we have*

$$O_{mo}(n) > E_{mo}(n), \text{if } m \text{ is odd},$$

and,

$$O_{mo}(n) < E_{mo}(n), \text{if } m \text{ is even}.$$

3. Proofs of Our Results

We use the standard q-series notation

$$(a)_n = (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}), \quad |q|<1,$$

and

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n.$$

Also recall Heine's transformation [GR04, Appendix III.1], which says for $|z|$, $|q|$, $|b| \leq 1$, we have

$$\sum_{n\geq 0} \frac{(a)_n(b)_n}{(q)_n(c)_n} z^n = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \sum_{n\geq 0} \frac{(z)_n(c/b)_n}{(q)_n(az)_n} b^n. \tag{4}$$

By appropriately iterating Heine's transformation, we obtain [GR04, Appendix III.3] what is sometimes called the q-analogue of Euler's transformation, which says that for $|z|$, $|\frac{abz}{c}| \leq 1$, we have

$$\sum_{n\geq 0} \frac{(a)_n(b)_n}{(q)_n(c)_n} z^n = \frac{(abz/c)_\infty}{(z)_\infty} \sum_{n\geq 0} \frac{(c/a)_n(c/b)_n}{(q)_n(c)_n} (abz/c)^n. \tag{5}$$

We also recall an identity of Sylvester [SF82, p. 281], which says for $|q| \leq 1$, we have

$$(-xq)_\infty = \sum_{n\geq 0} \frac{(-xq)_n}{(q)_n} (1+xq^{2n+1}) x^n q^{n(3n+1)/2}. \tag{6}$$

By standard combinatorial arguments, we have that $\dfrac{q^{bn}}{(q^2;q^2)_n}$ is the generating function for partitions with exactly n odd parts with minimum odd part b, as well as it is the generating function for partitions with exactly n even parts with minimum even part b. We will use this in the proofs below without commentary.

Proof of Theorem 2.1. Let $P_o(q)$ (resp. $P_e(q)$) be the generating functions of $q_o(n)$ (resp. $q_e(n)$. Then, we have

$$P_o(q) = \sum_{n\geq 0} \frac{q^{3n}}{(q^2;q^2)_n^2} - \sum_{n\geq 0} \frac{q^{5n}}{(q^2;q^2)_n^2} = q^3 + q^5 + q^6 + q^7 + 2q^8 \cdots,$$

and,

$$P_e(q) = \frac{1}{(q^2;q)_\infty} - \sum_{n\geq 0} \frac{q^{3n}}{(q^2;q^2)_n^2} = q^2 + 2q^4 + 3q^6 + q^7 + 5q^8 \cdots.$$

Substituting $c = q^4, a, b \to 0, z = q^3, q \to q^2$ in equation (5) we get

$$\begin{aligned}
P_o(q) &= \sum_{n\geq 1} \frac{q^{3n}}{(q^2;q^2)_n^2}(1-q^{2n}) \\
&= \frac{1}{(1-q^2)} \sum_{n\geq 1} \frac{q^{3n}}{(q^4;q^2)_{n-1}(q^2;q^2)_{n-1}} \\
&= \frac{q^3}{(1-q^2)} \sum_{n\geq 0} \frac{q^{3n}}{(q^4;q^2)_n(q^2;q^2)_n} \\
&= \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 0} \frac{q^{2n^2+5n+3}}{(q^2;q^2)_{n+1}(q^2;q^2)_n} \\
&= \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2+n}}{(q^2;q^2)_n(q^2;q^2)_{n-1}} \\
&= \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2+n}}{(q^2;q^2)_n^2}(1-q^{2n}).
\end{aligned}$$

Substituting $c = q^2, a, b \to 0, z = q^3, q \to q^2$ in equation (5) we get

$$\begin{aligned}
P_e(q) &= \frac{1}{(q^3;q^2)_\infty}\frac{1}{(q^2;q^2)_\infty} - \sum_{n\geq 0} \frac{q^{3n}}{(q^2;q^2)_n^2} \\
&= \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 0} \frac{q^{2n^2}}{(q^2;q^2)_n^2} - \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 0} \frac{q^{2n^2+3n}}{(q^2;q^2)_n^2} \\
&= \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2}}{(q^2;q^2)_n^2}(1-q^{3n}).
\end{aligned}$$

Now,

$$P_e(q) - P_o(q) = \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2}}{(q^2;q^2)_n^2}(1-q^n).$$

Clearly, for the summands from $n = 2$ onward the coefficients are positive, because if n is even, then $1-q^n$ will be cancelled by a factor of $(q^2;q^2)_n$ and if n is odd, then it will be cancelled by a factor of $(q^3;q^2)_\infty$.

For, $n = 1$, we do the following. Put $x = 1$ in equation (6) to get the following

$$\begin{aligned}
q^3 + q^5 + \frac{1}{(q^3;q^2)_\infty}\frac{q^2(1-q)}{(1-q^2)^2} &= q^3(1+q^2) + \frac{q^2(1-q)^2}{(1-q^2)^2}(-q)_\infty \\
&= q^3(1+q^2) + \frac{q^2}{(1+q)^2} \sum_{n\geq 0} \frac{(-q)_n}{(q)_n}(1+q^{2n+1})q^{\frac{3n^2+n}{2}} \\
&= q^3(1+q^2) + \frac{q^2}{(1+q)} + \frac{q^4(1+q^3)}{(1-q^2)} \\
&\quad + \frac{q^2}{(1+q)^2} \sum_{n\geq 2} \frac{(-q)_n}{(q)_n}(1+q^{2n+1})q^{\frac{3n^2+n}{2}} \\
&= \frac{q^2(1+q^2)}{(1-q^2)} + \frac{q^2}{(1+q)^2} \sum_{n\geq 2} \frac{(-q)_n}{(q)_n}(1+q^{2n+1})q^{\frac{3n^2+n}{2}},
\end{aligned}$$

which gives us

$$\frac{1}{(q^3;q^2)_\infty}\frac{q^2(1-q)}{(1-q^2)^2} = -q^3 - q^5 + \frac{q^2(1+q^2)}{(1-q^2)} + \frac{q^2}{1-q^2}\sum_{n\geq 2}\frac{(-q^2)_{n-1}}{(q^2)_{n-1}}(1+q^{2n+1})q^{\frac{3n^2+n}{2}}.$$

We see that the coefficients for all terms are nonnegative except for q^3 and q^5. The terms of the expansion of the third summand of the RHS consists of terms of the form q^{2i} for all $i \in N$. For $n = 2$ the fourth summand of the RHS gives a series where the terms are of the form q^{2i+1} for all $i \in N$ and $i \geq 4$. For all $n > 2$ the minimum power of q in the expansion of the fourth term of RHS is greater than 9. Also, for all $n > 1$ the minimum power of q in the expansion of $P_e(q) - P_o(q)$ is greater than or equal to 8. So, in each case the coefficient of q^7 is 0. This completes the proof. □

Proof of Theorem 2.2. Let $P_{1,0,m}(q)$ (resp. $P_{0,1m}(q)$) be the generating functions of $q_{1,0,m}(n)$ (resp. $q_{0,1,m}(n)$. Then, we have

$$P_{1,0,m}(q) = \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{q^{(m+1)n}}{(q^m;q^m)_n^2} - \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{q^{(m+1)n+mn}}{(q^m;q^m)_n^2},$$

and

$$P_{0,1,m}(q) = \frac{1}{(q^2;q)_\infty} - \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{q^{(m+1)n}}{(q^m;q^m)_n^2}.$$

Now,

$$\begin{aligned}
P_{1,0,m}(q) &= \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{q^{(m+1)n}}{(q^m;q^m)_n^2}(1-q^{mn}) \\
&= \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 1}\frac{q^{(m+1)n}}{(q^m;q^m)_n(q^m;q^m)_{n-1}} \\
&= \frac{(q^{m+1},q^m;q^m)_\infty}{(q^2;q)_\infty}\frac{q^{m+1}}{(1-q^m)}\sum_{n\geq 0}\frac{q^{(m+1)n}}{(q^m,q^{2m};q^m)_n}
\end{aligned}$$

by substituting, $q \to q^m, a, b \to 0, c \to q^{2m}$ and $z \to q^{m+1}$ in equation (5), we get

$$\begin{aligned}
&= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty}\frac{q^{m+1}}{(1-q^m)}\sum_{n\geq 0}\frac{q^{mn^2+2mn+n}}{(q^m,q^{2m};q^m)_n} \\
&= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 1}\frac{q^{mn^2+n}(1-q^{mn})}{(q^m;q^m)_n^2}. \qquad (7)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
P_{0,1,m}(q) &= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{q^{mn^2}}{(q^m;q^m)_n^2} - \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{q^{mn^2+(m+1)n}}{(q^m;q^m)_n^2} \\
&= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{q^{mn^2}}{(q^m;q^m)_n^2}(1-q^{(m+1)n}) \\
&= \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty}\sum_{n\geq 1}\frac{q^{mn^2}}{(q^m;q^m)_n^2}(1-q^{(m+1)n}). \qquad (8)
\end{aligned}$$

From equations (7) and (8), we get

$$P_{0,1,m}(q) - P_{1,0,m}(q) = \frac{(q^m;q^m)_\infty}{(q^2;q)_\infty} \sum_{n\geq 1} \frac{q^{mn^2}}{(q^m;q^m)_n^2}(1-q^n).$$

From Kim and Kim [KK21, Lemma 2.1], we see that the above has nonnegative coefficients for all q^k with $k > 2m+1$. The summand $n=2$ is $\frac{(q^m;q^m)_\infty q^{4m}}{(q^3;q)_\infty (q^m;q^m)_2^2}$. This shows that coefficients of q^k are positive for $k \geq 4m+3$. In fact, the coefficient of q^{4m} is also positive. So, we have our result. □

Proof of Theorem 2.4. Let the generating function of $\overline{p}_{bo}(n)$ (resp. $\overline{p}_{bu}(n)$) be $\overline{P}_{bo}(q)$ (resp. $\overline{P}_{bu}(q)$). Then we have,

$$\overline{P}_{bo}(q) = \sum_{n\geq 0} \overline{p}_{bo}(q) q^n = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1}{(q^b;q^b)_n} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n} \frac{q^{bn}}{(q^b;q^b)_n},$$

and

$$\overline{P}_{bu}(q) = \sum_{n\geq 0} \overline{p}_{bu}(q) q^n = \frac{(-q)_\infty}{(q^b;q^b)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1}{(q^b;q^b)_n}.$$

So,

$$\begin{aligned}
\overline{P}_{bu}(q) - \overline{P}_{bu}(q) &= \frac{(-q)_\infty}{(q^b;q^b)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1}{(q^b;q^b)_n} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1}{(q^b;q^b)_n}(1-q^{bn}) \\
&= \frac{(-q)_\infty}{(q^b;q^b)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1}{(q^b;q^b)_n} - \sum_{n\geq 1} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1}{(q^b;q^b)_{n-1}} \\
&= \frac{(-q)_\infty}{(q^b;q^b)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1}{(q^b;q^b)_n} \left(1 + \frac{q^{n+1}}{1-q^{n+1}}\right) \\
&= \frac{(-q)_\infty}{(q^b;q^b)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1}{(q^b;q^b)_n (1-q^{n+1})}.
\end{aligned}$$

Clearly the above sum is the generating function for overpartitions where the number of un-overlined parts is at most one more than the number of overlined parts. This completes the proof. □

Proof of Theorem 2.5. The proof of this result is exactly similar to the proof of Theorem 2.4, so we leave it to the reader. □

Proof of Theorem 2.6. Let the generating function of $E_e(n)$ (resp. $O_e(n)$) be $E_e(q)$ (resp. $O_e(q)$). Then, we have

$$E_e(q) = \sum_{\substack{n\geq 2 \\ \text{n is even}}} E_e(n) q^n = \sum_{\substack{n\geq 2 \\ \text{n is even}}} \frac{q^{2n}}{(q^2;q^2)_n} \left(\sum_{\substack{k=0 \\ \text{k is even}}}^{n-2} \frac{q^k}{(q^2;q^2)_k} \right),$$

and

$$O_e(q) = \sum_{\substack{n\geq 2 \\ \text{n is even}}} O_e(n) q^n = \sum_{\substack{n\geq 2 \\ \text{n is even}}} \frac{q^{n}}{(q^2;q^2)_n} \left(\sum_{\substack{k=0 \\ \text{k is even}}}^{n-2} \frac{q^{2k}}{(q^2;q^2)_k} \right).$$

So,

$$O_e(q) - E_e(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^n}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{2k}}{(q^2;q^2)_k}(1-q^{n-k})\right),$$

which clearly has positive coefficients as $n-k$ is even and its largest possible value is n, so $(1-q^{n-k})$ will be cancelled out by a factor in the denominator. This completes the proof. ☐

Proof of Theorem 2.7. The proof of this result is similar to the proof of Theorem 2.6, so we leave it to the reader. ☐

Proof of Theorem 2.8. Let the generating function of $E_{me}(n)$ (resp. $O_{me}(n)$) be $E_{me}(q)$ (resp. $O_{me}(q)$).

Case I: If m is odd, say $m = 2b-1$ for some $b \in \mathbb{Z}$. We have,

$$E_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} E_{me}(n)q^n = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{2bn}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{(2b-1)k}}{(q^2;q^2)_k}\right),$$

and,

$$O_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} O_{me}(n)q^n = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{(2b-1)n}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{2bk}}{(q^2;q^2)_k}\right).$$

Now,

$$O_{me}(q) - E_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{(2b-1)n}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{2bk}}{(q^2;q^2)_k}(1-q^{n-k})\right).$$

which clearly has non negative coefficients. This completes the proof of Case I.

Case II: If m is even, say $m = 2c$ for some $c \in \mathbb{Z}$. Then, we have

$$E_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} E_{me}(n)q^n = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{2cn}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{(2c-1)k}}{(q^2;q^2)_k}\right),$$

and,

$$O_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} O_{me}(n)q^n = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{(2c+1)n}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{2ck}}{(q^2;q^2)_k}\right).$$

Now,

$$E_{me}(q) - O_{me}(q) = \sum_{\substack{n\geq 2\\ \text{n is even}}} \frac{q^{(2c)n}}{(q^2;q^2)_n}\left(\sum_{\substack{k=0\\ \text{k is even}}}^{n-2} \frac{q^{(2c+1)k}}{(q^2;q^2)_k}(1-q^{n-k})\right),$$

which clearly has positive coefficients and it can be seen that the minimum power of q is $2m$. ☐

Proof of Theorem 2.7. The proof is similar to the proof of Theorem 2.8, so we leave it to the reader. □

4. A Few Related Results

In this short section, we mention two results which can be used in settings of the type that have been studied in the previous sections. We do not explore this direction any further.

Theorem 4.1. *Let b be an odd natural number. Then we have*

$$\sum_{n\geq 1}\frac{q^{bn}}{(q^2;q^2)_n}\frac{1}{(q^2;q^2)_{n-1}}=\sum_{n=0}^{\infty}\frac{q^{2n}}{(q^2;q^2)_n}\left(\sum_{k=n+1}^{\infty}\frac{q^{bk}}{(q^2;q^2)_k}\right).$$

Proof. If we consider the generating function of the number of partitions with odd parts greater then b and with odd parts greater then even parts, we get

$$\begin{aligned}\sum_{n\geq 1}\frac{q^{bn}}{(q^2;q^2)_n}\frac{1}{(q^2;q^2)_{n-1}}&=\frac{1}{(q^b;q^2)_\infty}\frac{1}{(q^2;q^2)_\infty}-\sum_{n\geq 0}\frac{q^{2n}}{(q^2;q^2)_n}\left(\sum_{k=0}^{n}\frac{q^{bk}}{(q^2;q^2)_k}\right)\\&=\left(\sum_{n\geq 0}\frac{q^{bn}}{(q^2;q^2)_n}\right)\left(\sum_{n\geq 0}\frac{q^{2n}}{(q^2;q^2)_n}\right)-\sum_{n\geq 0}\frac{q^{2n}}{(q^2;q^2)_n}\left(\sum_{k=0}^{n}\frac{q^{bk}}{(q^2;q^2)_k}\right)\\&=\sum_{n\geq 0}\frac{q^{2n}}{(q^2;q^2)_n}\left(\sum_{k\geq n+1}^{\infty}\frac{q^{bk}}{(q^2;q^2)_k}\right),\end{aligned}$$

which completes the proof. □

We can generalize the above result as well.

Theorem 4.2. *Let b and c be natural numbers with different parity. Then the following relation holds*

$$\sum_{n\geq 1}\frac{q^{bn}}{(q^2;q^2)_n}\left(\sum_{k=0}^{n-1}\frac{q^{ck}}{(q^2;q^2)_k}\right)=\sum_{n\geq 0}\frac{q^{cn}}{(q^2;q^2)_n}\left(\sum_{k=n+1}^{\infty}\frac{q^{bk}}{(q^2;q^2)_k}\right). \tag{9}$$

Proof. First, we assume that b is an odd natural number and c is an even natural number. We begin the proof by using similar concepts used earlier. We take the number of partitions of a natural number with minimum odd part b and minimum even part c. Then we consider the generating function of the number of partitions of a natural number with minimum odd part b and minimum even part c where odd parts are more than even parts. The generating function is

$$\begin{aligned}\sum_{n\geq 1}\frac{q^{bn}}{(q^2;q^2)_n}\left(\sum_{k=0}^{n-1}\frac{q^{ck}}{(q^2;q^2)_k}\right)&=\frac{1}{(q^b;q^2)_\infty}\frac{1}{(q^c;q^2)_\infty}-\sum_{n\geq 0}\frac{q^{cn}}{(q^2;q^2)_n}\left(\sum_{k=0}^{n}\frac{q^{bk}}{(q^2;q^2)_k}\right)\\&=\left(\sum_{n\geq 0}\frac{q^{bn}}{(q^2;q^2)_n}\right)\left(\sum_{n\geq 0}\frac{q^{cn}}{(q^2;q^2)_n}\right)-\sum_{n\geq 0}\frac{q^{cn}}{(q^2;q^2)_n}\left(\sum_{k=0}^{n}\frac{q^{bk}}{(q^2;q^2)_k}\right)\\&=\sum_{n\geq 0}\frac{q^{cn}}{(q^2;q^2)_n}\left(\sum_{k\geq n+1}^{\infty}\frac{q^{bk}}{(q^2;q^2)_k}\right).\end{aligned}$$

Next, we assume that b is an even natural number and c is an odd natural number. Then we compute the generating function of the number of partitions of a natural number with minimum parts b and c where the number of even parts is more than odd parts, which is

$$\sum_{n\geq 1}\frac{q^{bn}}{(q^2;q^2)_n}\left(\sum_{k=0}^{n-1}\frac{q^{ck}}{(q^2;q^2)_k}\right)=\frac{1}{(q^b;q^2)_\infty}\frac{1}{(q^c;q^2)_\infty}-\sum_{n\geq 0}\frac{q^{cn}}{(q^2;q^2)_n}\left(\sum_{k=0}^{n}\frac{q^{bk}}{(q^2;q^2)_k}\right)$$
$$=\left(\sum_{n\geq 0}\frac{q^{bn}}{(q^2;q^2)_n}\right)\left(\sum_{n\geq 0}\frac{q^{cn}}{(q^2;q^2)_n}\right)-\sum_{n\geq 0}\frac{q^{cn}}{(q^2;q^2)_n}\left(\sum_{k=0}^{n}\frac{q^{bk}}{(q^2;q^2)_k}\right)$$
$$=\sum_{n\geq 0}\frac{q^{cn}}{(q^2;q^2)_n}\left(\sum_{k\geq n+1}^{\infty}\frac{q^{bk}}{(q^2;q^2)_k}\right),$$

which completes the proof. □

In the last theorem, if we take $c = 2$, we get Theorem 4.1.

5. Concluding Remarks

There are several natural questions that arise from our study, including several avenues for further research. We list below a selection of such questions and comments.

(1) Experiments suggest that the inequality in Theorem 2.1 can be strengthened. We conjecture that, for all $n > 9$ we have
$$3q_o(n) < 2q_e(n).$$
(2) Chern [Che21, Theorem 1.3] has recently proved for $m \geq 2$ and for integers a and b such that $1 \leq a < b \leq m$, we have
$$p_{a,b,m}(n) \geq p_{b,a,m}(n),$$
thus generalizing the results of Kim and Kim [KK21]. Limited data suggests that this inequality is reversed if we consider $q_{j,k,m}(n)$ instead of $p_{j,k,m}(n)$. It would be interesting to get a unified proof of this observation.
(3) Kim, Kim and Lovejoy [KKL20] and Kim and Kim [KK21] also study asymptotics of some of their parity biases. It would be interesting to study such asymptotics for our cases as well.
(4) All the proofs in this paper are analytical. It would be interesting to get combinatorial proofs of some of these results.
(5) Analytical proofs of the inequalities (2) and (3) would also be of interest to see if we can get more generalized results of a similar flavour.
(6) It appears that there is a lot of interesting (parity) biases to be unearthed for different types of partition functions, a systematic study of such (parity) biases would also be of interest.

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