PROOF OF A NONNEGATIVITY CONJECTURE OF MERCA

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ABSTRACT. Merca recently studied the partitions of n in which no part appears d or more times, and connected an infinite family of linear inequalities related to these partitions to the nonnegativity of coefficients of a certain infinite series. In this paper, we prove Merca's conjecture.

1. INTRODUCTION

One of the central objects in number theory and combinatorics is integer partitions. A partition λ of n is a non-decreasing sequence of integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$, where the λ_i's are called the parts of the partition λ. Since the time of Euler, partitions and certain generalizations have been studied quite extensively. One such class of partitions consists of those in which only distinct parts are allowed. Here, we are concerned with a related class. Let $Q(d;n)$ denote the number of partitions of n where no part appears d times or more. Recently, Merca [Mer26] studied this class of partitions, and (among other things) connected an infinite family of linear inequalities involving the partition function $Q(d;n)$ to the question of nonnegativity of coefficients of a certain infinite sum.

In particular, Merca [Mer26] conjectured that the following series has nonnegative coefficients for $k \geq 1$ and $d \geq 2$:

$$(-1)^k \left((q^d;q^d)_\infty - \frac{(q^d;q^d)_\infty}{(q;q)_\infty} \sum_{n=1-k}^{k} (-1)^n q^{n(3n-1)/2} \right), \tag{1.1}$$

where we use the standard q-notation

$$(a;q)_\infty := \prod_{i=0}^{\infty} (1 - aq^i).$$

The coefficients of (1.1) were shown to be nonnegative for all $d \geq 3$ by the work of Yao [Yao24] and Ding & Sun [DS25], leaving the case $d = 2$ open. In this note, we prove that the $d = 2$ case is also true, thereby answering a question of Merca [Mer26, Conjecture 1].

We prove the following result.

Theorem 1.1. *For $n \geq 0$ and $k \geq 1$, the coefficient of q^n in*

$$(-1)^k \left((q^2;q^2)_\infty - \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \sum_{n=1-k}^{k} (-1)^n q^{n(3n-1)/2} \right) \tag{1.2}$$

is nonnegative.

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By the work of Ding & Sun [DS25, p.26], we can rewrite (1.2) as

$$(-1)^k\left((q^2;q^2)_\infty - \frac{(q^2;q^2)_\infty}{(q;q)_\infty}\sum_{n=1-k}^{k}(-1)^n q^{n(3n-1)/2}\right)$$
$$= q^{(3k^2+k)/2}\frac{(q^2;q^2)_\infty}{(1-q^3)(q^6;q)_\infty}\frac{\sum_{j=0}^{\infty}(-1)^j q^{(3j^2+6jk+j)/2}(1-q^{2j+2k+1})}{(1-q)(1-q^2)(1-q^4)(1-q^5)}$$
$$= q^{(3k^2+k)/2}\frac{(q^6;q^2)_\infty}{(q^6;q)_\infty}\frac{\sum_{j=0}^{\infty}(-1)^j q^{(3j^2+6jk+j)/2}(1-q^{2j+2k+1})}{(1-q)(1-q^3)(1-q^5)}.$$

Since $\frac{(q^6;q^2)_\infty}{(q^6;q)_\infty}$ clearly has nonnegative coefficients, we focus on proving the following result, from which Theorem 1.1 follows immediately.

Theorem 1.2. *For $n \geq 0, k \geq 1$, the coefficient of q^n in the series*

$$\sum_{n=0}^{\infty} M(n,k)q^n := \frac{1}{(1-q)(1-q^3)(1-q^5)}\sum_{j=0}^{\infty}(-1)^j q^{(3j^2+6jk+j)/2}(1-q^{2j+2k+1}) \tag{1.3}$$

is nonnegative.

We prove this result in Section 2, and hence Theorem 1.1 readily follows.

2. Proof of Theorem 1.2

Let $o_3(n)$ denote the number of partitions of n into odd parts chosen from the set $\{1,3,5\}$. The generating function for such partitions is

$$\sum_{n\geq 0} o_3(n)q^n = \frac{1}{(1-q)(1-q^3)(1-q^5)}. \tag{2.1}$$

We now find a partial fraction decomposition for the above generating function.

Lemma 2.1. *If $o_3(n)$ is defined as in* (2.1)*, then*

$$o_3(n) = \left\{\frac{(2n+9)^2}{120}\right\}, \tag{2.2}$$

where $\{N\}$ is the nearest integer to N. That is, $\{x\} = \left\lfloor x + \frac{1}{2}\right\rfloor$, for all $x \in \mathbb{R}$, where $\lfloor\cdot\rfloor$ is the greatest integer function. Also,

$$\frac{(2n+9)^2}{120} - \frac{49}{120} \leq o_3(n) \leq \frac{(2n+9)^2}{120} + \frac{13}{40}. \tag{2.3}$$

Proof. We have

$$\begin{aligned}
\sum_{n\geq 0} o_3(n)q^n &= \frac{1}{(1-q)(1-q^3)(1-q^5)}.\\
&= \frac{-1}{15(1-q)^3}+\frac{1}{5(1-q)^2}+\frac{14}{45(1-q)}+\frac{2+q}{9(1+q+q^2)}\\
&\quad+\frac{1+2q+q^2+q^3}{5(1+q+q^2+q^3+q^4)}\\
&= \frac{1}{15}\sum_{n=0}^{\infty}\binom{n+2}{n}q^n+\frac{1}{5}\sum_{n=0}^{\infty}(n+1)q^n+\frac{1}{3}\sum_{n=0}^{\infty}q^{3n}+\frac{2}{5}\sum_{n=0}^{\infty}q^{5n+1}\\
&\quad+\frac{1}{5}\sum_{n=0}^{\infty}q^{5n+3}+\frac{1}{5}\sum_{n=0}^{\infty}q^{5n}\\
&= \sum_{n=0}^{\infty}\left(\frac{\left(n+\frac{9}{2}\right)^2}{30}-\frac{49}{120}q^n+\frac{1}{3}q^{3n}+\frac{2}{5}q^{5n+1}+\frac{1}{5}q^{5n+3}+\frac{1}{5}\sum_{n=0}^{\infty}q^{5n}\right)\\
&= \sum_{n=0}^{\infty}\left(\frac{(2n+9)^2}{120}+\epsilon(n)\right)q^n,
\end{aligned}$$

where $\epsilon(n)\in\{-\frac{49}{120},-\frac{-5}{24},-\frac{1}{120},\frac{3}{40},\frac{1}{8},\frac{13}{40}\}$.

By the uniqueness of the Maclaurin series expansion, we conclude that

$$o_3(n)=\frac{(2n+9)^2}{120}+\epsilon(n).$$

From the above, we readily see that (2.3) is evident. Also, because $o_3(n)$ is an integer and $|\epsilon(n)|<\frac{1}{2}$, we consequently have (2.2). $\square$

From (1.3) and (2.1) we can write

$$\sum_{n=0}^{\infty}M(n,k)q^n=\sum_{m=0}^{\infty}o_3(m)q^m\sum_{j=0}^{\infty}\Big(q^{6j^2+6kj+j}-q^{6j^2+6kj+5j+2k+1}$$
$$-q^{6j^2+6kj+7j+3k+2}+q^{6j^2+6kj+11j+5k+5}\Big). \quad (2.4)$$

Here we have separated the sum in (1.3) into terms with even and odd indices. Comparing coefficients from both sides of the above, we have

$$M(n,k):=\sum_{j=0}^{\infty}\big(o_3(n-6j^2-6kj-j)-o_3(n-6j^2-6kj-5j-2k-1)$$
$$-o_3(n-6j^2-6kj-7j-3k-2)+o_3(n-6j^2-6kj-11j-5k-5)\big). \quad (2.5)$$

Thus, to prove Theorem 1.2, it is enough to show that $M(n,k)\geq 0$.

Let us now define the following partial sums

$$M(n,k,m_0):=\sum_{j=0}^{m_0-1}\big(o_3(n-6j^2-6kj-j)-o_3(n-6j^2-6kj-5j-2k-1)$$
$$-o_3(n-6j^2-6kj-7j-3k-2)+o_3(n-6j^2-6kj-11j-5k-5)\big). \quad (2.6)$$

We have the following lemma related to (2.6).

Lemma 2.2. *Let $m_0 \geq 1, k \geq 1, n \geq 6(m_0-1)^2 + 6k(m_0-1) + 11(m_0-1) + 5k + 5$ be integers, then we have*

$$M(n,k,m_0) \geq \frac{1}{15} m_0 \left(6k^2 + 18km_0 - k - 2n + 12m_0^2 - 33\right). \tag{2.7}$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}
M(n,k,m_0) &= \sum_{j=0}^{m_0-1} \Bigg(\left\{ \frac{(2n-12j^2-12kj-2j+9)^2}{120} \right\} \\
&\quad - \left\{ \frac{(2n-12j^2-12kj-10j-4k+7)^2}{120} \right\} \\
&\quad - \left\{ \frac{(2n-12j^2-12kj-14j-6k+5)^2}{120} \right\} \\
&\quad + \left\{ \frac{(2n-12j^2-12kj-22j-10k-1)^2}{120} \right\} \Bigg) \\
&\geq \sum_{j=0}^{m_0-1} \Bigg(\frac{(2n-12j^2-12kj-2j+9)^2}{120} - \frac{49}{120} \\
&\quad - \frac{(2n-12j^2-12kj-10j-4k+7)^2}{120} - \frac{13}{40} \\
&\quad - \frac{(2n-12j^2-12kj-14j-6k+5)^2}{120} - \frac{13}{40} \\
&\quad + \frac{(2n-12j^2-12kj-22j-10k-1)^2}{120} - \frac{49}{120} \Bigg) \\
&= \sum_{j=0}^{m_0-1} \frac{1}{15} \left(6\left(6jk + 6j(j+1) + k^2\right) + 17k - 2n - 21\right) \\
&= \frac{1}{15} m_0 \left(6k^2 + 18km_0 - k - 2n + 12m_0^2 - 33\right).
\end{aligned}$$

This proves the lemma. □

Remark 2.3. Note that we always assume $M(n,k,0) = 0$ which satisfies (2.6).

We have the following lemma related to (2.5).

Lemma 2.4. *Let $k \geq 1$ and $n \geq 0$ be integers, then $M(n,k)$ as defined in* (2.5) *satisfies $M(n,k) \geq 0$.*

Proof. Note that

$$\mathbb{N} = \bigcup_{j \geq 0} \{n \mid 6j^2 + 6kj + j \leq n \leq 6(j+1)^2 + 6k(j+1) + j\}.$$

So, for any integer $n \geq 0$, there exists an $m_0 \geq 0$ such that

$$6m_0^2 + 6km_0 + m_0 \leq n \leq 6(m_0+1)^2 + 6k(m_0+1) + m_0.$$

Before splitting into cases, we explain the structure. Since $o_3(N) = 0$ for $N < 0$, only finitely many summands in (2.5) contribute for fixed n. The four arguments appearing in the j-th quartet of (2.5): $n-6j^2-6kj-j, n-6j^2-6kj-5j-2k-1, n-6j^2-6kj-7j-3k-2,$

and $n-6j^2-6kj-11j-5k-5$, are nonnegative only once n exceeds the corresponding threshold, and within a fixed quartet these thresholds are listed in increasing order.

Consequently, for $n\in[\,6m_0^2+6km_0+m_0,\ 6(m_0+1)^2+6k(m_0+1)+m_0\,]$, every quartet with index $j\le m_0-1$ contributes all four of its o_3-values (captured by the partial sum $M(n,k,m_0)$ of (2.6)), while in the $j=m_0$ quartet exactly one, two, three, or four of the arguments are nonnegative depending on the value of n. The four cases below correspond precisely to these four sub-ranges, with one further term of the alternating sum becoming active each time n crosses the next threshold.

We will break our proof into four cases.

Case 1: $6m_0^2+6km_0+m_0\le n\le 6m_0^2+6m_0k+5m_0+2k$.

We have,

$$\begin{aligned}
M(n,k)&=M(n,k,m_0)+o_3(n-6m_0^2-6m_0k-m_0)\quad\text{(thanks to (2.5) and (2.6))}\\
&\ge\frac{1}{15}m_0\left(6k^2+18km_0-k-2n+12m_0^2-33\right)\\
&\quad+\frac{(2(n-6m_0^2-6m_0k-m_0)+9)^2}{120}-\frac{49}{120}\quad\text{(thanks to (2.7) and (2.3))}\\
&=\frac{n^2}{30}-\left(\frac{2km_0}{5}+\frac{2m_0^2}{5}+\frac{m_0}{5}-\frac{3}{10}\right)n+\frac{6k^2m_0^2}{5}+\frac{2k^2m_0}{5}+\frac{12km_0^3}{5}+\frac{8km_0^2}{5}\\
&\quad-\frac{28km_0}{15}+\frac{6m_0^4}{5}+\frac{6m_0^3}{5}-\frac{53m_0^2}{30}-\frac{5m_0}{2}+\frac{4}{15} \qquad (2.8)\\
&\ge\left(\frac{2k}{5}-\frac{4}{15}\right)m_0^2+\left(\frac{2k^2}{5}-\frac{k}{15}-\frac{8}{5}\right)m_0-\frac{49}{120}, \qquad (2.9)
\end{aligned}$$

where, in the last step, we have viewed the right-hand side of (2.9) as a quadratic polynomial in n with positive leading coefficient $\frac{1}{30}$, and used the fact that, if $a>0$, then

$$ax^2+bx+c\ge\frac{4ac-b^2}{4a}, \qquad (2.10)$$

the right-hand side being the value of the quadratic in x at the vertex of the parabola.

If $m_0=0$, then $0\le n\le 2k$, and from (2.8), we have

$$M(n,k)\ge\frac{n^2}{30}+\frac{3n}{10}+\frac{4}{15}\ge\frac{4}{15}.$$

If $m_0\ge1$ and $k\ge3$, we have from (2.9)

$$M(n,k)\ge\frac{93}{40}.$$

If $m_0\ge2$ and $k=2$, we have from (2.9)

$$M(n,k)\ge\frac{35}{24}.$$

If $m_0\ge10$ and $k=1$, we have from (2.9)

$$M(n,k)\ge\frac{31}{120}.$$

That is, to check the validity of the result for this case, we need to check the following subcases via Mathematica:

(1) $k=2$ and $m_0=1$, which implies $19\le n\le 27$;
(2) $k=1$ and $1\le m_0\le 9$, which implies $0\le n\le 587$.

This has been checked by the authors via Mathematica.

Thus, we see that $M(n,k) \geq 0$ in this case.

Case 2: $6m_0^2 + 6m_0k + 5m_0 + 2k + 1 \leq n \leq 6m_0^2 + 6m_0k + 7m_0 + 3k + 1$.

Now, by (2.5) and (2.6), we have

$$\begin{aligned}
M(n,k) &= M(n,k,m_0) + o_3(n - 6m_0^2 - 6m_0k - m_0) - o_3(n - 6m_0^2 - 6m_0k - 5m_0 - 2k - 1) \\
&\geq \frac{1}{15}m_0\left(6k^2 + 18km_0 - k - 2n + 12m_0^2 - 33\right) \\
&\quad + \left(\frac{(2(n - 6m_0^2 - 6m_0k - m_0) + 9)^2}{120} - \frac{49}{120}\right) \\
&\quad - \left(\frac{(2(n - 6m_0^2 - 6m_0k - 5m_0 - 2k - 1) + 9)^2}{120} + \frac{13}{40}\right) \\
&= \left(\frac{2k}{15} + \frac{2m_0}{15} + \frac{1}{15}\right)n - \frac{2k^2m_0}{5} - \frac{2k^2}{15} - \frac{6km_0^2}{5} - \frac{17km_0}{15} \\
&\quad + \frac{7k}{15} - \frac{4m_0^3}{5} - \frac{6m_0^2}{5} - \frac{4m_0}{3} - \frac{7}{15} \\
&\geq \frac{2k^2m_0}{5} + \frac{2k^2}{15} + \frac{2km_0^2}{5} + \frac{km_0}{5} + \frac{11k}{15} - \frac{2m_0^2}{15} - \frac{13m_0}{15} - \frac{2}{5} \qquad (2.11)
\end{aligned}$$

For $k \geq 2$, (2.11) gives us immediately $M(n,k) \geq 0$. And for $k = 1$, we have by (2.11)

$$M(n,1) \geq \frac{4m_0^2}{15} - \frac{4m_0}{15} + \frac{7}{15} \geq \frac{2}{5},$$

where, in the last inequality, we applied (2.10) to the right-hand side viewed as a quadratic polynomial in m_0 (with positive leading coefficient $\frac{4}{15}$); the bound $\frac{2}{5}$ is the value of the quadratic at the vertex of the parabola.

Thus, we see that $M(n,k) \geq 0$ in this case.

Case 3: $6m_0^2 + 6m_0k + 7m_0 + 3k + 2 \leq n \leq 6m_0^2 + 6m_0k + 11m_0 + 5k + 4$.

We have,

$$\begin{aligned}
M(n,k) &= M(n,k,m_0) + o_3(n-6m_0^2-6m_0k-m_0)\\
&\quad - o_3(n-6m_0^2-6m_0k-5m_0-2k-1) - o_3(n-6m_0^2-6m_0k-7m_0-3k-2)\\
&\geq \frac{1}{15}m_0\left(6k^2+18km_0-k-2n+12m_0^2-33\right)\\
&\quad + \left(\frac{(2(n-6m_0^2-6m_0k-m_0)+9)^2}{120} - \frac{49}{120}\right)\\
&\quad - \left(\frac{(2(n-6m_0^2-6m_0k-5m_0-2k-1)+9)^2}{120} + \frac{13}{40}\right)\\
&\quad - \left(\frac{(2(n-6m_0^2-6m_0k-7m_0-3k-2)+9)^2}{120} + \frac{13}{40}\right)\\
&= -\frac{n^2}{30} + \left(\frac{2km_0}{5} + \frac{k}{3} + \frac{2m_0^2}{5} + \frac{3m_0}{5} - \frac{1}{10}\right)n - \frac{6k^2m_0^2}{5} - \frac{8k^2m_0}{5} - \frac{13k^2}{30}\\
&\quad - \frac{12km_0^3}{5} - \frac{26km_0^2}{5} - \frac{23km_0}{15} + \frac{29k}{30} - \frac{6m_0^4}{5} - \frac{18m_0^3}{5} - \frac{11m_0^2}{6} - \frac{m_0}{6} - 1\\
&\geq \min\left\{\frac{1}{15}(6k+2)m_0^2 + \frac{1}{15}\left(6k^2+9k-9\right)m_0 + \frac{1}{15}\left(4k^2+14k-20\right),\right.\\
&\quad \left.\frac{1}{15}(6k+2)m_0^2 + \frac{1}{15}\left(6k^2+13k-27\right)m_0 + \frac{1}{15}\left(6k^2+7k-29\right)\right\}, \qquad (2.12)
\end{aligned}$$

where in the last step we have used the fact that if $f(x) = ax^2+bx+c$ is a function in $[x_0,x_1]$ with $a<0$, then we have

$$f(x) \geq \min\{f(x_0), f(x_1)\}.$$

It is clear that, if $k \geq 2$, then (2.12) immediately implies $M(n,k) \geq 0$. We now check the case when $k=1$. In this case, from (2.12), we have

$$\begin{aligned}
M(n,1) &\geq \frac{1}{15}\min\left\{8m_0^2+6m_0-2, 8m_0^2-8m_0-16\right\}\\
&= \frac{1}{15}(8m_0^2-8m_0-16) \quad \text{(since } m_0 \geq 0\text{)},
\end{aligned}$$

from which it is clear that, if $m_0 \geq 2$, then $M(n,1) \geq 0$. Finally, we check that if $0 \leq m_0 \leq 1$, $k=1$ then $5 \leq n \leq 32$, and the authors have verified using Mathematica that $M(n,k) \geq 0$ in these cases.

Thus, we have $M(n,k) \geq 0$ for this case as well.

Case 4: $6m_0^2+6m_0k+11m_0+5k+5 \leq n \leq 6(m_0+1)^2+6(m_0+1)k+m_0$.

We have

$$\begin{aligned}
M(n,k) &= M(n,k,m_0) + o_3(n-6m_0^2-6m_0k-m_0)\\
&\quad - o_3(n-6m_0^2-6m_0k-5m_0-2k-1)\\
&\quad - o_3(n-6m_0^2-6m_0k-7m_0-3k-2)\\
&\quad + o_3(n-6m_0^2-6m_0k-11m_0-5k-5)\\
&\geq \frac{1}{15}m_0\left(6k^2+18km_0-k-2n+12m_0^2-33\right)\\
&\quad + \left(\frac{(2(n-6m_0^2-6m_0k-m_0)+9)^2}{120}-\frac{49}{120}\right)\\
&\quad - \left(\frac{(2(n-6m_0^2-6m_0k-5m_0-2k-1)+9)^2}{120}+\frac{13}{40}\right)\\
&\quad - \left(\frac{(2(n-6m_0^2-6m_0k-7m_0-3k-2)+9)^2}{120}+\frac{13}{40}\right)\\
&\quad + \left(\frac{(2(n-6m_0^2-6m_0k-11m_0-5k-5)+9)^2}{120}-\frac{49}{120}\right)\\
&= -\left(\frac{2m_0}{15}+\frac{2}{15}\right)n+\frac{2k^2m_0}{5}+\frac{2k^2}{5}+\frac{6km_0^2}{5}+\frac{7km_0}{3}+\frac{17k}{15}+\frac{4m_0^3}{5}+\frac{12m_0^2}{5}\\
&\quad +\frac{m_0}{5}-\frac{7}{5}\\
&\geq \left(\frac{2k}{5}-\frac{2}{15}\right)m_0^2+\left(\frac{2k^2}{5}+\frac{11k}{15}-\frac{7}{3}\right)m_0+\frac{2k^2}{5}+\frac{k}{3}-\frac{11}{5}. \qquad (2.13)
\end{aligned}$$

For $k \geq 2$, it follows clearly from (2.13) that $M(n,k) \geq 0$. And, for $k=1$, we have

$$M(n,1) \geq \frac{4m_0^2}{15}-\frac{6m_0}{5}-\frac{22}{15}. \qquad (2.14)$$

For $m_0 \geq 6$, we have $M(n,1) \geq 0$ from (2.14). Finally, when $0 \leq m_0 \leq 5$ and $k=1$, then $10 \leq n \leq 258$, and we can easily check via Mathematica that, in this case $M(n,1) \geq 0$. This has been verified by the authors via Mathematica.

Thus, for this case as well we have $M(n,k) \geq 0$.

□

Proof of Theorem 1.2. The result now follows directly from Lemma 2.4. □

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