ON ODD DEFICIENT PERFECT NUMBERS WITH FOUR DISTINCT PRIME FACTORS

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Abstract. For a positive integer n, if $\sigma(n)$ denotes the sum of the positive divisors of n, then n is called a deficient perfect number if $\sigma(n) = 2n - d$ for some positive divisor d of n. In this paper, we prove that, if 5 does not divide an odd number with four distinct prime divisors, then there is only one such deficient perfect number. This extends previous work by various authors on characterizing odd deficient perfect numbers with a fixed number of distinct prime factors.

Disclaimer

This paper was finished in late 2018 and submitted to a journal which rejected it in May 2019 with a referee report which suggested to make the paper much shorter than it currently is. The referee made two suggestions on how to do this, one of which was incorrect. The author did not pursue this further and left the paper as it was. In August 2019, Sun and He [6] proved a stronger result than the one proved in this paper, using some similar techniques. They however do not cite the earlier work of the author and Dutta [1], which would have made their paper a bit shorter. This paper is available in the author's website and is not intended for publication as the main theorem is super-seeded by the work of Sun and He [6], as already pointed out.

1. Introduction

For a positive integer n, the function $\sigma(n)$ denotes the sum of the distinct positive prime divisors of n. A natural number n is called a perfect number if $\sigma(n) = 2n$. These type of numbers have been studied since antiquity and several generalizations of these numbers have appeared over the years (for instance, see [3] and the references therein for some of them).

Let d be a proper divisor of n. We call n, a near perfect number with redundant divisor d if $\sigma(n) = 2n + d$; and a deficient perfect number with deficient divisor d if $\sigma(n) = 2n - d$. If $d = 1$, then such a deficient perfect number is called an almost perfect number. Several results have been proved about these classes of numbers: for instance, Kishore [2] proved that if n is an odd almost perfect number then the number of distinct prime factors of n is at least 6, Pollack and Shevelev [4], Ren and Chen [5] found all near perfect numbers with two distinct prime factors; Tang, Ren and Li [8] showed that no odd near perfect number exists with three distinct prime factors and determined all deficient perfect numbers with two distinct prime factors. In a similar vein, Tang and Feng [7] showed that no odd deficient perfect number exists with three distinct prime factors; Tang, Ma and Feng [9] showed that there exists only one odd near perfect number with four distinct prime divisors.

The smallest known odd deficient perfect number with four distinct prime factors is $9018009 = 3^2.7^2.11^2.13^2$, and it is the only such number until 2.10^{12}. Recently, Dutta and Saikia [1] proved that any odd deficient perfect number with four distinct prime factors should have 3 as it's smallest prime factor, and either 5 or 7 as its second smallest prime factor. In this paper, we extend the work of Dutta and Saikia [1] and prove the following result.

Theorem 1.1. *If n is an odd deficient perfect number with four distinct prime factors, such that 5 does not divide n, then there is only one such n, which is $9018009 = 3^2.7^2.11^2.13^2$.*

We can rephrase the above result in the following way, keeping in mind the result proved in [1].

Theorem 1.2. *If n is an odd deficient perfect number with four distinct prime factors p_1, p_2, p_3 and p_4 such that $n = p_1^{a_1}.p_2^{a_2}.p_3^{a_3}.p_4^{a_4}$ with $p_1 < p_2 < p_3 < p_4$ and $a_1, a_2, a_3, a_4 \geq 1$, and if $p_2 \geq 7$ and $p_1 \geq 3$, then there is only one such n which is $9018009 = 3^2.7^2.11^2.13^2$.*

This paper is organized as follows: in Section 2 we state all the important notations that will be necessary for this paper; in Section 3 we prove Theorem 1.2 and we end the paper with some remarks in Section 4.

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2. Basic Setup

Before, we prove our results, we note the following result from Tang and Feng [7].

Lemma 2.1 (Lemma 2.1, [7]). *Let $n = \prod_{i=1}^{k} p_i^{a_i}$ be the canonical prime factorization of n. If n is an odd deficient perfect number, then all the a_i's are even for all i.*

Let us fix a few notations. Throughout this paper, unless otherwise mentioned we take $n = p_1^{a_1}.p_2^{a_2}.p_3^{a_3}.p_4^{a_4}$ with $p_1 < p_2 < p_3 < p_4$ distinct odd primes and a_i's to be natural numbers. In light of Lemma 2.1 all the a_i's are even. If a is any integer relatively prime to m such that k is the smallest positive integer for which $a^k \equiv 1 \pmod{m}$ then, we say that k is the order of a modulo m and denote it by $\mathrm{ord}_m(a)$. We also define the following function which we shall use very often in this paper

$$f(a_1, a_2, a_3, a_4) = \left(1 - \frac{1}{p_1^{a_1+1}}\right)\left(1 - \frac{1}{p_2^{a_2+1}}\right)\left(1 - \frac{1}{p_3^{a_3+1}}\right)\left(1 - \frac{1}{p_4^{a_4+1}}\right).$$

Assuming that n is an odd deficient perfect number with $p_1^{b_1}.p_2^{b_2}.p_3^{b_3}.p_4^{b_4}$ as the deficient divisor, we have

$$\sigma(p_1^{a_1}.p_2^{a_2}.p_3^{a_3}.p_4^{a_4}) = 2.p_1^{a_1}.p_2^{a_2}.p_3^{a_3}.p_4^{a_4} - p_1^{b_1}.p_2^{b_2}.p_3^{b_3}.p_4^{b_4}, \tag{2.1}$$

where $b_i \leq a_i$. Also write $D = p_1^{a_1-b_1}.p_2^{a_2-b_2}.p_3^{a_3-b_3}.p_4^{a_4-b_4}$. Then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} = \frac{\sigma(n)}{n} + \frac{1}{D}. \tag{2.2}$$

An inequality which we will use without commentary is the following

$$\frac{\sigma(n)}{n} < \frac{p_1 p_2 p_3 p_4}{(p_1 - 1)(p_2 - 1)(p_3 - 1)(p_4 - 1)}.$$

We also formally state the result from Dutta and Saikia [1].

Theorem 2.2 (Dutta-Saikia, [1]). *If n is an odd deficient perfect number with four distinct prime factors p_1, p_2, p_3 and p_4 such that $n = p_1^{a_1}.p_2^{a_2}.p_3^{a_3}.p_4^{a_4}$ with $p_1 < p_2 < p_3 < p_4$ and $a_1, a_2, a_3, a_4 \geq 1$, then $p_1 = 3$ and $p_2 \leq 7$.*

In light of Theorem 2.2, we need to only consider the cases $p_1 = 3, p_2 = 7$ to prove Theorem 1.2. So, for the remainder of the paper, we assume $p_1 = 3$ and $p_2 = 7$.

It so happens that, sometimes in several cases that we will analyze in the subsequent sections, more than one proof can be given. We will use the convention of using the methods used in the preceding results to prove any subsequent result if those methods can be used successfully. We will not mention other ways to prove a particular case, other than the one we present in this paper. It will also become clear to the reader that most of the proofs are similar, so we shall omit some of them for the sake of brevity. Sometimes, the arguments if spelled out exactly would be too tedious, so we write the details that needs to be verified[1].

3. Proof of Theorem 1.2

Lemma 3.1. *If $n = 3^{a_1}.7^{a_2}.p_3^{a_3}.p_4^{a_4}$ in Theorem 1.2 with $p_3 \geq 37$, then there is no odd deficient perfect number.*

Proof. Noting the elementary inequality $\frac{p}{p-1} > \frac{p+l}{p+l-1}$ for positive integers p and l we see that $D \geq 7$ cannot occur in this case, if $D \geq 7$ cannot occur when $p_3 = 37$. Indeed,

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.37.41}{2.6.36.40} + \frac{1}{7} < 2,$$

which is not possible. So, for all these cases we have $D = 3$.

From equation (2.1) we have

$$\sigma(3^{a_1}.7^{a_2}.p_3^{a_3}.p_4^{a_4}) = 5.3^{a_1-1}.7^{a_2}.p_3^{a_3}.p_4^{a_4}. \tag{3.1}$$

Let us use the function f defined earlier; which in this case is

$$f(a_1, a_2, a_3, a_4) = \left(1 - \frac{1}{3^{a_1+1}}\right)\left(1 - \frac{1}{7^{a_2+1}}\right)\left(1 - \frac{1}{p_3^{a_3+1}}\right)\left(1 - \frac{1}{p_4^{a_4+1}}\right).$$

We also introduce the following function

$$g(a_1, a_2, a_3, a_4) = \frac{20.(p_3 - 1).(p_4 - 1)}{21.p_3.p_4}.$$

[1]All the numerical calculations have been done on Mathematica 11.1 with 16 digits precision, but we have truncated the results to six decimal digits as it is sufficient for our purposes.

From equation (3.1), it is clear that in this case $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$.

In this case, it is clear that

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^3}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{37^3}\right)\left(1 - \frac{1}{41^3}\right) = 0.960123\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) \leq \frac{20}{21} = 0.952381\cdots.$$

Clearly, these two inequalities are not compatible with each. So, we have $p_3 \leq 31$.

□

Remark 3.2. *In this paper, we want to check inequalities of the form $f(a_1, a_2, a_3, a_4) \geq Q$ and $g(a_1, a_2, a_3, a_4) \leq R$ and then compare the values of Q and R, so we need to only verify for the smallest possible values of p_i's for $f(a_1, a_2, a_3, a_4)$ and the largest possible values of p_i's for $g(a_1, a_2, a_3, a_4)$. This was done in the case of $f(a_1, a_2, a_3, a_4)$ in the proof Lemma 3.1 above and will be done in the case of $g(a_1, a_2, a_3, a_4)$ in the proof of Lemma 3.3. We will not explicitly mention this in the verifications to follow from now on.*

Lemma 3.3. *If $n = 3^{a_1}.7^{a_2}.p_3^{a_3}.p_4^{a_4}$ in Theorem 1.2 with $19 \leq p_3 \leq 31$, then there is no odd deficient perfect number.*

Proof. If $D \geq 15$ here, we get a contradiction like the previous proof. So, the possible cases for D are $3, 7$ and 9.

Case 1. $D = 3$.

The proof is similar to the proof of Lemma 3.1, so we omit the details here.

Case 2. $D = 7$.

In this case, we get the following

$$\sigma(3^{a_1}.7^{a_2}.p_3^{a_3}.p_4^{a_4}) = 13.3^{a_1}.7^{a_2-1}.p_3^{a_3}.p_4^{a_4}. \tag{3.2}$$

We use the function f defined earlier; which in this case is

$$f(a_1, a_2, a_3, a_4) = \left(1 - \frac{1}{3^{a_1+1}}\right)\left(1 - \frac{1}{7^{a_2+1}}\right)\left(1 - \frac{1}{p_3^{a_3+1}}\right)\left(1 - \frac{1}{p_4^{a_4+1}}\right).$$

We also introduce the following function

$$g(a_1, a_2, a_3, a_4) = \frac{52.(p_3 - 1).(p_4 - 1)}{49.p_3.p_4}.$$

From equation (3.2), it is clear that, in this case $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$.

If $a_2 = 2$, then we have from equation (3.2) we have

$$19.\sigma(3^{a_1}.p_3^{a_3}.p_4^{a_4}) = 13.3^{a_1-1}.7.p_3^{a_3}.p_4^{a_4}. \tag{3.3}$$

If $p_3 \geq 23$ then this is not possible. If $p_3 = 19$, then we have from equation (3.3)

$$\sigma(3^{a_1}.19^{a_3}.p_4^{a_4}) = 13.3^{a_1-1}.7.19^{a_3-1}.p_4^{a_4}. \tag{3.4}$$

In this case, we have $23 \leq p_4 \leq 181$, otherwise we have

$$2 = \frac{\sigma(n)}{n} + \frac{1}{D} < \frac{3.7.19.191}{2.6.18.190} + \frac{1}{7} < 2,$$

which is impossible.

We now note that $\text{ord}_{19}(3) = 18$ and $\text{ord}_{19}(p_4)$ is even if p_4 is not one of the elements of $\{23, 43, 47, 61, 73, 83, 101, 131, 1$
So, we can exclude all the other cases as then 19 will not divide the left hand side of equation (3.4). Again, we note that $\text{ord}_7(3) = \text{ord}_7(19) = 6$ and $\text{ord}_7(p_4)$ is even for all elements of the above set except for $23, 43, 137, 149$ and 163. So, we can also exclude the rest of the cases. We also have $\text{ord}_{p_4}(3)$ and $\text{ord}_{p_4}(19)$ is even if p_4 equals $43, 137$ and 163, so we can exclude these cases as well.

If $p_4 = 23$, then using an order argument, we shall reach the conclusion that $3^{a_1-1} = \sigma(19^{a_3})$, $23^{a_4}.13 = \sigma(3^{a_1})$ and $7.19^{a_3-1} = \sigma(23^{a_4})$, solving this for a_3 then gives us $19^{a_3-1} < 1$ which is not possible.

If $p_4 = 149$, then using an order argument we reach the conclusion that $13 = \sigma(3^{a_1})$ which implies $a_1 = 2$, and then we have $3.149^{a_4} = \sigma(19^{a_3})$ and $7.19^{a_3-1} = \sigma(149^{a_4})$. Solving this for a_3 would again imply $19^{a_3-1} < 1$ which is not possible.

Hence, $a_2 \geq 4$.

If $a_1 = 2$, from equation (3.2) we get

$$\sigma(7^{a_2}.p_3^{a_3}.p_4^{a_4}) = 3^2.7^{a_2-1}.p_3^{a_3}.p_4^{a_4}. \tag{3.5}$$

If $p_4 \geq 29$, we have

$$2 = \frac{\sigma(n)}{n} + \frac{1}{D} < \frac{\sigma(3^2).7.19.29}{3^2.6.18.28} + \frac{1}{7} < 2,$$

which is impossible. So, $p_3 = 19, p_4 = 23$ and hence equation (3.5) becomes

$$\sigma(7^{a_2}.19^{a_3}.23^{a_4}) = 3^2 7^{a_2-1}.19^{a_3}.23^{a_4}. \tag{3.6}$$

Using a similar order argument, we can also conclude from here that $\sigma(19^{a_3})$ is either 1, 3 or 3^2, all of which are not possible. Hence, $a_1 \geq 4$.

If $a_1 = 4$, then 11 divides both sides of equation (3.2) which is not possible. Hence, $a_1 \geq 6$.

Let $a_1 \geq 6$ and $a_2 \geq 4$, then it is clear that

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{19^3}\right)\left(1 - \frac{1}{23^3}\right) = 0.999255\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) \leq \frac{52.30.36}{49.31.37} = 0.999235\cdots.$$

Both these inequalities cannot hold at the same time, so this case cannot occur. (We notice that if $p_3 = 31$, the largest possible in this Lemma, then $p_4 = 37$, and hence the choices for p_3 and p_4 in $g(a_1, a_2, a_3, a_4)$ is sufficient for our verification.)

Case 3. $D = 9$. In this case, we get the following

$$\sigma(3^{a_1}.7^{a_2}.p_3^{a_3}.p_4^{a_4}) = 17.3^{a_1-2}.7^{a_2}.p_3^{a_3}.p_4^{a_4}. \tag{3.7}$$

Let us introduce the function

$$g(a_1, a_2, a_3, a_4) = \frac{17.2^2.(p_3 - 1)(p_4 - 1)}{3^2.7.p_3.p_4}.$$

From equation (3.7), it is clear that

$$f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4).$$

Subcase 3.1. $p_3 = 19$.

If $p_4 \geq 47$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{1}{D} < \frac{3.7.19.47}{2.6.18.46} + \frac{1}{9} < 2,$$

which is impossible. So, $p_4 \leq 43$. Now, equation (3.7) in this case is

$$\sigma(3^{a_1}.7^{a_2}.19^{a_3}.p_4^{a_4}) = 17.3^{a_1-2}.7^{a_2}.19^{a_3}.p_4^{a_4}. \tag{3.8}$$

We now note that $\text{ord}_{17}(3) = \text{ord}_{17}(7) = 16, \text{ord}_{17}(19) = 8$ and $\text{ord}_{17}(p_4)$ are all even for the possible values of p_4, this means that 17 does not divide the left hand side of equation (3.8), which is a contradiction. Hence, this subcase cannot occur.

Subcase 3.2. $p_3 = 23$.

If $p_4 \geq 37$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{1}{D} < \frac{3.7.23.37}{2.6.22.36} + \frac{1}{9} < 2,$$

which is impossible. So, $p_4 \leq 31$. Now, equation (3.7) in this case is

$$\sigma(3^{a_1}.7^{a_2}.23^{a_3}.p_4^{a_4}) = 17.3^{a_1-2}.7^{a_2}.23^{a_3}.p_4^{a_4}. \tag{3.9}$$

We now note that $\text{ord}_{17}(3) = \text{ord}_{17}(7) = \text{ord}_{17}(23) = 16$ and $\text{ord}_{17}(p_4)$ are all even for the possible values of p_4, this means that 17 does not divide the left hand side of equation (3.9), which is a contradiction. Hence, this subcase cannot occur.

Subcase 3.3. $29 \leq p_3 \leq 31$.

We have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.29.31}{2.6.28.30} + \frac{1}{9} < 2,$$

which is not possible. Hence, this subcase is also not possible.

Combining all the above cases and subcases, we can conclude the validity of the lemma.

□

Lemma 3.4. *If $n = 3^{a_1}.7^{a_2}.17^{a_3}.p_4^{a_4}$ in Theorem 1.2, then there is no odd deficient perfect number.*

Proof. If $D \geq 27$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.17.19}{2.6.16.18} + \frac{1}{27} < 2,$$

which is not possible. So, $D \leq 26$, hence $D \in \{3, 7, 9, 17, 19, 21, 23\}$.

Case 1. $D = 3$.

The proof is similar to the proof of Lemma 3.1, so we will skip the details here.

Case 2. $D = 7$.

In this case, we have

$$\sigma(3^{a_1}.7^{a_2}.17^{a_3}.p_4^{a_4}) = 13.3^{a_1}.7^{a_2-1}.17^{a_3}.p_4^{a_4}. \tag{3.10}$$

Let us use the function f defined earlier; which in this case is

$$f(a_1, a_2, a_3, a_4) = \left(1 - \frac{1}{3^{a_1+1}}\right)\left(1 - \frac{1}{7^{a_2+1}}\right)\left(1 - \frac{1}{17^{a_3+1}}\right)\left(1 - \frac{1}{p_4^{a_4+1}}\right).$$

We also introduce the following function

$$g(a_1, a_2, a_3, a_4) = \frac{13.2^6.(p_4 - 1)}{7^2.17.p_4}.$$

From equation (3.10), it is clear that in this case $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$.

If $a_2 = 2$ from equation (3.10) we get

$$19.\sigma(3^{a_1}.17^{a_3}.p_4^{a_4}) = 13.3^{a_1-1}.7.17^{a_3}.p_4^{a_4}.$$

Clearly, 19 does not divide the right hand side of this equation, unless $p_4 = 19$. If $p_4 = 19$ then we have

$$\sigma(3^{a_1}.17^{a_3}.19^{a_4}) = 13.3^{a_1-1}.7.17^{a_3}.19^{a_4-1}. \tag{3.11}$$

We note that $\text{ord}_7(3) = \text{ord}_7(17) = \text{ord}_7(19) = 6$, so 7 does not divide the left hand side of equation (3.11), and hence $a_2 \geq 4$.

If $a_1 = 2$, from equation (3.10) we get

$$\sigma(7^{a_2}.17^{a_3}.p_4^{a_4}) = 3.7^{a_2}.17^{a_3}.p_4^{a_4}.$$

Here, if $p_4 \geq 29$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2).7.17.29}{3^2.6.16.28} + \frac{1}{7} < 2,$$

which is not possible. So, the possible cases for p_4 are 19 and 23. If $p_4 = 19$, then we note that $\text{ord}_7(17) = \text{ord}_7(19) = 6$, so 7 does not divide the left hand side of the above equation. If $p_4 = 23$, then we note that $\text{ord}_{17}(7) = \text{ord}_{17}(23) = 16$ and hence 17 does not divide the left hand side of the above equation. Thus, $a_1 \geq 4$.

If $a_1 = 4$, from equation (3.10) we get

$$11^2\sigma(7^{a_2}.17^{a_3}.p_4^{a_4}) = 13.3^3.7^{a_2}.17^{a_3}.p_4^{a_4}.$$

Clearly, 11 does not divide the right hand side of this equation. So, $a_1 \geq 6$.

If $a_1 \geq 6$, $a_2 \geq 4$, we have clearly,

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{17^3}\right)\left(1 - \frac{1}{19^3}\right) = 0.999134\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) \leq \frac{13.2^6}{7^2.17} = 0.9988\cdots.$$

Both of these inequalities cannot be true at the same time. So, this case cannot occur.

Case 3. $D = 9$.

If $p_4 \geq 67$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.17.67}{2.6.16.66} + \frac{1}{9} < 2,$$

which is not possible. So, $p_4 \in \{19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61\}$.

We have here

$$\sigma(3^{a_1}7^{a_2}17^{a_3}p_4^{a_4}) = 3^{a_1-2}.7^{a_2}.17^{a_3+1}.p_4^{a_4}.$$

This means, that 17 divides both sides of the above equation, however $\text{ord}_{17}(3) = \text{ord}_{17}(7) = 16$ and $\text{ord}_{17}(p_4)$ are all even for all choices of p_4 in the admissible cases; so 17 does not divide the left hand side of this equation, and hence this case is not possible.

Case 4. $D = 17$.

If $p_4 \geq 29$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.17.29}{2.6.16.28} + \frac{1}{17} < 2,$$

which is not possible. So, $p_4 = 19$ or $p_4 = 23$.

If $p_4 = 19$, then, we have

$$\sigma(3^{a_1}7^{a_2}17^{a_3}19^{a_4}) = 33.3^{a_1}7^{a_2}17^{a_3-1}19^{a_4}.$$

We note that $\text{ord}_7(3) = \text{ord}_7(17) = \text{ord}_7(19) = 6$ are all even, so 7 cannot divide the left hand side of the above equation, and hence this is not possible.

If $p_4 = 23$, then, we have

$$\sigma(3^{a_1}7^{a_2}17^{a_3}23^{a_4}) = 33.3^{a_1}7^{a_2}17^{a_3-1}23^{a_4}.$$

We note that $\text{ord}_{17}(3) = \text{ord}_{17}(23) = \text{ord}_{17}(7) = 16$ are all even, so 17 cannot divide the left hand side of the above equation, and hence this is not possible.

Case 5. $D = 19$. In this case, we must have $p_4 = 19$. Then, we have

$$\sigma(3^{a_1}7^{a_2}17^{a_3}19^{a_4}) = 37.3^{a_1}7^{a_2}17^{a_3}19^{a_4-1}.$$

We note that $\text{ord}_7(3) = \text{ord}_7(17) = \text{ord}_7(19) = 6$ are all even, so 7 cannot divide the left hand side of the above equation, and hence this case is not possible.

Case 6. $D = 21$.

If $p_4 \geq 23$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.17.23}{2.6.16.22} + \frac{1}{21} < 2,$$

which is not possible. So, $p_4 = 19$. Then, we have

$$\sigma(3^{a_1}7^{a_2}17^{a_3}19^{a_4}) = 41.3^{a_1-1}7^{a_2-1}17^{a_3}19^{a_4}.$$

We note that $\text{ord}_7(3) = \text{ord}_7(17) = \text{ord}_7(19) = 6$ are all even, so 7 cannot divide the left hand side of the above equation, and hence this case is not possible.

Case 7. $D = 23$.

In this case, we must have $p_4 = 23$. Then, we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.17.23}{2.6.16.22} + \frac{1}{23} < 2,$$

which is not possible.

Combining all the above cases, proves the lemma. □

Lemma 3.5. *If $n = 3^{a_1}.7^{a_2}.13^{a_3}.p_4^{a_4}$ in Theorem 1.2 with $p_4 \geq 31$, then there is no odd deficient perfect number.*

Proof. Let us use the function f defined earlier; which in this case is

$$f(a_1, a_2, a_3, a_4) = \left(1 - \frac{1}{3^{a_1+1}}\right)\left(1 - \frac{1}{7^{a_2+1}}\right)\left(1 - \frac{1}{13^{a_3+1}}\right)\left(1 - \frac{1}{p_4^{a_4+1}}\right).$$

In this case, if $D \geq 25$ then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.13.31}{2.6.12.30} + \frac{1}{25} < 2,$$

which is not possible. Hence, the choices of D are $3, 7, 9, 13, 21$.

Case 1. $D = 3$.

We have here the equation

$$\sigma(3^{a_1}.7^{a_2}.13^{a_3}.p_4^{a_4}) = 5.3^{a_1-1}.7^{a_2}.13^{a_3}.p_4^{a_4}. \tag{3.12}$$

Let us define the function

$$g(a_1, a_2, a_3, a_4) = \frac{5.2^2.12.(p_4 - 1)}{3.7.13.p_4}.$$

From equation (3.12), it is clear that $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$.

We have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^3}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{31^3}\right) = 0.959\cdots,$$

and

$$g(a_1,a_2,a_3,a_4) \le \frac{5.2^2.12}{3.7.13} = 0.879121\cdots.$$

Clearly, both these inequalities are not compatible with each other. Hence, this case is not possible.

Case 2. $D = 7$.

We have here the equation

$$\sigma(3^{a_1}.7^{a_2}.13^{a_3}.p_4^{a_4}) = 3^{a_1}.7^{a_2-1}.13^{a_3+1}.p_4^{a_4}. \tag{3.13}$$

Let us define the function

$$g(a_1,a_2,a_3,a_4) = \frac{2^4.3.(p_4-1)}{7^2.p_4}.$$

From equation (3.13), it is clear that $f(a_1,a_2,a_3,a_4) = g(a_1,a_2,a_3,a_4)$.

If $a_1 = 2$, then equation (3.13) becomes

$$\sigma(7^{a_2}.13^{a_3}.p_4^{a_4}) = 3^2.7^{a_2-1}.13^{a_3}.p_4^{a_4}. \tag{3.14}$$

We note that $\text{ord}_7(13) = 2$ and $\text{ord}_{13}(7) = 12$; this gives us the relations $\sigma(7^{a_2}.13^{a_3}) = 9.p_4^{a_4}$ and $\sigma(p_4^{a_4}) = 7^{a_2-1}.13^{a_3}$. After simplification we get,

$$7^{a_2-1}.13^{a_3}(-11p_4+648) = p_4(7^{a_2+1}+13^{a_3+1}-1)+648. \tag{3.15}$$

If $p_4 \ge 59$, then the right hand side of equation (3.15) is always positive, while the left hand side is always negative. Hence, this is a contradiction, so in this case $31 \le p_4 \le 53$. Now, we note that $\text{ord}_{13}(31) = \text{ord}_{13}(47) = 4, \text{ord}_{13}(37) = \text{ord}_{13}(41) = 12$ and $\text{ord}_{13}(43) = 6$, so p_4 cannot be equal to these values, otherwise 13 will not divide the left hand side of equation (3.14). If $p_4 = 53$, then equation (3.15) becomes

$$5.7^{a_2-1}.13^{a_3+1} - 7^{a_2+1}.53 - 5.7.17 = 13^{a_3+1}.53.$$

Clearly, 7 divides the left hand side of this equation, but not the right hand side. Hence, $p_4 \ne 53$. So, in conclusion, we have $a_1 \ge 4$.

If $a_1 \ge 4$, we have

$$f(a_1,a_2,a_3,a_4) \ge \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{13^3}\right)\left(1-\frac{1}{31^3}\right) = 0.992496\cdots,$$

and

$$g(a_1,a_2,a_3,a_4) \le \frac{2^4.3}{7^2} = 0.979592\cdots.$$

Clearly, both these inequalities are not compatible with each other. So, this case cannot occur.

Case 3. $D = 9$.

We have here the equation

$$\sigma(3^{a_1}.7^{a_2}.13^{a_3}.p_4^{a_4}) = 17.3^{a_1-2}.7^{a_2}.13^{a_3}.p_4^{a_4}. \tag{3.16}$$

Let us define the function

$$g(a_1,a_2,a_3,a_4) = \frac{17.2^4.(p_4-1)}{3.7.13.p_4}.$$

From equation (3.16), it is clear that $f(a_1,a_2,a_3,a_4) = g(a_1,a_2,a_3,a_4)$.

If $a_2 = 2$, then we get that 19 divides both sides of equation (3.16), which is not possible. So, $a_2 \ge 4$. If $a_1 = 4$, then 11 divides both sides of equation (3.16), which is not possible. If $a_1 = 2$, then we have from equation (3.16)

$$\sigma(7^{a_2}.13^{a_3}.p_4^{a_4}) = 17.7^{a_2}.13^{a_3-1}.p_4^{a_4}. \tag{3.17}$$

We have $\text{ord}_{17}(7) = 16$, $\text{ord}_{17}(13) = 4$, $\text{ord}_7(13) = 2$ and $\text{ord}_{13}(7) = 12$; so we must have $p_4^{a_4} = \sigma(7^{a_2}.13^{a_3})$ and $17.7^{a_2}.13^{a_3-1} = \sigma(p_4^{a_4})$. From these two relations we get

$$7^{a_2}.13^{a_3-1}(-41p_4+1224) = p_4(7^{a_2+1}+13^{a_3+1}-1)+84.$$

Since, $p_4 \ge 31$ and $a_2 \ge 4, a_3 \ge 2$, so the right hand side of the above equation is always positive, while the left hand side is always negative. Hence, this is a contradiction, so $a_1 \ge 6$.

If $a_1 \geq 6$ and $a_2 \geq 4$, we have

$$f(a_1,a_2,a_3,a_4) \geq \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{7^5}\right)\left(1-\frac{1}{13^3}\right)\left(1-\frac{1}{31^3}\right) = 0.998995\cdots,$$

and

$$g(a_1,a_2,a_3,a_4) \leq \frac{17.2^4}{3.7.13} = 0.996337\cdots.$$

Clearly, both these inequalities are not compatible with each other. Hence, this case cannot occur.

Case 4. $D = 13$.

If $p_4 \geq 71$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.13.71}{2.6.12.70} + \frac{1}{13} < 2,$$

which is not possible. So, $31 \leq p_4 \leq 67$.

In this case, we have the equation

$$\sigma(3^{a_1}.7^{a_2}.13^{a_3}.p_4^{a_4}) = 5^2.3^{a_1}.7^{a_2}.13^{a_3-1}.p_4^{a_4}.$$

We note that $\text{ord}_5(3) = \text{ord}_5(7) = \text{ord}_5(13) = 4$, so 5 divides only $\sigma(p_4^{a_4})$. But if $p_4 \in \{37,43,47,53,59,67\}$, then $\text{ord}_5(p_4)$ is even. So, the only possibilities are $p_4 \in \{31,41,61\}$.

We note that $\text{ord}_7(3) = 6$, $\text{ord}_7(13) = 2$, $\text{ord}_7(31) = 6$ and $\text{ord}_7(61) = 6$, so $p_4 \neq 31, 61$.

We note that, $\text{ord}_{41}(3) = 8, \text{ord}_{41}(7) = \text{ord}_{41}(13) = 40$, so $p_4 \neq 41$.

Thus, this case cannot occur.

Case 5. $D = 21$.

If $p_4 \geq 37$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.13.37}{2.6.12.36} + \frac{1}{21} < 2,$$

which is not possible. So, $p_4 = 31$.

In this case, we have the equation

$$\sigma(3^{a_1}.7^{a_2}.13^{a_3}.31^{a_4}) = 41.3^{a_1-1}.7^{a_2-1}.13^{a_3}.31^{a_4}.$$

We note that $\text{ord}_7(3) = 6$, $\text{ord}_7(13) = 2$ and $\text{ord}_7(31) = 6$, so 7 divides the right hand side of the above equation, but not the left hand side. Hence, this case is not possible.

□

Lemma 3.6. *If $n = 3^{a_1}.7^{a_2}.13^{a_3}.p_4^{a_4}$ in Theorem 1.2 with $17 \leq p_4 \leq 29$, then there is no odd deficient perfect number.*

Proof. In this case, we have for some $b_i \geq 0$, $(i = 1,2,3,4)$

$$\sigma(3^{a_1}.7^{a_2}.13^{a_3}.p_4^{a_4}) = 2.3^{a_1}.7^{a_2}.13^{a_3}.p_4^{a_4} - 3^{b_1}.7^{b_2}.13^{b_3}.p_4^{b_4} \tag{3.18}$$

Let us define the function

$$g(a_1,a_2,a_3,a_4) = \frac{2^5.3.(p_4-1)}{7.13.p_4} - \frac{2^4.3.(p_4-1)}{D_0},$$

where $D_0 = 3^{a_1-b_1}.7^{a_2-b_2+1}.13^{a_3-b_3+1}.p_4^{a_4-b_4+1}$. So, $D_0 > 7.13.p_4$. From equation (3.18), it is clear that $f(a_1,a_2,a_3,a_4) = g(a_1,a_2,a_3,a_4) < 1$.

Case 1. $p_4 = 29$.

If $D \geq 28$ then we have,

$$2 = \frac{\sigma(n)}{n} + \frac{1}{D} < \frac{3.7.13.29}{2.6.12.28} + \frac{1}{28} < 2,$$

which is not possible. So, the possible values of D are $3, 7, 9, 13, 21$ and 27.

If, $D = 3$ and 7, then the argument is exactly similar to the proof of Lemma 3.1, so we omit the details here.

If $D = 9$, then we have from equation (3.18),

$$\sigma(3^{a_1}.7^{a_2}.13^{a_3}.29^{a_4}) = 17.3^{a_1-2}.7^{a_2}.13^{a_3}.29^{a_4}. \tag{3.19}$$

We now note that $\text{ord}_{17}(3) = \text{ord}_{17}(7) = \text{ord}_{17}(29) = 16$ and $\text{ord}_{17}(13) = 4$, so 17 does not divide the left hand side of equation (3.19), so this is not possible.

In a similar way, we can prove that $D = 13$ and 21 are also not possible, where the role of 17 is replaced by 5 and 41 respectively.

If $D = 27$, then from equation (3.18), we have

$$\sigma(3^{a_1}.7^{a_2}.13^{a_3}.29^{a_4}) = 53.3^{a_1-3}.7^{a_2}.13^{a_3}.29^{a_4}.$$

From the above equation, it is clear that $a_1 \neq 4, 6$, $a_2 \neq 2$ and $a_3 \neq 2$, otherwise there will be some prime factors on the left hand side of the above equation, which will not be in the right hand side. So, we have two possibilities $a_1 = 2$ or $a_1 \geq 8, a_2, a_3 \geq 4$. For the second possibility, we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^9}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{13^5}\right)\left(1 - \frac{1}{29^3}\right) = 0.999846\cdots .$$

We also define the function

$$g(a_1, a_2, a_3, a_4) = \frac{53.2^6}{3^2.13.29} = 0.999705\cdots .$$

From equation (3.18), it is clear that

$$f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4),$$

but in this case the values are incompatible. So, we see that only $a_1 = 2$ is possible. But then, $D \neq 27$, so we can conclude that none of the possibilities hold.

Case 2. $p_4 = 23$.

If $D \geq 56$ then we have,

$$2 = \frac{\sigma(n)}{n} + \frac{1}{D} < \frac{3.7.13.23}{2.6.12.22} + \frac{1}{56} < 2,$$

which is not possible. So, the possible values of D are $3, 7, 9, 13, 21, 23, 27, 39$ and 49.

If, $D = 3, 7$ and 9, then the argument is exactly similar to the proof of Lemma 3.1, so we omit the details here.

If $D = 13, 21, 23$ and 49, then the argument is exactly similar to the argument for $D = 13$ in the previous case, so we omit the details here.

If $D = 27$, then the argument is exactly similar to the argument for $D = 27$ in the previous case, so we omit the details here.

So, we have to check only the case $D = 39$. Then, we have from equation (3.18)

$$\sigma(3^{a_1}.7^{a_2}.13^{a_3}.23^{a_4}) = 11.3^{a_1-1}.7^{a_2+1}.13^{a_3-1}.23^{a_4}.$$

Using a similar argument like in the previous case for $D = 27$, we can conclude that either $a_1 = 2$ or $a_1 \geq 6, a_2, a_3 \geq 4$, and then reach a contradiction for the second possibility. So, $a_1 = 2$ in this case, and then the previous equation becomes

$$\sigma(7^{a_2}.13^{a_3}.23^{a_4}) = 11.3.7^{a_2+1}.13^{a_3-2}.23^{a_4}.$$

We now note that $\mathrm{ord}_{13}(7) = 12$ and $\mathrm{ord}_{13}(23) = 6$, so 13 does not divide the left hand side of the above equation, and hence $D \neq 39$.

Case 3. $p_4 = 19$.

We note here $\mathrm{ord}_7(3) = 2$ and $\mathrm{ord}_7(13) = \mathrm{ord}_7(19) = 6$, so 7 does not divide the left hand side of equation (3.18), and hence $b_2 = 0$, which means $D_0 \geq 7^3.13.19 = 84721$ in this case. This implies $D \geq 49$.

If $a_1 = 2$ and $D \geq 14$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2).7.13.19}{3^2.6.12.18} + \frac{1}{14} < 2,$$

which is not possible. So, $D \leq 13$, which is not possible as well from the preceding discussion. Hence, $a_1 \geq 4$ and we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{19^3}\right) = 0.992385\cdots .$$

If $D_0 \leq 3^2.7^2.13.19$, then we have from equation (3.18)

$$g(a_1, a_2, a_3, a_4) \leq \frac{2^5.3.18}{7.13.19} - \frac{2^4.3.18}{3^2.7^2.13.19} = 0.991490\cdots ,$$

which is not possible. So, $D \geq 3^2.7 = 63$.

If $a_1 = 4$ and $D \geq 142$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^4).7.13.19}{3^4.6.12.18} + \frac{1}{142} < 2,$$

which is not possible. So, we must have $63 \leq D \leq 141$, and this means the possibilities are $D = 63, 81, 91, 117$ or 133.

If $D = 63$, then we have from equation (3.18)

$$11^2.\sigma(7^{a_2}.13^{a_3}.19^{a_4}) = 5^3.3^3.7^{a_2}.13^{a_3-1}.19^{a_4}.$$

We now note that 11 does not divide the right hand side of the above equation, which is not possible. Hence, $D \neq 63$.

In fact, in exactly a similar way, we can prove that $D \neq 81, 91, 117$ and 133. This means, $a_1 \geq 6$ and we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{19^3}\right) = 0.996030\cdots.$$

If $D - 0 \leq 2.3^6.13^2$, then we have

$$g(a_1, a_2, a_3, a_4) = \frac{2^5.3.18}{7.13.19} - \frac{2^4.3.18}{2.3^6.13^2} = 0.995915\cdots,$$

which is not possible. So, we have $D \geq 143$.

If $a_2 = 2$ and $D \geq 214$, then we reach a contradiction like before. So, we should have in this case $143 \leq D \leq 213$. Thus, the possible values of D are 169 and 189. It is not difficult to show that these are not possible, by following the method similar to the case for $D = 63$ above. So, we can conclude that $a_2 \geq 4$.

If $(a_3, a_4) = (2, 2)$, then from equation (3.18), we get after simplification

$$-69723(3^{a_1+1} + 7^{a_2+1} - 1) = 33.3^{a_1}.7^{a_2} - 12.3^{b_1}.13^{b_3}.19^{b_4}.$$

The left hand side of the above equation is clearly negative, but the right hand side is positive for $b_3, b_4 \leq 1$. Let $b_3, b_4 = 2$, then we notice that 13 (resp. 19) divides the left hand side of equation (3.18) only when 13 (resp. 19) divides $\sigma(3^{a_1})$ (resp. $\sigma(7^{a_2})$). The lowest such admissible values are $a_1 = a_2 = 8$, in which case, the right hand side of the above satisfies

$$33.3^{a_1}.7^{a_2} - 12.3^{b_1}.13^{b_3}.19^{b_4} > 3^{a_1}(33.7^8 - 12.13^2.19^2) > 0,$$

which gives us a contradiction. Thus, $(a_3, a_4) \neq (2, 2)$.

Again, note that if $a_3, a_4 \geq 4$, then we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{13^5}\right)\left(1 - \frac{1}{19^5}\right) = 0.999480\cdots.$$

But,

$$g(a_1, a_2, a_3, a_4) = \frac{2^5.3.18}{7.13.19} = 0.999422\cdots,$$

which is not possible. So, one of a_3 or a_4 has to be 2.

Let $a_3 = 2$ and $a_4 \geq 4$. If $a_1 \geq 10$ then

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^{11}}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{19^5}\right) = 0.999479\cdots,$$

which is not possible. So, either $a_1 = 6$ or $a_1 = 8$ in this case. If $a_1 = 6$ we must have $b_3 = 0$, otherwise 13 will not divide the right hand of equuation (3.18), then from equation (3.18) we have

$$-20019(7^{a_2+1} + 19^{a_4+1} - 1) = 23948889.7^{a_2}.19^{a_4} - 108.3^{b_1}.19^{b_4}.$$

Clearly, the left hand side of the above equation is negative, but the right hand side is positive, so this cannot happen. If $a_1 = 8$ and $a_2 \geq 6$, then we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^9}\right)\left(1 - \frac{1}{7^7}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{19^5}\right) = 0.999492\cdots,$$

which is not possible. So, we have $a_2 = 4$, and then this implies $b_4 = 0$. Further, since $\mathrm{ord}_{13}(19) = 12$, so we can also conclude that $b_3 = 1$ in this case. Now, from equation (3.18) we have

$$-388025331 = 88468533435.19^{a_4} - 18.3^{b_1}.$$

Clearly, the left hand side is negative, but not the right hand side, which gives us a contradiction. Combining all the arguments in this paragraph, we can conclude that $a_3 \neq 2$. So, we must have $a_4 = 2$.

Let $a_4 = 2$ and $a_3 \geq 4$. If $a_1 \geq 8$ then

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^9}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{13^5}\right)\left(1 - \frac{1}{19^3}\right) = 0.999741\cdots,$$

which is not possible. So, $a_1 = 6$, which would then imply $b_3 = 0$, otherwise 13 will not divide the left hand side of equation (3.18), but only the right hand side. So, we get from (3.18), after simplification

$$-416433(7^{a_2+1} + 13^{a_3+1} - 1) = 933.7^{a_2}.13^{a_3} - 72.3^{b_1}.19^{b_4}.$$

Since, here $a_2, a_3 \geq 4$ and $b_1 \leq 6$, $b_4 \leq 2$, so the right hand side of the above is always positive, while the left hand is always negative, which gives us a contradiction.

Combining all the arguments in the above case, we can conclude that $p_4 \neq 19$.

Case 4. $p_4 = 17$.

If $a_1 \geq 6$, then we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{17^3}\right) = 0.995972\cdots.$$

However in this case we have

$$g(a_1, a_2, a_3, a_4) \leq \frac{2^5.3.16}{7.13.17} = 0.992889\cdots.$$

Both these inequalities cannot be true at the same time. So, $a_1 \leq 4$.

If $a_1 = 2$, and $D \geq 17$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2).7.13.17}{3^2.6.12.16} + \frac{1}{17} < 2,$$

which is not possible. So, the possible choices for D are $3, 7, 9, 11$ and 13.

If $D = 3$, then from equation (3.18) we have

$$\sigma(7^{a_2}.13^{a_3}.17^{a_4}) = 5.3.7^{a_2}.13^{a_3-1}.17^{a_4}.$$

Noting now, $\mathrm{ord}_5(7) = \mathrm{ord}_5(13) = \mathrm{ord}_5(17) = 4$, we see that 5 does not divide the left hand side of the above eqaution and hence this case is not possible.

If $D = 7$, we have

$$\sigma(7^{a_2}.13^{a_3}.17^{a_4}) = 3^2.7^{a_2-1}.13^{a_3-1}.17^{a_4}.$$

Note now, $\mathrm{ord}_{17}(7) = 16$ and $\mathrm{ord}_{17}(13) = 4$, so 17 does not divide the left hand side of the above equation and hence this is not possible.

In a similar way, we can also show that the other values of D are not possible in this case. So, $a_1 \neq 2$.

If $a_1 = 4$ and $a_2 \geq 4$ then we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{17^3}\right) = 0.99517\cdots.$$

But then this is not compatible with the value of $g(a_1, a_2, a_3, a_4)$. So, in this case $a_2 = 2$. Further, from equation (3.18), we can see that $b_2 = b_3 = b_4 = 0$ in this case, and hence we shall have from equation (3.18), after simplification

$$47.13^{a_3}.17^{a_4} + 64.3^{b_1} + 2299 = 2299(13^{a_3+1} + 17^{a_4+1}).$$

We can rewrite the previous equation as

$$13^{a_3}(47.17^{a_4} - 29897) = 29897.17^{a_4} - 2299 - 64.3^{b_1},$$

and looking at this equation modulo 13, we can conclude that $b_1 = 3$. Similarly, rewriting this equation now with $b_1 = 3$ and looking at it modulo 17, we can conclude that this is not possible.

□

Lemma 3.7. *If* $n = 3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}$ *in Theorem 1.2 with* $p_4 \geq 41$*, then there is no odd deficient perfect number.*

Proof. Let $D \geq 38$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.11.41}{2.6.10.40} + \frac{1}{38} < 2,$$

which is not possible. So, in this case $D \in \{3, 7, 9, 11, 21, 27, 33\}$.

Let us use the function f defined earlier; which in this case is

$$f(a_1, a_2, a_3, a_4) = \left(1 - \frac{1}{3^{a_1+1}}\right)\left(1 - \frac{1}{7^{a_2+1}}\right)\left(1 - \frac{1}{11^{a_3+1}}\right)\left(1 - \frac{1}{p_4^{a_4+1}}\right).$$

Case 1. $D = 3$.

The proof is similar to the proof of Lemma 3.1, so we will skip the details here.

Case 2. $D = 7$.

We have the equation

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}) = 13.3^{a_1}.7^{a_2-1}.11^{a_3}.p_4^{a_4}. \tag{3.20}$$

We introduce the following function

$$g(a_1,a_2,a_3,a_4)=\frac{13.2^3.5.(p_4-1)}{7^2.11.p_4}.$$

From equation (3.20), it is clear that in this case $f(a_1,a_2,a_3,a_4)=g(a_1,a_2,a_3,a_4)$.

If $a_1\geq 4$, then we have,

$$f(a_1,a_2,a_3,a_4)\geq\left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{11^3}\right)\left(1-\frac{1}{41^3}\right)=0.992221\cdots,$$

and

$$g(a_1,a_2,a_3,a_4)\leq\frac{13.2^3.5}{7^2.11}=0.96475\cdots.$$

Both of these inequalities cannot be true at the same time. So, $a_1=2$, in which case equation (3.20) becomes

$$\sigma(7^{a_2}.11^{a_3}.p_4^{a_4})=3^2.7^{a_2-1}.11^{a_3}.p_4^{a_4}.$$

If, $a_2=2$, then the above becomes

$$19.\sigma(11^{a_3}.p_4^{a_4})=3.7.11^{a_3}.p_4^{a_4},$$

which would imply $p_4=19$, which is not possible. So, $a_2\geq 4$. If $p_4\geq 541$, then we have

$$2=\frac{\sigma(n)}{n}+\frac{d}{n}<\frac{\sigma(3^2).\sigma(7^4).11.541}{3^2.7^4.10.540}+\frac{1}{7}<2,$$

which is not possible. So, $41\leq p_4\leq 523$.

If p_4 is one of $41,53,61,67,71,79,83,89,97,109,113,127,131,137,151,157,163,167,173,179,$ $181,191,211,223,227,233,241,257,263,271,277,281,283,307,313,317,337,347,359,367,379,$ $389,397,421,431,433,439,443,449,457,461,467,487,491,499,503,509,521$, then $\text{ord}_{p_4}(7)$ and $\text{ord}_{p_4}(11)$ are both even, which is not possible. So, the possible choices of p_4 are $43,47,59,73,101,$ $103,107,139,149,193,197,199,229,239,251,269,293,311,331,349,353,373,383,401,409,419,$ $463,479$, and 523.

Now, we note that $\text{ord}_{11}(7)=10$ and $\text{ord}_{11}(p_4)$ is even if p_4 is one of $43,73,101,107,139,149,$ $193,197,239,293,349,373,409,479$ and 523; so these cases for p_4 are not possible, otherwise 11 will not divide the left hand side of equation (3.20). So, the least possible value for p_4 is 47 and the highest possible value is 463.

If $a_3=2$, then equation (3.20) in this case becomes

$$19.\sigma(7^{a_2}p_4^{a_4})=3^2.7^{a_2-2}.11^2.p_4^{a_4}.$$

This would again imply that $p_4=19$, which is not possible, so $a_3\geq 4$. In this case we have

$$f(a_1,a_2,a_3,a_4)\geq\left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{7^5}\right)\left(1-\frac{1}{11^5}\right)\left(1-\frac{1}{47^3}\right)=0.96289\cdots,$$

and

$$g(a_1,a_2,a_3,a_4)\leq\frac{13.2^3.5.462}{7^2.11.463}=0.962666\cdots.$$

Both of these inequalities cannot be true at the same time. Hence, for the remaining chocies for p_4, we reach a contradiction. This, $a_2=2$ is also not possible in this case.

Combining the above arguments, we reach the conclusion that this case is not possible.

Case 3. $D=9$.

We have the equation

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4})=17.3^{a_1-2}.7^{a_2}.11^{a_3}.p_4^{a_4}. \tag{3.21}$$

We introduce the following function

$$g(a_1,a_2,a_3,a_4)=\frac{17.2^3.5.(p_4-1)}{3^2.7.11.p_4}.$$

From equation (3.21), it is clear that in this case $f(a_1,a_2,a_3,a_4)=g(a_1,a_2,a_3,a_4)$.

If $a_1\geq 4$, then we have,

$$f(a_1,a_2,a_3,a_4)\geq\left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{11^3}\right)\left(1-\frac{1}{41^3}\right)=0.992221\cdots,$$

and

$$g(a_1,a_2,a_3,a_4)\leq\frac{17.2^3.5}{3^2.7.11}=0.981241\cdots.$$

Both of these inequalities cannot be true at the same time. So, $a_1 = 2$. But, then 13 divides left hand side of equation (3.21), but not the right hand side, which is a contradiction.

Case 4. $D = 11$.

We have the equation

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}) = 3^{a_1+1}.7^{a_2+1}.11^{a_3-1}.p_4^{a_4}. \tag{3.22}$$

We introduce the following function

$$g(a_1, a_2, a_3, a_4) = \frac{2^3.3.5.(p_4 - 1)}{11^2.p_4}.$$

From equation (3.22), it is clear that in this case $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$.

If $a_1 \geq 4$, then we have,

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{41^3}\right) = 0.992221\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) \leq \frac{2^3.3.5}{11^2} = 0.991736\cdots.$$

Both of these inequalities cannot be true at the same time. So, $a_1 = 2$. But, then 13 divides left hand side of equation (3.22), but not the right hand side, which is a contradiction.

Case 5. $D = 21$.

We have the equation

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}) = 41.3^{a_1-1}.7^{a_2-1}.11^{a_3}.p_4^{a_4}. \tag{3.23}$$

If $p_4 \geq 73$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.11.73}{2.6.10.72} + \frac{1}{21} < 2,$$

which is not possible. So, in this case $41 \leq p_4 \leq 71$.

We note that $\text{ord}_{41}(3) = 8, \text{ord}_{41}(7) = \text{ord}_{41}(11) = 40$ and $\text{ord}_{41}(p_4)$ is even when $p_4 \in \{43, 47, 53, 61, 67, 71\}$. This means that if p_4 is one of these values, then 41 does not divide the left hand side of equation (3.23). So, the only possibility is $p_4 = 59$.

We introduce the following function

$$g(a_1, a_2, a_3, a_4) = \frac{2^3.5.41.(p_4 - 1)}{3.7^2.11.p_4}.$$

From equation (3.23), it is clear that in this case $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$.

If $a_2 = 2$, then from equation (3.23) we have 19 divides both side of the equation, but it cannot divide the right hand side of the above equation. So, $a_2 \geq 4$. If $a_1 = 2$, then from equation (3.23) we have 13 divides both side of the equation, but it cannot divide the right hand side of the above equation. If $a_1 = 4$, then we have

$$\sigma(7^{a_2}.11^{a_3}.59^{a_4}) = 41.3^3.7^{a_2-1}.11^{a_3-2}.59^{a_4}. \tag{3.24}$$

From equation (3.24), we can deduce by an order argument the relations $41.11^{a_3-2} = \sigma(59^{a_4})$, $27.59^{a_4} = \sigma(7^{a_2})$ and $7^{a_2-1} = \sigma(11^{a_3})$. Solving these relations for a_2, we will arrive at $7^{a_2-1} < 0$, which is not possible for $a_2 \geq 4$. Hence $a_1 \geq 6$.

If $a_1 \geq 6, a_2 \geq 4$, then we have,

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{59^3}\right) = 0.998727\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) = \frac{2^3.5.41.58}{3.7^2.11.59} = 0.9970346\cdots.$$

Both of these cannot be true at the same time. Thus, this case is not possible.

Case 6. $D = 27$.

We have the equation

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}) = 53.3^{a_1-3}.7^{a_2}.11^{a_3}.p_4^{a_4}. \tag{3.25}$$

If $p_4 \geq 53$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.11.53}{2.6.10.52} + \frac{1}{27} < 2,$$

which is not possible. So, in this case $41 \leq p_4 \leq 47$.

We note that $\text{ord}_{53}(3) = \text{ord}_{53}(41) = 52, \text{ord}_{53}(7) = \text{ord}_{53}(11) = \text{ord}_{53}(43) = 26$; so $p_4 \neq 41, 43$. Thus, the only possibility is $p_4 = 47$.

We introduce the following function

$$g(a_1, a_2, a_3, a_4) = \frac{53.2^3.5.(p_4 - 1)}{3^3.7.11.p_4}.$$

From equation (3.25), it is clear that in this case $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$.

If $a_2 = 2$, then from equation (3.25) we see that 19 divides both sides of the equation, which is not possible. So, $a_2 \geq 4$. If $a_1 = 2$, then we see that 13 divides both sides of the equation (3.25) which is not possible. If $a_1 = 4$, then we have from equation (3.25)

$$\sigma(7^{a_2}.11^{a_3}.47^{a_4}) = 53.3.7^{a_2}.11^{a_3-2}.47^{a_4}. \tag{3.26}$$

Like earlier, using an order argument we can get the following relations from equation (3.26): $53.11^{a_3-2} = \sigma(47^{a_4})$, $3.47^{a_4} = \sigma(7^{a_2})$ and $7^{a_2} = \sigma(11^{a_3})$. Solving these for a_4, we shall get $47^{a_4} < 1$, which is not possible for $a_4 \geq 2$. Hence, $a_1 \geq 6$.

If $a_1 \geq 6, a_2 \geq 4$, then we have,

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{47^3}\right) = 0.998723\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) = \frac{2^3.5.53.46}{3^3.7.11.47} = 0.998025\cdots.$$

Both of these cannot be true at the same time.

Case 7. $D = 33$.

We have the equation

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}) = 5.13.3^{a_1-1}.7^{a_2}.11^{a_3-1}.p_4^{a_4}. \tag{3.27}$$

If $p_4 \geq 47$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.11.47}{2.6.10.46} + \frac{1}{33} < 2,$$

which is not possible. So, in this case p_4 is either 41 or 43.

If $p_4 = 41$, we note that $\text{ord}_{41}(3) = 8, \text{ord}_{41}(11) = \text{ord}_{41}(7) = 40$, so 41 does not divide the left hand side of equation (3.27), which is not possible.

Let $p_4 = 43$. If, either a_2 or a_3 equals 2, then in this case 19 will divide the left hand side of equation (3.27), but not the right hand side of it. So, $a_2, a_3 \geq 4$.

If $a_1 = 6$, then 1093 divides the left hand side of equation (3.27), but not the right hand side of it, so $a_1 \neq 6$. If $a_1 = 4$, then equation (3.27) becomes

$$\sigma(7^{a_2}.11^{a_3}.43^{a_4}) = 5.13.3^3.7^{a_2}.11^{a_3-3}.43^{a_4}.$$

However, we note that $\text{ord}_{13}(7) = \text{ord}_{13}(11) = 12$ and $\text{ord}_{13}(43) = 6$, so 13 does not divide the left hand side of the above equation. Hence, $a_1 \neq 4$.

If $a_1 = 2$, then equation (3.27) becomes

$$\sigma(7^{a_2}.11^{a_3}.43^{a_4}) = 5.3.7^{a_2}.11^{a_3-1}.43^{a_4}.$$

However, we note that $\text{ord}_{11}(7) = 10$ and $\text{ord}_{11}(43) = 2$, so 11 does not divide the left hand side of the above equation. Hence, $a_1 \neq 2$. Thus, $a_1 \geq 6$, if we combine all the above arguments.

Let

$$g(a_1, a_2, a_3, a_4) = \frac{5^2.13.2^3.42}{3.7.11^2.43} = 0.999423\cdots.$$

From equation (3.27), it is clear that for this case

$$f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4).$$

We already have $a_1 \geq 6$ and $a_2, a_3 \geq 4$, so

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{11^5}\right)\left(1 - \frac{1}{43^3}\right) = 0.999465\cdots,$$

which is not compatible with the value of $g(a_1, a_2, a_3, a_4)$. Hence, this case is also not possible.

Combining all the above cases completes the proof of the lemma. □

Lemma 3.8. *If* $n = 3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}$ *in Theorem 1.2 with* $29 \leq p_4 \leq 37$*, then there is no odd deficient perfect number.*

Proof. In this case, we have for some $b_i \geq 0$, $(i = 1, 2, 3, 4)$

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}) = 2.3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4} - 3^{b_1}.7^{b_2}.11^{b_3}.p_4^{a_4}. \tag{3.28}$$

Let us use the function f defined earlier; which in this case is

$$f(a_1, a_2, a_3, a_4) = \left(1 - \frac{1}{3^{a_1+1}}\right)\left(1 - \frac{1}{7^{a_2+1}}\right)\left(1 - \frac{1}{11^{a_3+1}}\right)\left(1 - \frac{1}{p_4^{a_4+1}}\right).$$

We also introduce the function

$$g(a_1, a_2, a_3, a_4) = \frac{2^4.5.(p_4 - 1)}{7.11.p_4} - \frac{2^3.5.(p_4 - 1)}{D_0},$$

where $D_0 = 3^{a_1-b_1}.7^{a_2-b_2+1}.11^{a_3-b_3+1}.p_4^{a_4-b_4+1} > 7.11.p_4$. Clearly from equation (3.28), we have

$$f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4).$$

Case 1. $p_4 = 37$.

Let $D \geq 47$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.11.37}{2.6.10.36} + \frac{1}{47} < 2,$$

which is not possible. So, the possible choices for D are $3, 7, 9, 11, 21, 27, 33$ and 37.

If $D = 3$ and 7, then the argument is exactly similar to the proof of Lemma 3.1, so we omit the details here.

If $D = 9$ and 27, then the argument is exactly similar to the arguments for $D = 9$ in Case 1 of the proof of Lemma 3.6, so we omit the details here.

We now need to check for the following values of D: $11, 21, 33$ and 37.

If $a_1 = 2$ and $D \geq 11$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2).7.11.37}{3^2.6.10.36} + \frac{1}{11} < 2,$$

which is not possible. So, for all the admissible values of D, we have $a_1 \geq 4$, and

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{37^3}\right) = 0.992216\cdots.$$

If $D = 11$, then we have from equation (3.28)

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.37^{a_4}) = 3^{a_1+1}.7^{a_2+1}.11^{a_3-1}.37^{a_4}.$$

In this case,

$$g(a_1, a_2, a_3, a_4) = \frac{2^5.3^3.5}{11^2.37} = 0.964932\cdots.$$

but then the above value is not compatible with the value of $f(a_1, a_2, a_3, a_4)$. So, $D \neq 11$.

In a similar way, we can also prove that $D \neq 21$.

If $D = 33$, then from equation (3.28), we have

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.37^{a_4}) = 5.13.3^{a_1-1}.7^{a_2}.11^{a_3-1}.37^{a_4}.$$

If in the above $a_1 = 4$, and taking note of $\text{ord}_{13}(7) = \text{ord}_{13}(11) = \text{ord}_{13}(37) = 12$, we see that 13 will not divide the left hand side of the above equation. So $a_1 \geq 6$. But, if $a_1 = 6$, then 1093 divides the left hand side of the above equation, but not the right hand side, hence, $a_1 \geq 8$, in which case we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^9}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{37^3}\right) = 0.996265\cdots.$$

In this case,

$$g(a_1, a_2, a_3, a_4) = \frac{2^5.3.5^2.13}{7.11^2.37} = 0.995565\cdots,$$

but then this values is not incompatible with the value of $f(a_1, a_2, a_3, a_4)$, hence $D \neq 33$.

Now, let $D = 37$. If $a_1 = 4$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^4).7.11.37}{3^4.6.10.36} + \frac{1}{37} < 2,$$

which is not possible. So, $a_1 \geq 6$.

Now from equation (3.28) we have

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.37^{a_4}) = 73.3^{a_1}.7^{a_2}.11^{a_3}.37^{a_4-1}.$$

Clearly $a_2 \neq 2$, otherwise 19 will divide the left hand side of the above equation, but not the right hand side. So, $a_2 \geq 4$.

Thus, we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{37^3}\right) = 0.998713\cdots.$$

Here,

$$g(a_1, a_2, a_3, a_4) = \frac{2^5.3^2.5.73}{7.11.37^2} = 0.99722\cdots,$$

but then this value is incompatible with the value of $f(a_1, a_2, a_3, a_4)$, hence $D \neq 37$.

Case 2. $p_4 = 31$.

Let $D \geq 93$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3.7.11.31}{2.6.10.30} + \frac{1}{93} < 2,$$

which is not possible. So, the possible choices for D are $3, 7, 9, 11, 21, 27, 31, 33, 49, 63, 77$ and 81.

If $D = 3, 7$ and 9, then the argument is exactly similar to the proof of Lemma 3.1, so we omit the details here.

If $D = 21, 27, 31, 49$ and 77, then the argument is exactly similar to the arguments for $D = 9$ in Case 1 of the proof of Lemma 3.6, the only difference is the role of the appropriate prime in taking the orders, so we omit the details here.

We now need to check for the following values of D: $11, 33, 63$ and 81.

If $D = 11$, then from equation (3.28), we have

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.31^{a_4}) = 3^{a_1+1}.7^{a_2+1}.11^{a_3-1}.31^{a_4}.$$

Clearly, $a_1 \neq 2$, otherwise 13 will divide the left hand side of the above equation, but not the right hand side. So, $a_1 \geq 4$. Thus, we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{31^3}\right) = 0.992202\cdots.$$

Also, in this case

$$g(a_1, a_2, a_3, a_4) = \frac{2^4.3^2.5^2}{11^2.31} = 0.959744\cdots,$$

and from equation (3.28), it is clear that

$$f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4),$$

which is not compatible with the bounds found in this case. So, $D \neq 11$.

If $a_1 = 2$ and $D \geq 12$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2).7.11.31}{3^2.6.10.30} + \frac{1}{12} < 2,$$

which is not possible.

So, for all the remaining admissible values of D, we have $a_1 \geq 4$, and

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{31^3}\right) = 0.992202\cdots.$$

If $D = 33$, then we have from equation (3.28)

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.31^{a_4}) = 5.13.3^{a_1-1}.7^{a_2}.11^{a_3-1}.31^{a_4}.$$

In this case,

$$g(a_1, a_2, a_3, a_4) = \frac{5^3.13.2^4}{11^2.7.31} = 0.990212\cdots.$$

but then the above value is not compatible with the value of $f(a_1, a_2, a_3, a_4)$. So, $D \neq 33$.

The case for $D = 63$ is done similarly.

If $D = 81$, then we have from equation (3.28)

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.31^{a_4}) = 23.3^{a_1-4}.7^{a_2+1}.11^{a_3-1}.31^{a_4}.$$

If $a_1 = 4$, in this case we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^4).7.11.31}{3^4.6.10.30} + \frac{1}{81} < 2,$$

which is not possible. If $a_1 = 6$, then 1093 divides the left hand side, but not the right hand side of the above equation. So, $a_1 \geq 8$. In a similar way, we can prove $a_2 \geq 4$ and $a_3 \geq 4$. Hence, we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^9}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{11^5}\right)\left(1 - \frac{1}{31^3}\right) = 0.999850\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) = \frac{2^4.5^2.23}{3^3.11.31} = 0.999240\cdots.$$

But, from equation (3.28) we have

$$f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4),$$

which is not compatible with the bounds above. Hence, $D \neq 81$.

Case 3. $p_4 = 29$.

If $a_1 = 2$ and $D \geq 13$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2).7.11.29}{3^2.6.10.28} + \frac{1}{13} < 2,$$

which is not possible. So, in this case the possible values of D are $3, 7, 9$ and 11.

If $D = 3$, then equation (3.28) becomes

$$13.\sigma(7^{a_2}.11^{a_3}.29^{a_4}) = 5.3.7^{a_2}.11^{a_3}.29^{a_4}.$$

Clearly 13 does not divide the right hand side of the above equation, so this is not possible. In a similar way, we can show that $D = 9$ and $D = 11$ are not possible.

If $D = 7$, then equation (3.29) becomes

$$\sigma(7^{a_2}.11^{a_3}.29^{a_4}) = 3^2.7^{a_2-1}.11^{a_3}.29^{a_4}.$$

We now note that $\text{ord}_{11}(7) = \text{ord}_{11}(29) = 10$, so this case is also not possible.

In conclusion from the above discussion, we have $a_1 \geq 4$, in which case we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{29^3}\right) = 0.992195\cdots.$$

If $D_0 \leq 3^3.7.11.29$, then we have

$$g(a_1, a_2, a_3, a_4) \leq \frac{2^4.5.36}{7.11.37} - \frac{2^3.5.36}{3^3.7.11.29} = 0.986997\cdots.$$

This is not compatible with the value of $f(a_1, a_2, a_3, a_4)$, so $D_0 > 3^3.7.11.29$, which means $D > 27$. But, if $D \geq 161$, then we get a contradiction. So, $28 \leq D \leq 160$, which means that the possible choices of D are $29, 33, 49, 77, 81, 87, 99$ and 121.

If $D = 29$, then from equation (3.28) we have

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.29^{a_4}) = 19.3^{a_1+1}.7^{a_2}.11^{a_3}.29^{a_4},$$

Using an order argument we can conclude that $19 \mid \sigma(7^{a_2}.11^{a_3})$, $3^{a_1+1} \mid \sigma(7^{a_2})$, $11^{a_3} \mid \sigma(3^{a_1})$, $7^{a_2} \mid \sigma(11^{a_3}.29^{a_4})$ and $29^{a_4} \mid \sigma(7^{a_2})$. We shall have two cases, depending on whether 19 divides $\sigma(7^{a_2})$ or whether 19 divides $\sigma(11^{a_3})$.

If it is the former case, then after simplifying all the relations we shall get

$$61607.11^{a_3}.29^{a_4} + 31920.29^{a_4} + 287 = -7(11^{a_3+1} + 29^{a_4+1}),$$

which is not possible, because both the sides are of different signs for $a_i \geq 2$. We shall reach a similar contradiction if we take the later case as well. So, $D \neq 29$.

If $D = 33$, then from equation (3.28) we have

$$g(a_1, a_2, a_3, a_4) = \frac{2^5.5^2.13}{3.11^2.29} = 0.987936\cdots,$$

which is not compatible with the value of $f(a_1, a_2, a_3, a_4)$ in this case. So, $D \neq 33$.

If $D = 49$, then, we have from equation (3.28)

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.29^{a_4}) = 97.3^{a_1}.7^{a_2-2}.11^{a_3}.29^{a_4},$$

from which using an order argument we can conclude that 97 does not divide the left hand side of the above equation, and hence this is not possible.

The cases for $D = 77$ and 121 are dealt with similarly.

The cases for $D = 81, 87$ and 99 are exactly similar to the $D = 81$ case when $p_4 = 31$ above, so we omit the details here.

Combining now, all the cases we conclude the proof of this lemma.

□

Lemma 3.9. *If $n = 3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}$ in Theorem 1.2 with $13 \leq p_4 \leq 23$, then there is only one odd deficient perfect number.*

Proof. In this case, we have for some $b_i \geq 0$, $(i = 1, 2, 3, 4)$

$$\sigma(3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4}) = 2.3^{a_1}.7^{a_2}.11^{a_3}.p_4^{a_4} - 3^{b_1}.7^{b_2}.11^{b_3}.p_4^{a_4}. \tag{3.29}$$

Let us use the function f defined earlier; which in this case is

$$f(a_1, a_2, a_3, a_4) = \left(1 - \frac{1}{3^{a_1+1}}\right)\left(1 - \frac{1}{7^{a_2+1}}\right)\left(1 - \frac{1}{11^{a_3+1}}\right)\left(1 - \frac{1}{p_4^{a_4+1}}\right).$$

We also introduce the function

$$g(a_1, a_2, a_3, a_4) = \frac{2^4.5.(p_4 - 1)}{7.11.p_4} - \frac{2^3.5.(p_4 - 1)}{D_0},$$

where $D_0 = 3^{a_1-b_1}.7^{a_2-b_2+1}.11^{a_3-b_3+1}.p_4^{a_4-b_4+1} > 7.11.p_4$. Clearly from equation (3.29), we have

$$f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4).$$

Case 1. $p_4 = 23$.

If $a_1 = 2$ and $D \geq 17$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2).7.11.23}{3^2.6.10.22} + \frac{1}{17} < 2,$$

which is not possible. So, possible choices for D in this case are $3, 7, 9$ and 11.

If $D = 3$, then equation (3.29) becomes

$$13.\sigma(7^{a_2}.11^{a_3}.23^{a_4}) = 5.3.7^{a_2}.11^{a_3}.23^{a_4}, \tag{3.30}$$

clearly this is not possible as 13 does not divide the left hand side of this equation. The cases for $D = 9$ and 11 are similarly done.

If $D = 7$, then we have similar to equation (3.30), the following

$$\sigma(7^{a_2}.11^{a_3}.23^{a_4}) = 3^2.7^{a_2-1}.11^{a_3}.23^{a_4}.$$

We now note that, $\text{ord}_{23}(7) = \text{ord}_{23}(11) = 22$, hence 23 does not divide the right hand side of the above equation, so this case is not possible. Thus, we have $a_1 \geq 4$.

If $a_1 \geq 6$, we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{23^3}\right) = 0.995798\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) \leq \frac{2^4.5.22}{7.11.23} = 0.993789\cdots.$$

Both of these cannot be true at the same time. So, $a_1 = 4$.

If $a_1 = 4, a_2 \geq 4$, we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^5}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{23^3}\right) = 0.994996\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) \leq \frac{2^4.5.22}{7.11.23} = 0.993789\cdots.$$

Both of these cannot be true at the same time. So, $a_1 = 4, a_2 = 2$. In this case, it is not difficult to see from equation (3.29) that $b_1 = 1$ and $b_4 = 0$. Then from equation (3.29), after simplification we shall arrive at

$$473.11^{a_3}.23^{a_4} + 2299(23^{a_4+1} + 11^{a_3+1}) = 220.7^{b_2}.11^{b_3} - 2299.$$

We note that b_2 equals either 1 or 2 and $b_3 \leq a_3$, so in this scenario the right hand side of the above equation will always be less than the left hand side, and hence this is not possible.

Case 2. $p_4 = 19$.

If $a_1 \geq 4$, then we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{19^3}\right) = 0.992091\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) \leq \frac{2^4.5.18}{7.11.19} = 0.984279\cdots.$$

Both of these cannot be true at the same time. So, $a_1 = 2$.

If $a_1 = 2$ and $D \geq 24$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2).7.11.19}{3^2.6.10.18} + \frac{1}{24} < 2,$$

which is not possible. So, possible choices for D in this case are $3, 7, 9, 11, 19$ and 21.

If $D = 3$, we have from equation (3.29),

$$13.\sigma(7^{a_2}.11^{a_3}.19^{a_4}) = 5.3.7^{a_2}.11^{a_3}.19^{a_4}.$$

It is clear that 13 divides the left hand side of the above equation, but not the right hand side, hence this is not possible.

The proofs for the cases $D = 9, 11, 19$ and 21 are exactly the same, so we omit the details here.

If $D = 7$, then we have from equation (3.29),

$$\sigma(7^{a_2}.11^{a_3}.19^{a_4}) = 3^2.7^{a_2-1}.11^{a_3}.19^{a_4}.$$

We note that $\mathrm{ord}_{11}(7) = \mathrm{ord}_{11}(19) = 10$ and hence 11 does not divide the left hand side of the above equation, so this is not possible.

Case 3. $p_4 = 17$.

If $a_1 \geq 4$, then we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{17^3}\right) = 0.992033\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) \leq \frac{2^4.5.16}{7.11.17} = 0.977846\cdots.$$

Both of these cannot be true at the same time. So, $a_1 = 2$.

If $a_1 = 2$ and $D \geq 33$, then we have

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2).7.11.17}{3^2.6.10.16} + \frac{1}{33} < 2,$$

which is not possible. So, possible choices for D in this case are $3, 7, 9, 11, 17$ and 21.

If $D = 3$, we have from equation (3.29),

$$13.\sigma(7^{a_2}.11^{a_3}.17^{a_4}) = 5.3.7^{a_2}.11^{a_3}.17^{a_4}.$$

It is clear that 13 divides the left hand side of the above equation, but not the right hand side, hence this is not possible.

The proofs for the cases $D = 9, 11, 17$ and 21 are exactly the same, so we omit the details here.

If $D = 7$, then we have from equation (3.29),

$$\sigma(7^{a_2}.11^{a_3}.17^{a_4}) = 3^2.7^{a_2-1}.11^{a_3}.17^{a_4}.$$

We note that $\mathrm{ord}_{11}(7) = \mathrm{ord}_{11}(17) = 10$ and hence 11 does not divide the left hand side of the above equation, so this is not possible.

Case 4. $p_4 = 13$.

If $a_1 \geq 4$, then we have

$$f(a_1, a_2, a_3, a_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{13^3}\right) = 0.991784\cdots,$$

and

$$g(a_1, a_2, a_3, a_4) \leq \frac{2^4.5.12}{7.11.13} = 0.959041\cdots.$$

Both of these cannot be true at the same time. So, $a_1 = 2$.

We can use a similar argument to get $a_2 = a_3 = a_4 = 2$. So the only possibility is that, $n = 3^2.7^2.11^2.13^2$ is an odd deficient perfect number, which is known to be true.

Combining all the above cases, we conclude the proof of the lemma.

□

Proof of Theorem 1.2. Combining all the lemmas of this section, proves the result. □

4. Remarks

In [1], Dutta and Saikia has conjectured that there exists only finitely many odd deficient perfect numbers with k distinct prime factors, when $k \geq 2$. The result presented in this paper give evidence to support this conjecture for $k = 4$. It is the belief of the author that there exists only one odd deficient perfect number with four distinct prime factors, which was also conjectured in [1]. It might also be possible to prove this conjecture using the techniques used here, but the calculations are very tedious. In fact, the author has been able to make some progress on this, namely if 5 divides a deficient perfect number with four distinct prime factor, then the third smallest prime factor must be less than 32. This work might be reported in another paper.

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