Graphical Condensation and Aztec Rectangles

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9 May, 2016

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$$\mathrm{M}(G)\,\mathrm{M}(G-\{w,x,y,z\}) + \mathrm{M}(G-\{w,y\})\,\mathrm{M}(G-\{x,z\})$$

$$= \mathrm{M}(G-\{w,x\})\,\mathrm{M}(G-\{y,z\}) + \mathrm{M}(G-\{w,z\})\,\mathrm{M}(G-\{x,y\}).$$

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- $w, x \in V_1, y, z \in V_2, |V_1| = |V_2|$; third term vanishes
- $w, x, y, z \in V_1, |V_1| = |V_2| + 2$; first term vanishes

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- Superimpose a perfect matching of G (blue) and a perfect matching of $G - \{w, x, y, z\})$ (red) on the same copy of G

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- There is a blue-red alternating path from w to one of x, y, z
- Two such paths cannot cross, so w does not connect to y
- Switch the edges in the path of w and get a pair of matchings of $G - \{w, x\})$ and $G - \{y, z\})$ or of $G - \{w, z\}$ and $G - \{x, y\})$

Pfaffians

Let $A = (a_{i,j})$ be a $2n \times 2n$ antisymmetric matrix and Γ_n be the set of all perfect matchings of K_{2n}.

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$$\mathrm{Pf}(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}.$$

Ciucu's Extension of Kuo's Condensation

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Then we have that

$$\mathrm{M}(G \setminus \{a_1, a_2, \ldots, a_{2k}\}) = \frac{\mathrm{Pf}(A)}{[\mathrm{M}(G)]^{k-1}}. \tag{1.2}$$

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Our generalization of Ciucu's result

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$$\mathrm{M}(G - \{w\})\,\mathrm{M}(G - \{x, y, z\}) + \mathrm{M}(G - \{y\})\,\mathrm{M}(G - \{w, x, z\})$$

$$= \mathrm{M}(G - \{x\})\,\mathrm{M}(G - \{w, y, z\}) + \mathrm{M}(G - \{z\})\,\mathrm{M}(G - \{w, x, y\}).$$

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$$\mathrm{M}(G)\,\mathrm{M}(G + \{a_1, \dots, a_{2k}\}) + \sum_{l=2}^{k} \mathrm{M}(G + \{a_1, a_{2l-1}\})\,\mathrm{M}(G + \overline{\{a_1, a_{2l-1}\}}) = \sum_{l=1}^{k} \mathrm{M}(G + \{a_1, a_{2l}\})\,\mathrm{M}(G + \overline{\{a_1, a_{2l}\}}),$$

where $\overline{\{a_i, a_j\}}$ stands for the complement of $\{a_i, a_j\}$ in the set $\{a_1, \dots, a_{2k}\}$.

Aztec Diamonds

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- The Aztec Diamond of order n (denoted by AD(n)) is the union of all unit squares inside the contour $|x| + |y| = n + 1$

Aztec Diamonds

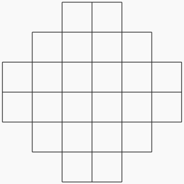

Figure: AD(3), Aztec Diamond of order 3

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Theorem (Elkies–Kuperberg–Larsen–Propp)

The number of domino tilings of an Aztec Diamond of order n is $2^{n(n+1)/2}$.

Aztec Rectangles

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Aztec Rectangle

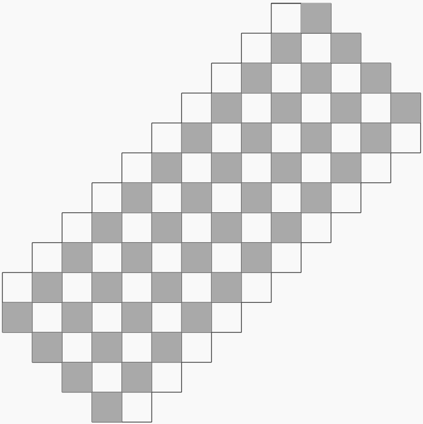

Figure: Checkerboard representation of an Aztec Rectangle with $a = 4, b = 10$

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Our goal here is to extend this result.

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- But the problem is, if $k > 0$ in Ciucu's result, then our graph G has no matchings.
- So, we modify our region suitably and try to use condensation results.

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- If a_i, a_j are on adjacent sides, then the defects are of different type and we get matchings.

Aztec Diamond with defects on adjacent sides

Proposition

The number of domino tilings of AD(a) *with one defect on the southeastern side at the* i*-th position counted from the south corner and one defect on the northeastern side on the* j*-th position counted from the north corner is given by*

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$$\sum_{l=1}^{\min\{i,j\}} 2^{(a-l)(a-l+1)/2+\sum_{k=0}^{l-2}(a-k)} \binom{a-l}{i-l}\binom{a-l}{j-l}.$$

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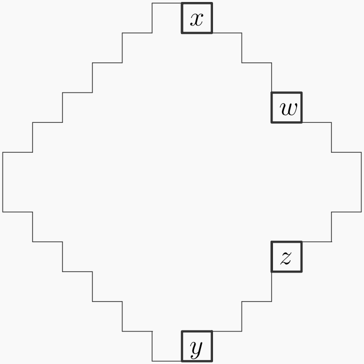

Figure: Aztec Diamond with some marked squares; here $a = 6$

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We note that when either i or j is 1 or a, some dominoes are forced in any tiling and hence we are reduced to an Aztec rectangle of size $(a-1) \times a$. It is easy to see that our formula is correct for this.

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$$\mathrm{M}(\mathrm{AD}_a(i,j))\,\mathrm{M}(\mathrm{AD}(a-1)) = \mathrm{M}(\mathrm{AD}(a))\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1))$$

$$+\,\mathrm{M}(\mathcal{AR}_{a-1,a}(j))\,\mathrm{M}(\mathcal{AR}_{a-1,a}(i)).$$

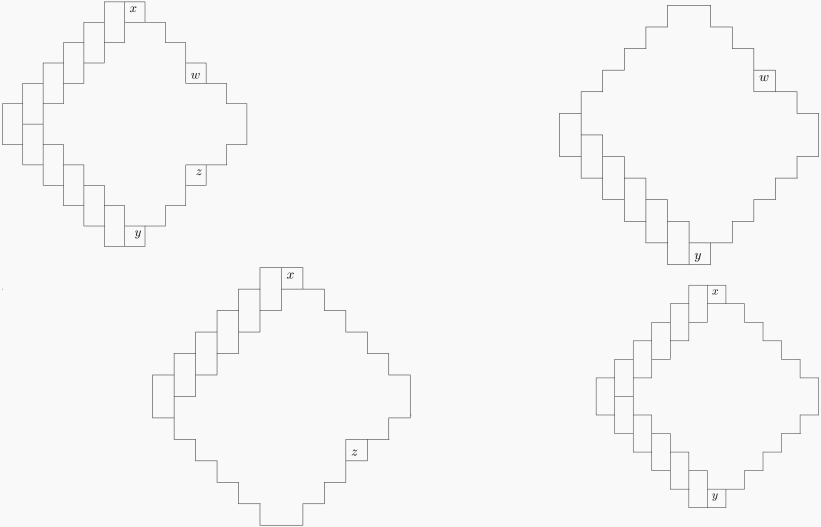

Figure: Forced dominoes, where the vertices we remove are marked

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Simplifying the previous equation, we get the following

$$\mathrm{M}(\mathrm{AD}_a(i,j)) = 2^a\,\mathrm{M}(\mathrm{AD}_{a-1}(i-1,j-1)) + 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1} \tag{2.1}$$

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The above follows from using the theorems of Elkies et. al. and Heffgott–Gessel.

Proof contd.

Simplifying the previous equation, we get the following

$$M(AD_a(i,j)) = 2^a\, M(AD_{a-1}(i-1,j-1)) + 2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1} \tag{2.1}$$

The above follows from using the theorems of Elkies et. al. and Heffgott–Gessel.

Now, using our inductive hypothesis on this equation and making a change of label $l+1 \mapsto w$ completes the proof.

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- We consider regions of the type $\mathcal{AR}_{a,b} \setminus \{\beta_1, \dots, \beta_{n+k}, \alpha_1, \dots, \alpha_n\}$, which are more general than the type considered by Heffgott–Gessel.

Preliminaries

We define the region $\mathcal{AR}^k_{a,b}$ to be the region obtained from $\mathcal{AR}_{a.b}$ by adding a string of k unit squares along the boundary of the southeastern side (γ defects) as shown in the figure below.

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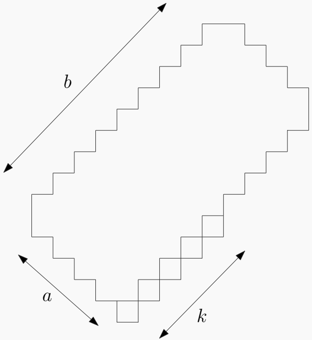

Figure: $\mathcal{AR}^k_{a,b}$ with $a = 4, b = 8, k = 4$

Main Theorem

Theorem (S.)

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Then we have

$$\mathrm{M}(\mathcal{AR}_{a,b} \setminus \{\beta_1, \ldots, \beta_{n+k}, \alpha_1, \ldots, \alpha_n\}) =$$
$$\frac{1}{[\mathrm{M}(\mathcal{AR}^k_{a,b})]^{n-k+1}} \mathrm{Pf}[(\mathrm{M}(\mathcal{AR}^k_{a,b} \setminus \{\delta_i, \delta_j\}))_{1 \leq i < j \leq 2n+2k}],$$

where all the terms on the right hand side are given by explicit formulas.

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$M(\mathcal{AR}_{a,b}^{k} \setminus \{\beta_i, \alpha_j\})$

- It is given by the previous proposition (Aztec Diamond with defects on adjacent sides) if β_i is on the south-eastern side and not above a γ defect;

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- It is given by the next proposition if the β defect is in the northwestern side at a distance of more than $k-1$ from the western corner,

Regions with defects

Proposition

Let $1 \leq a \leq b$ be positive integers with $k = b - a > 0$, then the number of domino tilings of $\mathcal{AR}_{a,b}(2, 3, \dots, k)$ with a defect on the northwestern side in the i-th position counted from the west corner as shown in the next figure is given by

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$$2^{a(a+1)/2} \sum_{l=0}^{\min\{i-1,k-1\}} \binom{a-1+l}{l} \binom{a}{a+1-i+l}.$$

Regions with defects contd.

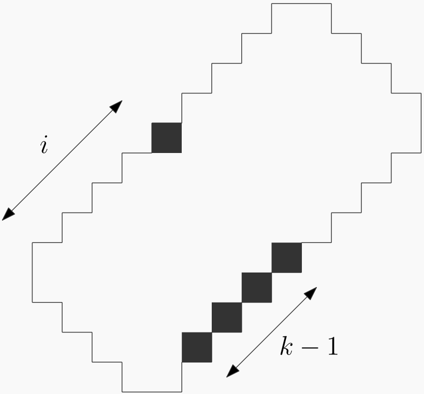

Figure: An $a \times b$ Aztec rectangle with defects marked in black; here $a = 4, b = 9. k = 5, i = 5$

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Proposition

Let $1 \leq a \leq b$ be positive integers with $k = b - a > 0$, then the number of domino tilings of $\mathcal{AR}_{a,b}(j)$ with $k - 1$ squares added to the southeastern side starting at the second position (and not at the bottom) as shown in the following figure is given by

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$$\frac{2^{a(a+1)/2}}{(j-k-1)!} \sum_{l=0}^{k-2} \left[\binom{b-l-1}{b-j} \prod_{i=l+2}^{j-k+l} (j-i) \right]. \tag{2.2}$$

Regions with defects contd.

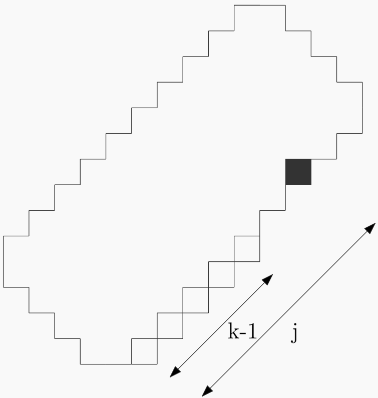

Figure: Aztec rectangle with $k-1$ squares added on the southeastern side and a defect on the j-th position shaded in black; here $a=4, b=10, k=6, j=8$

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- In their paper, Ciucu and Fischer find tilings of a hexagon with dents on adjacent and opposite sides, they use some heavy machinery to derive the results. We can do it in a simpler way by using Kuo condensation in a clever manner.

General Case

Theorem (S.)

Let $\beta_1, \dots, \beta_{n+k}$ *be arbitrary defects of type* β *and* $\alpha_1, \dots, \alpha_n$ *be arbitrary defects of type* α *along the boundary of* $\mathcal{AR}_{a,b}$*. Then* $\mathrm{M}(\mathcal{AR}_{a,b} \setminus \{\beta_1, \dots, \beta_{n+k}, \alpha_1, \dots, \alpha_n\})$ *is equal to the Pfaffian of a* $2n \times 2n$ *matrix whose entries are Pfaffians of* $(2k+2) \times (2k+2)$ *matrices of the type in the statement of main theorem.*

Questions?

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Thank you for your attention.