Refined Enumeration of Symmetry Classes of Alternating Sign Matrices

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26 November, 2020

$1, 1, 2, 7, 42, 429, 7436, 218348, \ldots$

The sequence in the title is given by the following 'nice' formula

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}$$

or, in product notation

$$\prod_{j=0}^{n-1}\frac{(3j+1)!}{(n+j)!}.$$

This formula was conjectured by Mills, Robbins and Rumsey (JCTA, 1983) to count what are called *alternating sign matrices* (ASMs).

ASMs

An alternating sign matrix (ASM) of size n is an $n \times n$ matrix with entries in the set $\{0, 1, -1\}$ such that

- all row and column sums are equal to 1,
- and the non-zero entries alternate in each row and column.

Clearly all permutation matrices are also ASMs.
For instance, there are 7 ASMs of order 3, these are the six permutation matrices and the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

ASMs Enumeration

The ASM conjecture was proved by Doron Zeilberger (EJC, 1996) and independently by Greg Kuperberg (IMRN, 1996).

Zeilberger proved a stronger statement using constant term identities, while Kuperberg exploited a connection with certain models from statistical mechanics.

How were they first defined?

Given a matrix A, we let A^i_j denote the matrix that remains when the ith row and jth column of A are deleted. If we remove more than one row or column, then the indices corresponding to those are added to the super- and sub- scripts.

Theorem (Desnanot-Jacobi adjoint matrix theorem)

If A is an $n \times n$ matrix, then

$$\det(A)\det(A^{1,n}_{1,n}) = \det(A^1_1)\det(A^n_n) - \det(A^1_n)\det(A^n_1)$$

or

$$\det(A) = \frac{1}{\det(A^{1,n}_{1,n})} \times \det\begin{pmatrix} \det(A^1_1) & \det(A^1_n) \\ \det(A^n_1) & \det(A^n_n) \end{pmatrix}.$$

This gives us a way of evaluating determinants, in terms of smaller determinants.

More determinants

Reverend Charles L. Dodgson, better known by his pen name of Lewis Carroll used Desnanot-Jacobi theorem to give an algorithm for evaluating determinants in terms of 2×2 determinants.
For instance, we get

$$\det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \frac{1}{a_{2,2}} \times \det \begin{pmatrix} \det \begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} & \det \begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} \\ \det \begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} & \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \end{pmatrix}.$$

Generalizing the determinant

In the 1980s, Robbins and Rumsey looked at a generalization of the 2×2 determinant, which they called the λ-determinant.
They defined

$$\det_{\lambda} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = a_{1,1}a_{2,2} + \lambda a_{2,1}a_{1,2}.$$

Using the previous observations, they generalized it to an $n \times n$ determinant.

ASMs and Determinants

Their main result in this direction was

Theorem (Robbins-Rumsey (Adv. Math., 1986))

Let A be an $n \times n$ matrix with entries $a_{i,j}$, $\mathcal{A}_n$ be the set of all ASMs, $\mathcal{I}(B)$ be the inversion number of B and $\mathcal{N}(B)$ be the number of -1's in B. Then

$$\det_{\lambda}(A) = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}}.$$

This was the first appearance of an ASM in the literature.

What's the inversion number?

An easy way to calculate the inversion number is to take products of all pairs of entries for which one of them lies to the right and above the other, and then adding them all up.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

There are seven pairs here whose product is $+1$ and two pairs whose product is -1. So the inversion number is 5.

How does one get a formula from this?

First some observations

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

There can be only one 1 in the top row (or, first column). Let $A_{n,k}$ be the number of $n \times n$ ASMs with a 1 at the top of top kth column. Some thought will give us, $A_{n,k} = A_{n,n+1-k}$ (symmetry).

Further, if A_n is the number of $n \times n$ ASMs, then $A_{n,1} = A_{n,n} = A_{n-1}$.

Guessing the formula

This allows one to check small values to get a formula. Mills, Robbins and Rumsey did exactly that.

They first conjectured

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}.$$

This means that the $A_{n.k}$'s are uniquely determined by the $A_{n,k-1}$'s when $k > 1$ and by $A_{n,1} = \sum_{k=1}^{n-1} A_{n-1,k}$.

Towards a final formula

The above conjecture can be reformulated as

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}.$$

From here, knowing that $A_n = A_{n+1,1}$ allows one to conjecture the ASM enumeration formula.

The Many Faces of ASMs

The sequence $1, 1, 2, 7, 42, 429, 7436, 218348, \ldots$ actually counts several combinatorial objects.

Theorem (Andrews (Invent. Math., 1979), Zeilberger, Ayyer-Behrend-Fischer (Adv. Math., 2020))

The following combinatorial objects are counted by the formula

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

- *Alternating Sign Matrices,*
- *Descending Plane Partitions,*
- *Totally Symmetric Self-Complementary Plane Partitions,*
- *Alternating Sign Triangles,*
- *...and a couple more.*

Missing Bijections

There are other combinatorial objects which are equinumerous with ASMs, and one of the major open problems in enumerative combinatorics is to find bijections between such objects.

Recently, Fischer and Konvalinka (PNAS, 2020) have given a bijective proof of the DPP-ASM part.

Does the story end here?

In the late 1980's Richard Stanley (Lec. Notes. Math., 1986) suggested the study of various symmetry classes of ASMs; this let Robbins to conjecture formulas for many of these classes.

It turned out to be as difficult as enumerating ASMs, and this study was only recently completed in 2016.

Symmetry Classes

- Vertically Symmetric ASMs: $a_{i,j} = a_{i,n+1-j}$, n odd (Kuperberg Annals, 2002)
- Half-turn Symmetric ASMs: $a_{i,j} = a_{n+1-i,n+1-j}$, n odd (Razumov-Stroganov Teoret. Mat. Fiz., 2005), n even (Kuperberg Annals, 2002)
- Diagonally Symmetric ASMs: $a_{i,j} = a_{j,i}$, no 'nice' formula
- Quarter-turn Symmetric ASMs: $a_{i,j} = a_{j,n+1-i}$, n odd (Razumov-Stroganov Teoret. Mat. Fiz., 2005), n even (Kuperberg Annals, 2002)
- Horizontally and vertically Symmetric ASMs: $a_{i,j} = a_{i,n+1-j} = a_{n+1-i,j}$, n odd (Okada JACO, 2004)
- Diagonally and Antidiagonally Symmetric ASMs: $a_{i,j} = a_{j,i} = a_{n+1-j,n+1-i}$, n odd (Behrend-Fischer-Konvalinka Adv. Math., 2017)
- All symmetries: $a_{i,j} = a_{j,i} = a_{i,n+1-j}$, no 'nice' formula.

Refined Enumeration of ASMs

Some observations are in order.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

- There is only one 1 in any boundary row/column of an ASM.
- This suggests the question: how many ASMs with the position of the 1 fixed at a certain row/column exist?
- These are called refined enumeration of ASMs.

Refined Enumeration of ASMs

The study began with conjectures by Robbins about the number of ASMs of order n with the position of the 1 in the first row at the kth column is given by

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{3j+1)!}{(n+j)!}.$$

This was proved by Zeilberger (NYJ Math., 1996).

Several people have worked on the refined enumration of ASMs as well as their symmetry classes: Behrend (Adv. Math., 2013), Fischer (JCTA, 2007), Romik-Fischer (Adv. Math., 2009), Razumov-Stroganov (Teoret. Mat. Fiz., 2004), Ayyer-Romik (Adv. Math., 2013), Romik-Karlinsky (Adv. Appl. Math., 2010, etc.

Some (ex-) conjectures

Several conjectures on refined enumeration of ASMs existed.

- Fischer (JCTA, 2009) conjectured a formula for the number of VSASMs with the position of the 1's in the second row fixed.
- Robbins (late 1980s) conjectured formulas for refined enumeration of QTSASMs.
- Duchon (FPSAC, 2008) conjectured a formula for the refined enumeration of quasi-QTSASMs.
- We proved these conjectures – and more – in joint work with Ilse Fischer (JCTA, 2021) and in some recent work (in progress, 2020).

VSASMs

In the case of VSASMs, we can make the following observations:

- They exist for odd order.
- The middle column is always $(1, -1, 1, \ldots, -1, 1)^T$.
- The second row has exactly two 1's.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

So we can ask for refined enumeration w.r.t. the position of the 1's in the second row.

Fischer's Conjecture

Razumov and Stroganov (Teoret. Mat. Fiz., 2004) has a formula counting the number of VSASMs with a fixed one in the first column.

$$\mathsf{A}_{\mathsf{VC}}(2n+1,i) = \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!}$$
$$\times \sum_{k=1}^{i-1} (-1)^{i+k-1} \frac{(2n+k-2)!(4n-k-1)!}{(4n-2)!(k-1)!(2n-k)!}$$

Ilse Fischer (JCTA, 2009) had conjectured that the number of $(2n+1)\times(2n+1)$ VSASMs, where the first one in the second row is in the ith column is equal to

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!} \left(\prod_{j=1}^{n-1} \frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!} \right).$$

Fischer's Conjecture

Theorem (Fischer-S., 2019)

The number of $(2n+1) \times (2n+1)$ VSASM with a 1 in the i-th position in it's second row is given by

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!}\left(\prod_{j=1}^{n-1}\frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!}\right)$$
$$= \mathsf{A}_{\mathsf{VC}}(2n+1,i) + \mathsf{A}_{\mathsf{VC}}(2n+1,i+1).$$

A bijective proof of the last relation would be of interest.

Other Symmetry Classes?

- We have formulas for many of the symmetry classes as well as other type of ASMs.
- In some cases the results are in terms of generating functions.

Vertically and Horizontally Symmetric ASMs

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

VHSASMs

Theorem (Fischer-S., 2019)

Let $\mathsf{A}_{\mathsf{VH}}(4n+3,i)$ *denote the number of VHSASMs of order* $4n+3$*, with the first occurrence of a* 1 *in the second row be in the* i*-th column. Then, for all* $n \geq 1$ *the following is satisfied*

$$
\begin{aligned}
&3^{-n^2}(1-z^2)\left(\sum_{i=2}^{2n}\mathsf{A}_{\mathsf{VC}}(2n+1,i)z^{-i}\right)\\
&\qquad\left(\sum_{1\leq j\leq i\leq n+1} Q_{n,i}(zq-1)^{n+i-2j+1}(q-z)^{n-i+2j-1}(-q)^{-n}\right)\\
&=\sum_{i=1}^{2n+1}\left(\mathsf{A}_{\mathsf{VH}}(4n+3,i+1)-\mathsf{A}_{\mathsf{VH}}(4n+3,i)\right)\left(z^{i-2n-1}-z^{-i+2n+1}\right),
\end{aligned}
$$

where every quantity appearing on the left-hand side is explicitly known.

A similar result holds for VHSASMs of order $4n+1$.

Values

$$\mathrm{A_{VC}}(2n+1,i)=\frac{1}{2^{n-1}}\prod_{k=1}^{n-1}\frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!}$$

$$\times\sum_{k=1}^{i-1}(-1)^{i+k-1}\frac{(2n+k-2)!(4n-k-1)!}{(4n-2)!(k-1)!(2n-k)!}$$

$$Q_{n,i}=\frac{3^{n(n-1)}}{2^{n-1}(4n-1)!}\prod_{j=0}^{n-1}\frac{(4j+3)(6j+6)!}{(2n+2j+1)!}$$

$$\times\sum_{j=0}^{n}\left[\frac{27^j(3j-2n-i+2)_{4n-3}(3n-3j+1)}{(3j)!(n-j)!(3j+1)_{3n}}\right.$$

$$\left.\times\left(\frac{\left(n-j+\frac{4}{3}\right)_{2j}(2n+3j-i-1)_2}{(3n+3j+1)_2}-\frac{\left(n-j+\frac{2}{3}\right)_{2j}(-2n+3j-i)_2}{(3n-3j+1)_2}\right)\right]$$

Vertically and Horizontally Perverse ASM

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & \star & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

VHPASMs

Theorem (Fischer-S., 2019)

The number of order $4n+2$ VHPASMs with the leftmost occurrence of 1 in the second row in i-th column is

$$\sum_{k=0}^{i-2} \mathrm{A}_{\mathrm{VC}}(2n{+}1, k{+}2)\left(\mathrm{A}_{\mathrm{VC}}(2n+1, i-k) + \mathrm{A}_{\mathrm{VC}}(2n+1, i-3-k)\right).$$

Theorem (Fischer-S., 2019)

The number of order $4n+2$ VHPASMs with the topmost occurrence of 1 in the second column in the i-th row is

$$\sum_{k=0}^{i-2} \mathrm{A}_{\mathrm{VC}}(2n{+}1, k{+}2)\left(\mathrm{A}_{\mathrm{VC}}(2n+1, i-k) + \mathrm{A}_{\mathrm{VC}}(2n+1, i-1-k)\right).$$

Relation between the refined enumeration numbers

From the previous theorems it follows that

$$A^{R}_{VHP}(4n+2,i) = A^{C}_{VHP}(4n+2,i) + A^{C}_{VHP}(4n+2,i-2) - A^{C}_{VHP}(4n+2,i-1).$$

where

- $A^{R}_{VHP}(4n+2,i)$ is the row refinement number,
- $A^{C}_{VHP}(4n+2,i)$ is the column refinement number.

Off-diagonally and off-antidiagonally symmetric ASMs

- An ASM of order $2n+1$ symmetric w.r.t. reflection along the diagonal and antidiagonal.
- With all entries 0 along the diagonal and the antidiagonal, except for the central entry which is $(-1)^n$.

Off-diagonally and off-antidiagonally symmetric ASMs

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

OOSASMs

Theorem (Fischer-S., 2019)

A refined enumeration result similar to the case of VHSASMs holds for OOSASMs.

From our results we get the following relations

$$\mathsf{A}_{\mathsf{VH}}(4n+1, i) = \mathsf{A}_{\mathsf{OO}}(4n-1, i) + \mathsf{A}_{\mathsf{OO}}(4n-1, i-1),$$

and

$$\mathsf{A}_{\mathsf{VH}}(4n+3, i) = \mathsf{A}_{\mathsf{OO}}(4n+1, i) + \mathsf{A}_{\mathsf{OO}}(4n+1, i-1).$$

Bijective proofs of these relations would be of interest.

Similar results hold for vertically and off-diagonally symmetric ASMs.

Quarter turn symmetric ASMs

- ASMs that are invariant under a 90° rotation are called quarter turn symmetric ASMs (QTSASMs).
- As a first observation, we see that these ASMs cannot occur for order $4n+2$, consider the QTSASM of order $2n$ where the entries are given by $a_{i,j}$ $(1 \leq i, j \leq 2n)$. Then we have

$$2n = \sum_{1\leq i,j\leq 2n} a_{i,j} = 4 \sum_{1\leq i,j\leq n} a_{i,j},$$

  and this implies that $2|n$. So for the even case they occur only for order $4n$.
- Robbins (late 1980s) conjectured refined enumeration for QTSASMs of order $4n, 4n+1$ and $4n+3$.
- We proved all these conjectures.

Quarter turn symmetric ASMs

Theorem (Fischer-S., 2019)

Let $\mathsf{A}_{\mathsf{QT}}(n,i)$ *be the number of QTSASMs of order* n *with the position of the unique* 1 *in the first row in the* i*-th column. Then*

$$\sum_{i=2}^{4n-1} \mathsf{A}_{\mathsf{QT}}(4n,i) z^{i-2} = \left(\sum_{i=1}^{n} A_{n,i} z^{i-1}\right)^2 \left(\sum_{i=1}^{2n} \mathsf{A}_{\mathsf{HT}}(2n,i) z^{i-1}\right),$$

$$\sum_{i=2}^{4n} \mathsf{A}_{\mathsf{QT}}(4n+1,i) z^{i-2} = \left(\sum_{i=1}^{n} A_{n,i} z^{i-1}\right)^2 \left(\sum_{i=1}^{2n+1} \mathsf{A}_{\mathsf{HT}}(2n+1,i) z^{i-1}\right),$$

and

$$\sum_{i=2}^{4n+2} \mathsf{A}_{\mathsf{QT}}(4n+3,i) z^{i-2} = \left(\sum_{i=1}^{n+1} A_{n+1,i} z^{i-1}\right)^2 \left(\sum_{i=1}^{2n+1} \mathsf{A}_{\mathsf{HT}}(2n+1,i) z^{i-1}\right),$$

where all quantities appearing in the right-hand side are known.

Values

$$A_{n,i} = \binom{n+i-2}{n-1} \frac{(2n-i-1)!}{(n-i)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}$$

and

$$\mathrm{A}_{\mathrm{HT}}(2n,i) = \frac{(2n-1)!^2}{(n-1)!^2(3n-3)!(3n-1)!} \prod_{j=0}^{n-1} \frac{(3j+2)(3j+1)!^2}{(3j+1)(n+j)!^2}$$
$$\times \sum_{j=1}^{i} \left(\frac{(n^2-nj+(j-1)^2(n+j-3)!)(2n-j-1)!(n+i-j-1)!(2n-i+j-2)!}{(j-1)!(n-j+1)!(i-j)!(n-i+j-1)!} \right).$$

quasi-Quarter turn symmetric ASMs

- As pointed out, there are no even order QTSASMs of order $4n + 2$.
- However, Duchon introduced a new type of ASM, called quasi-QTSASMs (qQTSASMs) which follows all the conditions of an ASM and has quarter turn symmetry for all entries except the middle 2×2 square, which can be either $\{1, 0, 0, 1\}$ or $\{0, -1, -1, 0\}$.

qQTSASMs

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Duchon (2008) also conjectured refined enumeration for these type of matrices, which we also prove.

qQTSASMs

Theorem (Fischer-S., 2019)

Let $\mathrm{A}_{\mathrm{qQT}}(n, i)$ *denote the number of order* n *qQTSASMs with the unique* 1 *in the first row in the* i*-th column,then we have*

$$\begin{aligned}\sum_{i=2}^{4n+1} \mathrm{A}_{\mathrm{qQT}}(4n+2, i) z^{i-2} &= \left(\sum_{i=1}^{n} \mathrm{A}(n, i) z^{i-1} \right) \left(\sum_{i=1}^{n+1} \mathrm{A}(n+1, i) z^{i-1} \right) \\ &\qquad \times \left(\sum_{i=1}^{2n+1} \mathrm{A}_{\mathrm{HT}}(2n+1, i) z^{i-1} \right),\end{aligned}$$

where all the quantities appearing in the right-hand side are known.

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Half turn symmetric ASMs

- ASMs that are invariant under a 180° rotation are called half turn symmetric ASMs (HTSASMs).
- They exist for both odd and even order.
- The refined enumeration of HTSASMs w.r.t. the position of the 1 in the first row was already done by Razumov & Stroganov (Teoret. Mat. Fiz., 2006).
- One can also ask for doubly refined enumeration of HTSASMs w.r.t. the position of the 1's in the first row and first column.

Theorem (S, 2020)

A doubly refined generating function for HTSASMs exist where the quantities on one side are explicitly known.

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$1/N$ phenomenon of QTSASMs

The central entry of a QTSASM of order $4n+1$ is always 1.

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,2n} & a_{1,2n+1} & a_{1,2n+2} & \cdots & a_{1,4n+1} \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{2n+1,1} & \cdots & a_{2n+1,2n} & \star & a_{2n+1,2n+2} & \cdots & a_{2n+1,4n+1} \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{4n+1,1} & \cdots & a_{4n+1,2n} & a_{4n+1,2n+1} & a_{4n+1,2n+2} & \cdots & a_{4n+1,4n+1} \end{pmatrix}$$

- This follows from $4\left(\sum_{1\le i,j\le 2n} a_{i,j} + \sum_{1\le j\le 2n} a_{2n+1,j}\right) + \star = 4n+1$.
- Similarly, the central entry of a QTSASM of order $4n+3$ is always -1.
- Because of the symmetry, all adjacent entries of this central entry are the same.

$1/N$ phenomenon of ASMs

- The proportion of QTSASMs of appropriate order with fixed entries adjacent to the central entry was conjectured by Stroganov to be $\frac{n+1}{n}$.
- This phenomenon was already observed in the case of qQTSASMs by Aval and Duchon (EJC, 2010), in case of HTSASMs by Razumov and Stroganov (Teoret. Mat. Fiz., 2006), and in case of odd order DADSASMs by Behrend, Fischer and Konvalinka (Adv. Math., 2017).
- This is called $1/N$ phenomenon in ASMs.
- Recently, we proved Stroganov's conjecture for QTSASMs.

QTSASMs

Theorem (S, 2020)

Let the number of QTSASMs of order n with the entries adjacent to its central entry being $\star$'s be $\mathsf{A}_{\mathsf{QT}}^{(\star)}(n)$. Then the following equations are true

$$\frac{\mathsf{A}_{\mathsf{QT}}^{(0)}(4n+1)}{\mathsf{A}_{\mathsf{QT}}^{(\text{-}1)}(4n+1)} = \frac{n+1}{n},$$

and

$$\frac{\mathsf{A}_{\mathsf{QT}}^{(1)}(4n+3)}{\mathsf{A}_{\mathsf{QT}}^{(0)}(4n+3)} = \frac{n+1}{n}.$$

Contributions

- We proved singly refined enumeration results for almost all symmetry classes of ASMs.
- This proved conjectures of Fischer, Robbins and Duchon.
- We proved singly refined enumeration results for related classes of ASMs like OOSASMs and Vertically and off-diagonally symmetric ASMs.
- This actually completes the singly refined enumeration of ASMs.
- We proved doubly refined enumeration results for HTSASMs.
- We proved conjectures of Stroganov about the $1/N$ phenomenon of QTSASMs.

Thank you for your attention!

Bijection between ASMs and Six Vertex Model

Kuperberg's proof of the ASM conjecture was by exploiting a bijection between the ASMs and a model in statistical physics, called the six-vertex model.

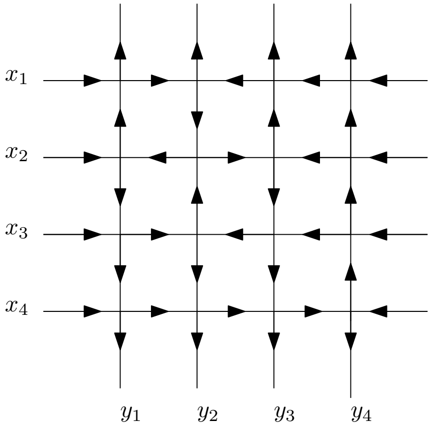


Figure: Six Vertex Model with Domain Wall Boundary Condition.

Bijection between ASMs and Six Vertex Model

A state of a corresponding six-vertex model is an orientation on the edges of this graph, such that both the in-degree and the out-degree of each vertex with degree 4 is 2.

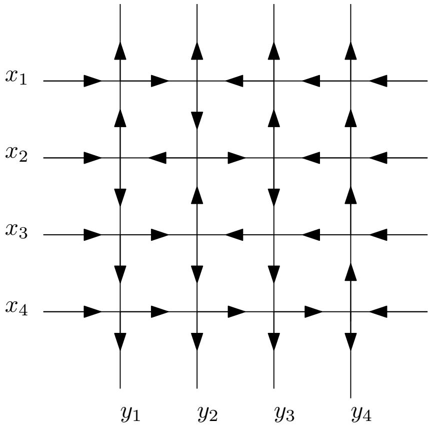


Figure: Six Vertex Model with Domain Wall Boundary Condition.

The Bijection

If we associate to each of the degree 4 vertex in a six-vertex state with a number, as given in the figure below

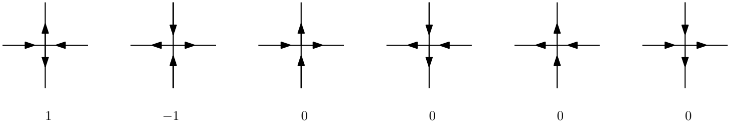


Figure: The corresponding states of the six-vertex model and the entries of an ASM.

then we obtain a matrix with entries in the set $\{0, 1, -1\}$.

Such a matrix will be an ASM, and we get a bijection between ASMs and states of the six-vertex model.

Example

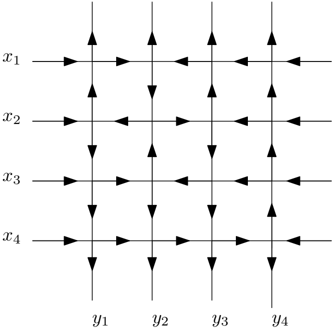


Figure: Six Vertex Model with Domain Wall Boundary Condition.

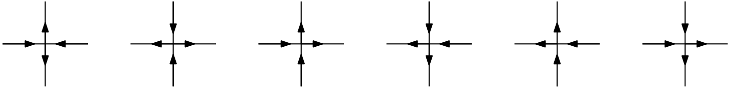
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Weighted Enumeration

- We assign to each vertex v, a weight $w(v)$.
- Weight of a state C is $W(C) = \prod_{v \in C} w(v)$.
- Generating function or the partition function $Z_n = \sum_C W(C)$.
- Specializing the parameters in Z_n, we get enumeration results.

Our Weights

q is a parameter, which we will specialize later, $\bar{x} = \dfrac{1}{x} = x^{-1}$ and $\sigma(x) = x - \bar{x}$.

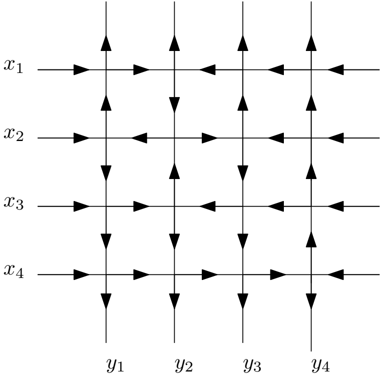


$\sigma(q^2)$ $\sigma(q^2)$ $\sigma(qu)$ $\sigma(qu)$ $\sigma(q\bar{u})$ $\sigma(q\bar{u})$

Figure: The weights of the vertices of an ASM with spectral parameter u.

We normalize further by $\sigma(q^2)$ so that we have all entries 1 and -1 to have weight 1.

How are weights assigned?



A vertex lying at the intersection of a vertical line with parameter y_j and a horizontal line with parameter x_i is assigned the label $\dfrac{x_i}{y_j}$.

Another type of ASM

In order to study VSASMs we need what are called U-turn domain boundary wall conditions.

The set of VASAMs is a subset of what are called U-turn ASMs or ASMs with U-turn boundary (UASMs).

We will explain this connection shortly.

UASMs

An U-turn ASM is an $2n \times n$ array which satisfies the usual properties of ASMs if one looks at it vertically.

However, if one looks at it horizontally then the 1's and -1's alternate if we start along an odd numbered row from left to right and then continue along the next even numbered row from right to left.

$$\begin{array}{cc} 0 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{array}$$

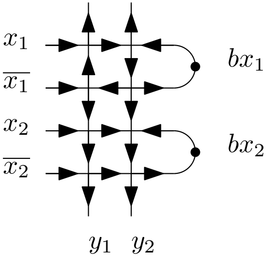


Figure: An U-turn ASM with the corresponding six-vertex state.

New weights

As can be seen from the figure, we add an additional parameter on the U-turns.

This gives rise to two new type of vertices whose corresponding weights are given below.

$\sigma(bu)$ $\sigma(b\overline{u})$

Figure: Weights of the new vertices.

Partition Function

Tsuchiya was the first to consider a U-turn domain wall boundary condition, and gave a partition function for them.

$$Z_U(n;\mathbf{x},\mathbf{y}) = \frac{\sigma(q^2)^n \prod_i (\sigma(b\overline{y_i})\sigma(q^2x_i^2)) \prod_{i,j}(\sigma'(x_i\overline{y_j})\sigma'(x_iy_j))}{\prod_{i<j}(\sigma(\overline{x_i}x_j)\sigma(y_i\overline{y}_j)) \prod_{i\leq j}(\sigma(\overline{x_i x_j})\sigma(y_iy_j))} \times \det M_U(n;\mathbf{x},\mathbf{y}), \tag{1}$$

where $\sigma'(x) = \sigma(qx)\sigma(q\overline{x})$ and M_U is an $n \times n$ matrix defined as

$$M_U(n;\mathbf{x},\mathbf{y})_{i,j} = \frac{1}{\sigma'(x_i\overline{y}_j)} - \frac{1}{\sigma'(x_iy_j)}.$$

Some observations

- VSASMs occur only for odd order.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Some observations

- VSASMs occur only for odd order.
- We need only the first $n+1$ columns of the VSASM to know the full matrix.
- The middle column is an alternating row with 1 and -1.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Some observations

- VSASMs occur only for odd order.
- We need only the first $n+1$ columns of the VSASM to know the full matrix.
- The middle column is an alternating row with 1 and -1.
- So, n columns are sufficient to know the whole matrix.
- Moreover, the first and last rows are always the same.

Transformation

We can transform a VSASM into an USASM in two steps:

- Delete the last row.
- Connect pairwise the alternating edges on the right most column of the $2n \times n$ matrix.

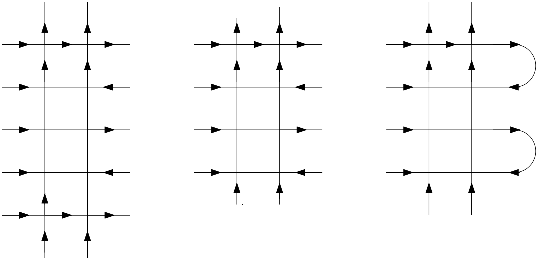


Figure: Transformation of a VSASM into an UASM.

Notice that all U-turns are downward pointing.

Proof of Fischer's Conjecture

Theorem (Fischer-S., 2019)

The number of $(2n+1) \times (2n+1)$ *VSASM with a* 1 *in the* i*-th position in it's second row is given by*

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!}\left(\prod_{j=1}^{n-1}\frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!}\right). \quad (2)$$

How to transfer to the partition function?

We will specialize

$$(x_1, \ldots, x_n) = (x, 1, \ldots, 1) \quad \text{and} \quad (y_1, \ldots, y_n) = (1, \ldots, 1),$$

as well as

$$b = q \quad \text{and} \quad q + \bar{q} = 1, \tag{3}$$

in the partition function.

When $b = q$ and $x_i = 1$ for $i > 1$ implies that the configurations that have at least one up-pointing U-turn in positions $2, 3, \ldots, n$ have weight

$$\sigma(b\bar{u}) = \sigma(q \times \bar{q}) = 0$$

and can therefore be omitted.

Proof

For the remaining configurations we can distinguish between the cases where the topmost U-turn is down-pointing (Case 1) or not (Case 2).

Let us denote the numbers that we are interested in by $A_V(2n+1, i)$.

Case 1. If the topmost U-turn is down-pointing, then the top row is forced and all vertex configurations are of type .

In the second row, there is precisely one configuration of type , say in position i counted from the left, and the configurations right of it are all of type , while the configurations left of it are of type .

Case 1

Such configurations correspond to $(2n+1) \times (2n+1)$ VSASMs that have the first 1 in the second row in the i-th column.

The top U-turn contributes $\sigma(q^2x)$, while all other $n-1$ U-turns contribute $\sigma(q^2)$.

In total such a configuration has the following weight

$$\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2n-i} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{i-1} \sigma(q^2x)\sigma(q^2)^{n-1}.$$

This case contributes the following term towards the partition function

$$\sum_{i=1}^{n} \mathsf{A}_{\mathsf{V}}(2n+1,i) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{i} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2n} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-1} \sigma(q^2x)\sigma(q^2)^{n-1}.$$

Case 2

Case 2. If the topmost U-turn is up-pointing, there is a unique occurrence of  in the top row, say in position i. There is either one occurrence of  in the second row, say in position j with $1 \leq j < i$, or no such occurrence.

In the first case, the weight is

$$\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-j-2}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-2i+j-1}\sigma(\bar{x})\sigma(q^2)^{n-1}.$$

We notice that for fixed i, these configurations give rise to all the configurations counted by $\mathrm{A_V}(2n+1, j)$.

So, this case contributes the following term towards the partition function

$$\sum_{j=1}^{n}\mathrm{A_V}(2n+1,j)\left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{j}\sum_{i=j+1}^{n}\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2}\left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-2i-2}\sigma(\bar{x})\sigma(q^2)^{n-1}.$$

Case 2. contd.

In the second case the weight is

$$\left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^i \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-i-1} \sigma(\bar{x})\sigma(q^2)^{n-1}.$$

We notice that such configurations are essentially the same as the ones counted by $\mathsf{A_V}(2n+1, i)$ with just the first U-turn reversed,so this contributes the following term towards the partition function

$$\sum_{i=1}^{n} \mathsf{A_V}(2n+1, i) \left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^i \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-1} \sigma(\bar{x})\sigma(q^2)^{n-1}.$$

Partition function relation

Combining the three cases that we got, we have

$$
\begin{aligned}
Z_U(n; x, \underbrace{1, \dots, 1}_{n-1}; \underbrace{1, \dots, 1}_{n}) = \sum_{i=1}^{n} \mathrm{A_V}(2n+1, i) \left(\frac{\sigma(qx)}{\sigma(q\bar{x})}\right)^{i} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-1} \sigma(\bar{x})\sigma(q^2)^{n-1} \\
+ \sum_{j=1}^{n} \mathrm{A_V}(2n+1, j) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{j} \sum_{i=j+1}^{n} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2i-2} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{2n-2i-2} \sigma(\bar{x})\sigma(q^2)^{n-1} \\
+ \sum_{i=1}^{n} \mathrm{A_V}(2n+1, i) \left(\frac{\sigma(q\bar{x})}{\sigma(qx)}\right)^{i} \left(\frac{\sigma(qx)}{\sigma(q^2)}\right)^{2n} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)}\right)^{-1} \sigma(q^2 x)\sigma(q^2)^{n-1},
\end{aligned}
\quad (4)
$$

Partition function relation

We replace

$$z = \frac{\sigma(q\bar{x})}{\sigma(qx)}$$

and eliminate x.

A tedious but straight forward computation shows that

$$- \sigma(q^2)^n \sigma(q\bar{x})^{-2n} \frac{1+z}{1-2z} Z_U(n; x, \underbrace{1, \dots, 1}_{n-1}; \underbrace{1, \dots, 1}_{n})$$

$$= \sum_{i=1}^{n} \mathsf{A}_{\mathsf{V}}(2n+1, i) \left(z^{i-2n-1} + z^{-i} \right). \quad (5)$$

Final calculations

Using results by Okada with some simplifications we would also get the following in our case.

$$
\begin{aligned}
&-\sigma(q^2)^{2n-1}\sigma(q\bar{x})^{-2n}\frac{1+z}{1-2z}\sigma(q^2x^2)3^{-n(n-1)} \\
&\qquad\times Sp_{4n}(n-1,n-1,\dots,0,0;x^2,1,\dots,1) \\
&\qquad\qquad = \sum_{i=1}^{n} \mathsf{A}_{\mathsf{V}}(2n+1,i)\left(z^{i-2n-1}+z^{-i}\right). \quad (6)
\end{aligned}
$$

Here

$$Sp_{2n}(\lambda_1,\dots,\lambda_n;x_1,\dots,x_n) = \frac{W^-(\lambda_1+n,\lambda_2+n-1,\dots,\lambda_n+1;x_1,\dots,x_n)}{W^-(n,n-1,\dots,1;x_1,\dots,x_n)},$$

and

$$W^-(\alpha_1,\dots,\alpha_n;x_1,\dots,x_n) = \det_{1\le i,j\le n}\left(x_i^{\alpha_j}-x_i^{-\alpha_j}\right).$$

Final Calculations contd.

Again using results of Razumov and Stroganov with lots of simplifications we shall arrive at

$$Sp_{4n}(n-1, n-1, n-2, n-2, \ldots, 0, 0; x^2, 1, \ldots, 1) \\ = 3^{n(n-1)} \left(\frac{\sigma(q\bar{x})}{\sigma(q^2)} \right)^{2n-2} \sum_{i=2}^{2n} \mathrm{A_O}(2n, i) z^{-i+2},$$

where

$$\mathrm{A_O}(2n, i) = \begin{cases} 0, & \text{if } i = 0, 1; \\ \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!} \sum_{k=1}^{i-1} (-1)^{i+k-1} \frac{(2n+k-2)!(4n-k-1)!}{(4n-2)!(k-1)!(2n-k)!}, & \text{for } i \geq 2. \end{cases} \tag{7}$$

Final Step

All that is now left to be done is to combine all of the previous equations to get the following relation

$$\mathsf{A}_{\mathsf{V}}(2n+1,i) = \mathsf{A}_{\mathsf{O}}(2n,i) + \mathsf{A}_{\mathsf{O}}(2n,i+1). \tag{8}$$

Putting in the values of $\mathsf{A}_{\mathsf{O}}(2n,i)$ will now give us the theorem

$$\mathsf{A}_{\mathsf{V}}(2n+1,i) = \frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!} \prod_{k=1}^{n-1} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!}.$$

Descending Plane Partitions

A *descending plane partition* is an array of positive integers $(d_{i,j})_{1\leq i\leq r, i\leq j\leq \lambda_i+i-1}$ of the form

$$\begin{array}{ccccccc} d_{1,1} & d_{1,2} & d_{1,3} & & \cdots & & d_{1,\lambda_1} \\ & d_{2,2} & d_{2,3} & & \cdots & & d_{2,\lambda_2+1} \\ & & \ddots & & & \iddots & \\ & & & d_{r,r} & \cdots & d_{r,\lambda_r+r-1} & \end{array}$$

such that

- all row entries are weakly decreasing,
- all column entries are strictly decreasing,
- the number of entries in each row is strictly less than the first entry in the same row and is at least as large as the first entry in the following row.

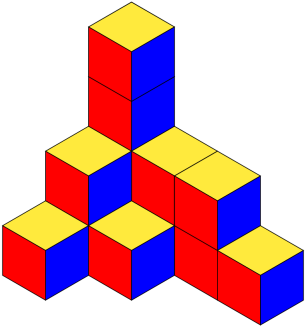
It is known that the number of descending plane partitions with entries at most n are equinumerous with order n ASMs.

Plane Partitions

A *plane partition* in an $a \times b \times c$ box is a subset

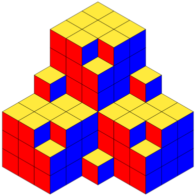
$$PP \subset \{1, 2, \cdots, a\} \times \{1, 2, \cdots, b\} \times \{1, 2, \cdots, c\}$$

with $(i', j', k') \in PP$ if $(i, j, k) \in PP$ and $(i', j', k') \leq (i, j, k)$.



Totally Symmetric Self-Complementary Plane Partitions

If a plane partition has all the symmetries (that is, $(i, j, k) \in PP$ if and only if all six permutations of (i, j, k) are also in PP)and is its own complement (that is, if $(i, j, k) \in PP$ then $(2n + 1 - i, 2n + 1 - j, 2n + 1 - k) \notin PP$), then it is called *totally symmetric self-complementary plane partitions* (TSSCPP).



The class of TSSCPPs inside a $2n \times 2n \times 2n$ box are equinumerous with $n \times n$ ASMs.

Alternating Sign Triangles (ASTs)

An AST of size n is a triangular array

$$\begin{array}{ccccc} a_{1,1} & a_{1,2} & \dots & a_{1,2n-2} & a_{1,2n-1} \\ & a_{2,2} & \dots & a_{2,2n-2} & \\ & & \vdots & & \\ & & a_{n,n} & & \end{array}$$

such that

- the entries are either $1, -1$ or 0,
- along the columns and rows the non-zero entries alternate,
- the first non-zero entry from the top is a 1 and the rowsums are equal to 1.

Example

Following is an AST of order 3.

$$\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ & 1 & -1 & 1 & \\ & & 1 & & \end{array}$$

ASTs of order n are equinumerous with ASMs of order n.

There are other combinatorial objects which are equinumerous with ASMs, and one of the major open problems in enumerative combinatorics is to find bijections between such objects.

Recently, Fischer and Konvalinka have given a bijective proof of the DPP-ASM part.

Thank you for your attention!