Hard and Easy Instances of L-Tromino Tilings [1]

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30 June 2020
ICMMAS, Dibrugarh University

[1]Lecture Notes in Comp. Sc. (LNCS), Vol. 11355 (Springer) (2019).
Theoretical Computer Science (Elsevier), Vol. 815 (2020).

Outline

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1. Introduction
   - Polyominoes
   - L-Tromino Tiling Problem
   - Computational Complexity
2. Tiling of the Aztec Rectangles
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   - Aztec Rectangle with a single defect
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3. 180-Tromino Tiling
   - A rotation constraint
   - Forbidden Polyominoes
4. Open Problems

Polyominoes

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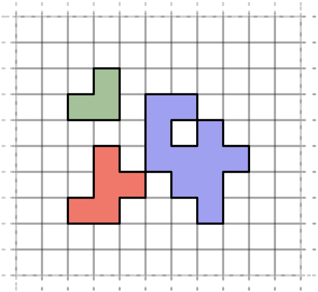

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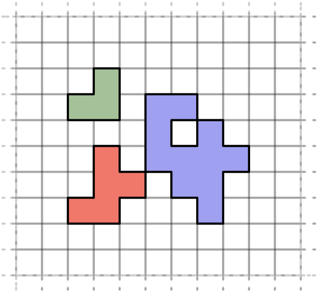

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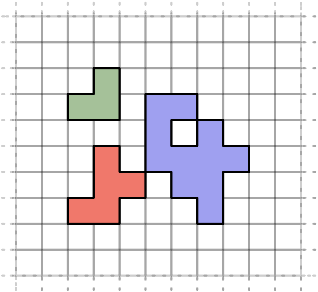

- Every cell (square) is fixed in a square lattice.
- Two cell are adjacent if the Manhattan distance is 1.

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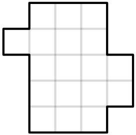
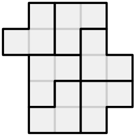


(a) A region R



(b) A tiling of region R

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- Complexity classes are concerned with the rate of growth of the requirement in resources as the input increases.
- Time complexity is commonly estimated by counting the number of elementary operations performed by the algorithm, supposing that each elementary operation takes a fixed amount of time to perform.
- If the time complexity is polynomial in the input parameters, then we say that a problem can be solved in Polynomial time.

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- Example: Subgraph isomorphism problem.

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- T. Horiyama, T. Ito, K. Nakatsuka, A. Suzuki and R. Uehara (2012) constructed a one-one reduction from 1-in-3 SAT.

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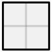

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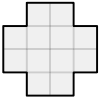
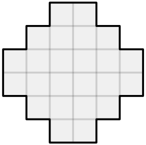
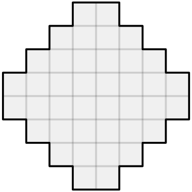


(b) $AD(2)$



(c) $AD(3)$



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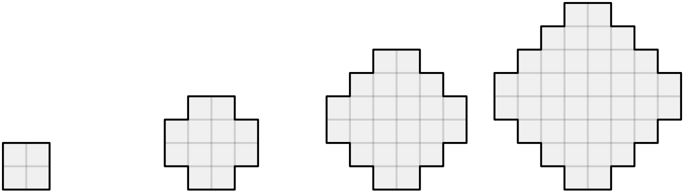

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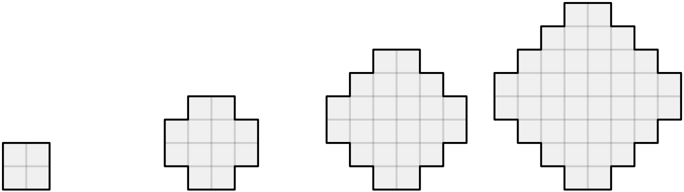

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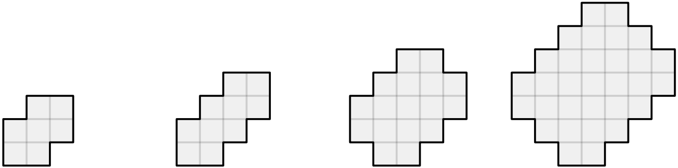

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Tiling Aztec Rectangle (cont'd)

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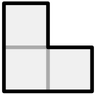
Each piece of L-tromino **covers 3 cells**.



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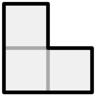

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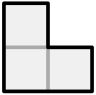

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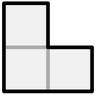

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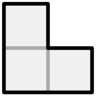

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Theorem

An Aztec rectangle $\mathcal{AR}_{a,b}$ has a tiling with L-trominoes
$\iff |\mathcal{AR}_{a,b}| \equiv 0 \pmod 3$
$\iff (a,b)$ *is equal to* $(3k, 3k')$ *or* $(3k-1, 3k'-1)$ *for some* $k, k' \in \mathbb{N}$.

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A decision problem is P-complete if it is in P and every problem in P can be reduced to it by an appropriate reduction.

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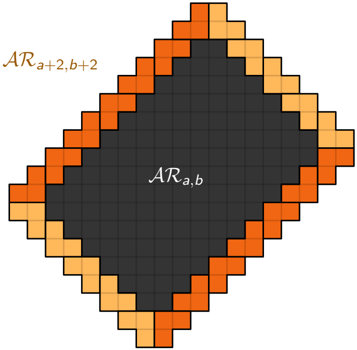

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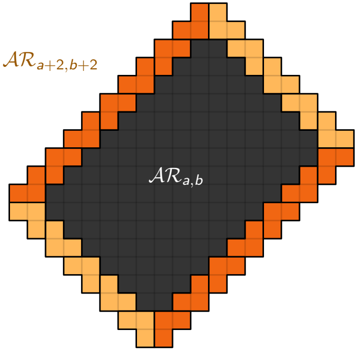

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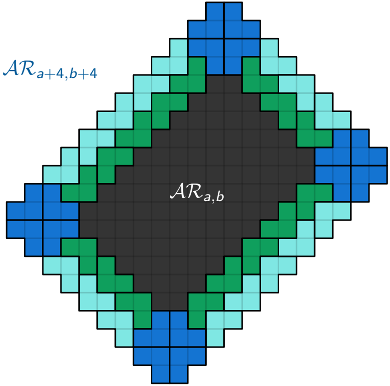

(b) Pattern 2

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- If (a, b) equals $(3, 3k')$, use pattern 3.
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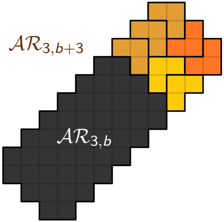

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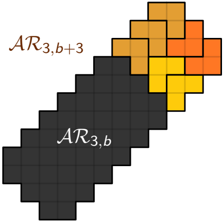

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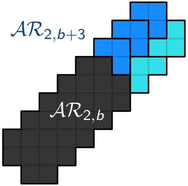

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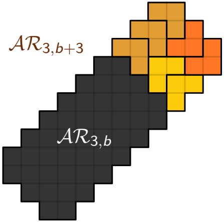

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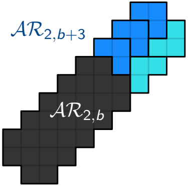

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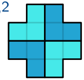

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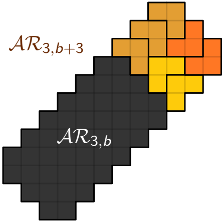

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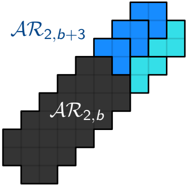

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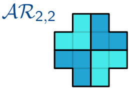

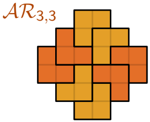

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5. Return R.

$\square$

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- Steps 3.2 and 4.2 can be done in time $O(b)$.
- Giving a total time complexity of $O(b^2)$.

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Tiling Aztec Rectangle with a single defect

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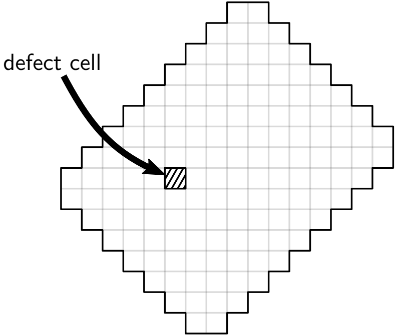

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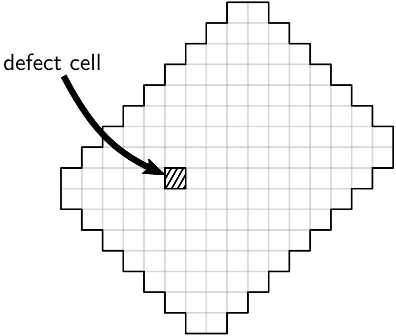


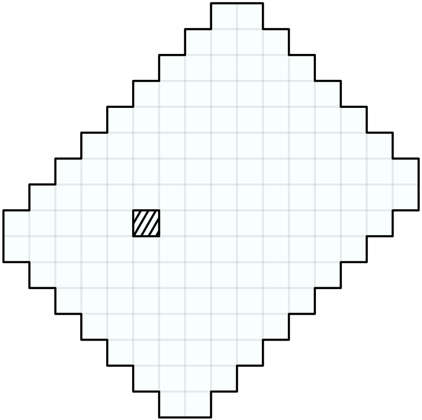
Theorem

An Aztec rectangle $\mathcal{AR}_{a,b}$ with one defect has a tiling with L-trominoes

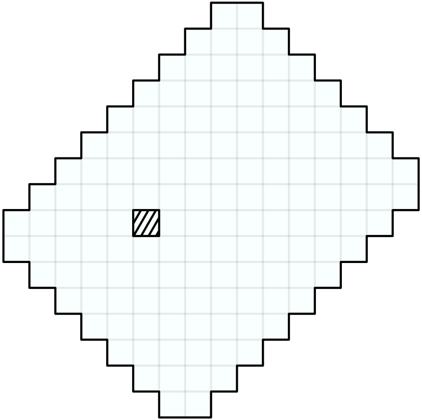
$\iff$ $|\mathcal{AR}_{a,b}| \equiv 1 \pmod 3$

$\iff$ *a or b is equal to $3k - 2$ for some $k \in \mathbb{N}$.*

Tiling Aztec Rectangle with a single defect (cont'd)



- Place a *fringe* where it covers the defect.



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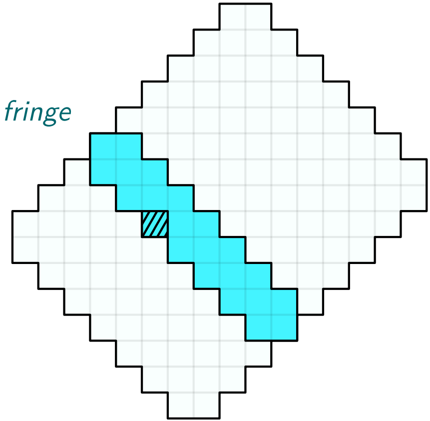

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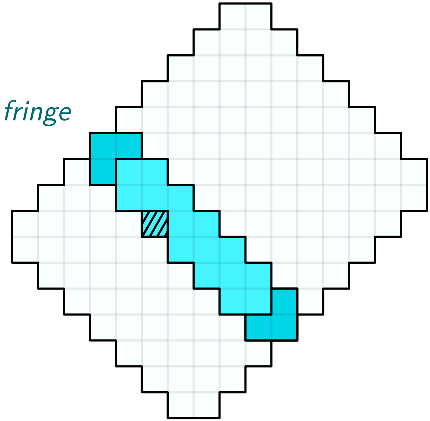

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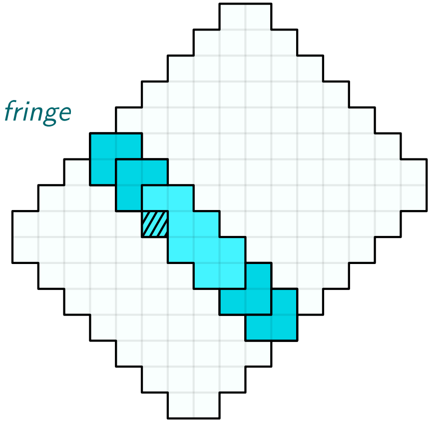

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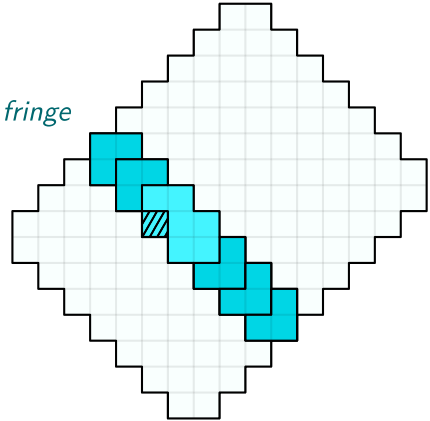

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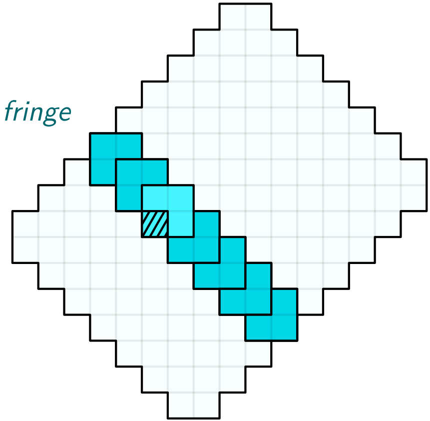

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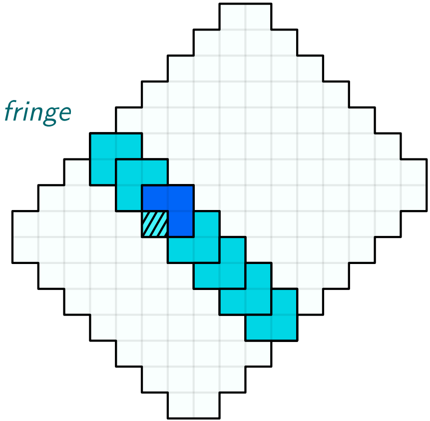

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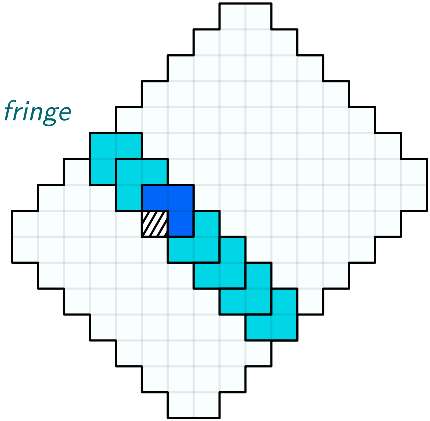

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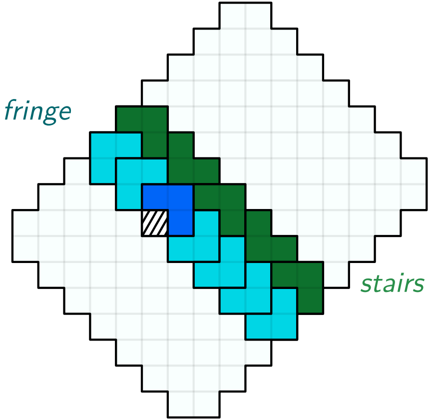

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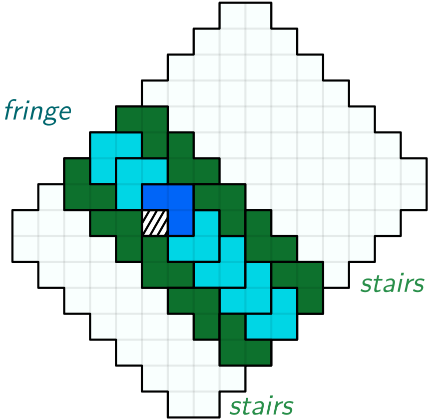

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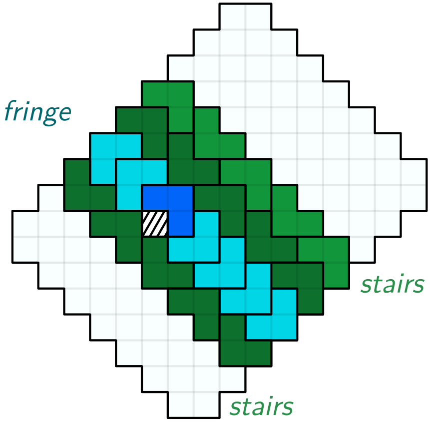

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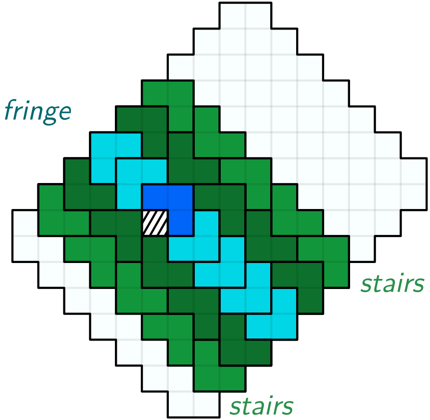

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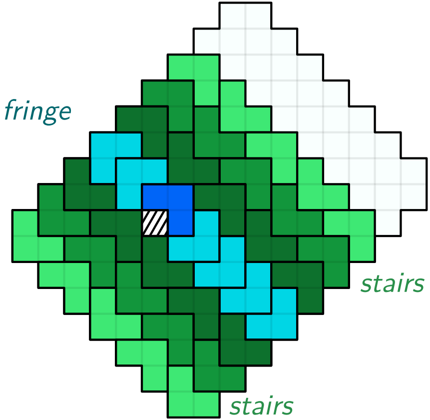

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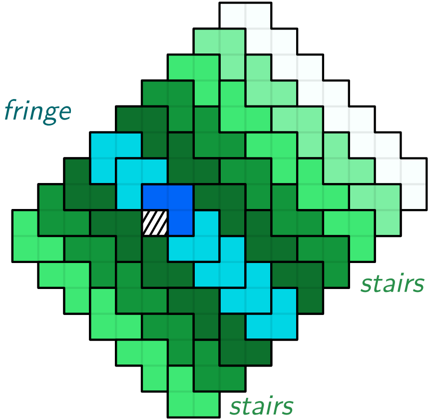

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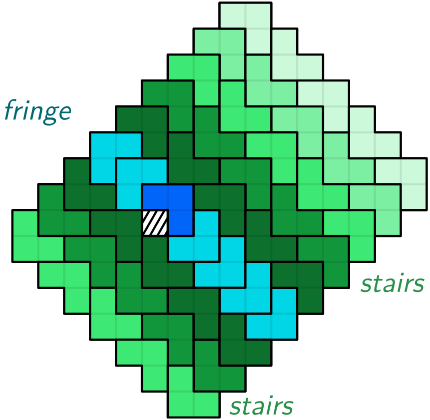

Outline

Tiling Aztec Rectangle with an unbounded number of defects

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Given a region R', we can embed R' inside a sufficiently large Aztec Rectangle $\mathcal{AR}_{a,b}$.

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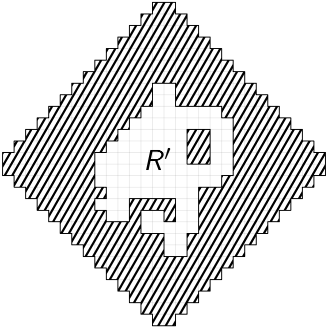

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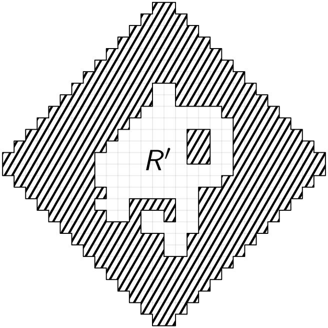

Theorem

The problem of tiling Aztec Rectangle $\mathcal{AR}_{a,b}$ with an unbounded number of defects is ***NP-complete****.*

Outline

180°L-Tromino Tiling

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Definition

The 180-tromino tiling problem only allows 180° rotations of L-trominoes, i.e., the tile set can be

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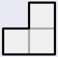
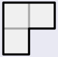
$\Sigma = \{$ **right-oriented** 180-trominoes $\} = \{$ ,  $\}$

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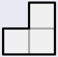
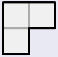
$\Sigma = \{$ **right-oriented** 180-trominoes $\} = \{$ ,  $\}$

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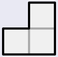
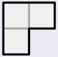
$\Sigma = \{$ **right-oriented** 180-trominoes $\} = \{$ ,  $\}$

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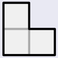
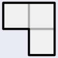
$\Sigma = \{$ **left-oriented** 180-trominoes $\} = \{$ ,  $\}$.

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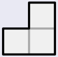
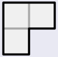
$\Sigma = \{$ **right-oriented** 180-trominoes $\} = \{$ ,  $\}$

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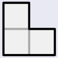
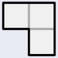
$\Sigma = \{$ **left-oriented** 180-trominoes $\} = \{$ ,  $\}$.

With no loss of generality, we will only consider **right-oriented 180-trominoes**.

180°L-Tromino Tiling (cont'd)

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Theorem

There is a one-one correspondence between 180-tromino tiling and the triangular trihex tiling [Conway and Lagarias, (1990)].

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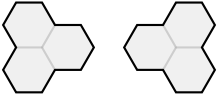

Two **triangular trihex**.

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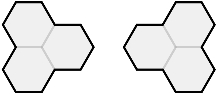

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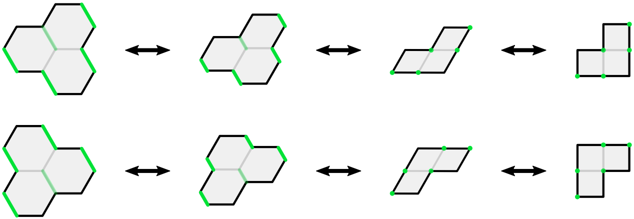

Transformation from **triangular trihex** to **180-tromino**

180°L-Tromino Tiling (cont'd)

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If there is a I-tromino tiling for some R, then there is also a 180-tromino tiling for $R^{\boxplus}$.

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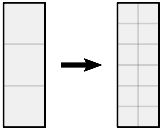

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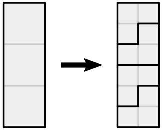

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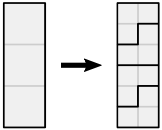

However, it is not known if the converse statement is true or false.

180°L-Tromino Tiling (cont'd)

Horiyama et al. also proved that the I-tromino tiling problem is **NP-Complete**.

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Theorem [Horiyama, Ito, Nakatsuka, Suzuki and Uehara (2012)]

1-in-3 SAT $\leq_P$ I-tromino Tiling

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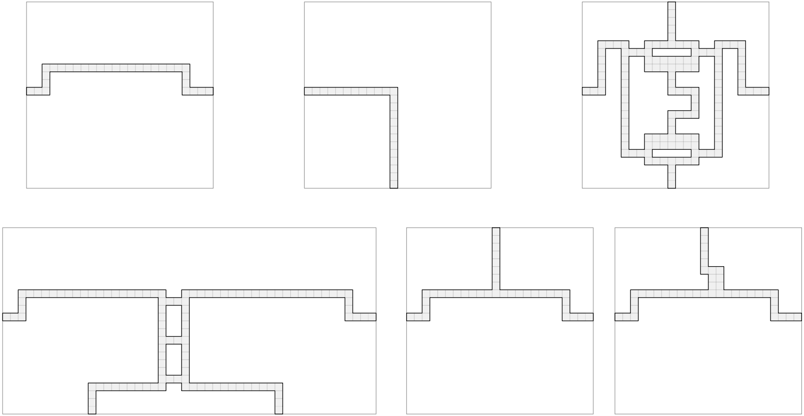

180°L-Tromino Tiling (cont'd)

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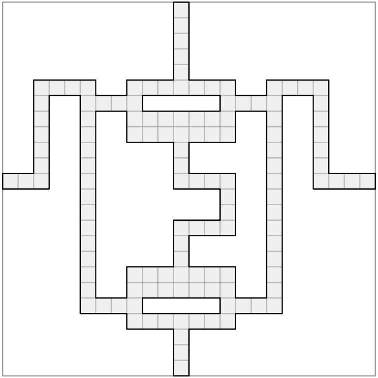

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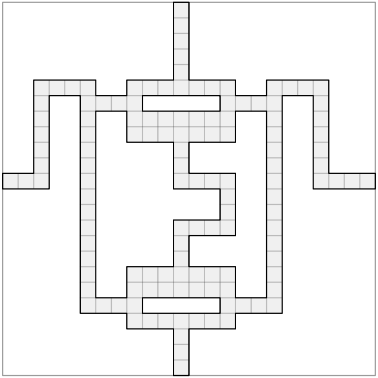

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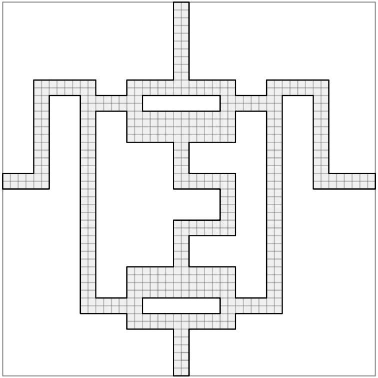

(b) Tetrasected gadget $G^{\boxplus}$.

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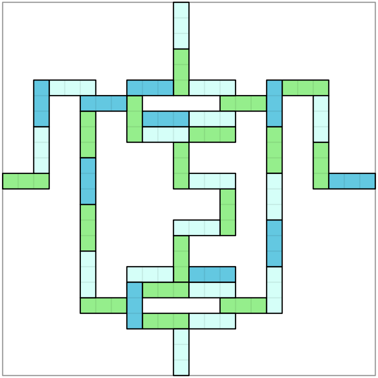

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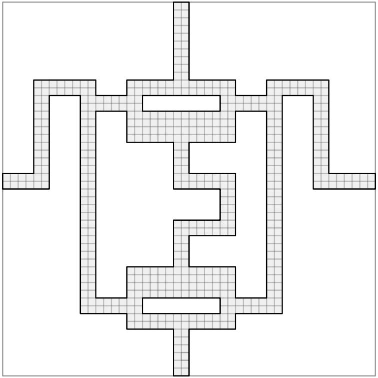

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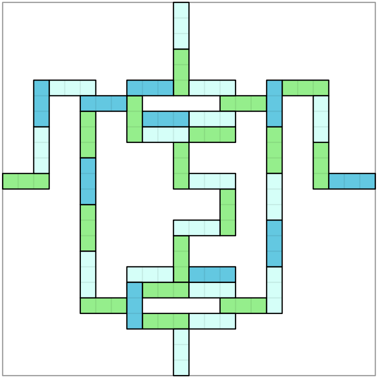

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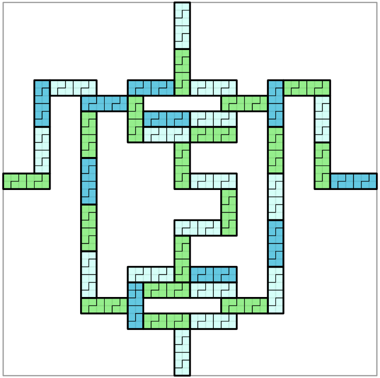

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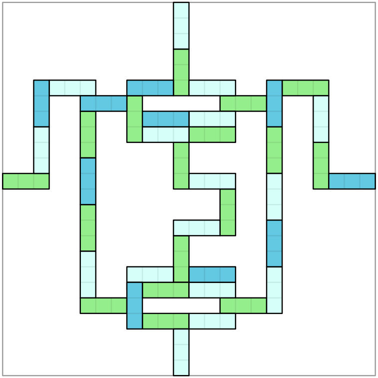

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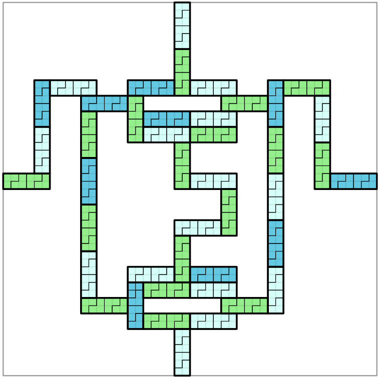

(b) Tetrasected gadget $G^{\boxplus}$.

Theorem

180-tromino tiling is ***NP-complete****.*

Outline

The 180-tromino tiling can also be reduced to the **Maximum Independent Set** problem.

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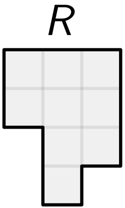

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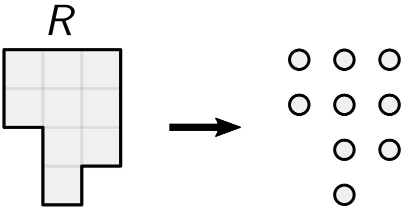
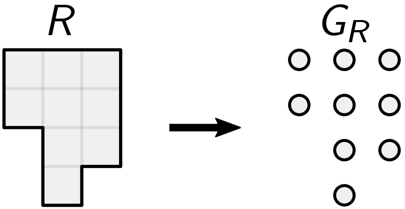


- Transformation from R to G_R:
  - Transform every cell of R to vertices of G_R.
  - Add horizontal, vertical and northeast-diagonal edges.

The 180-tromino tiling can also be reduced to the **Maximum Independent Set** problem.



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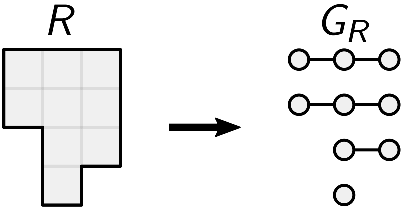

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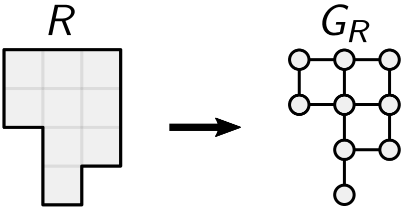

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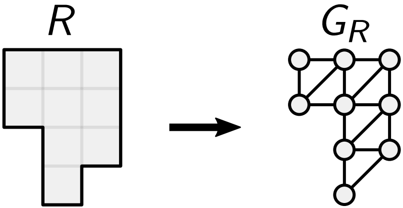

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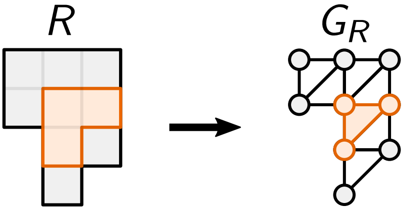

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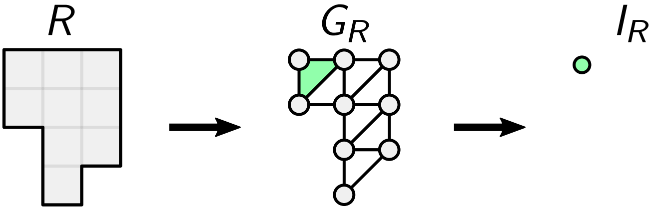

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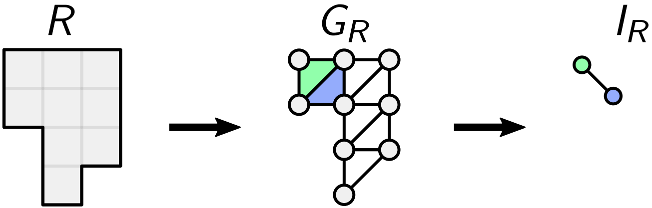

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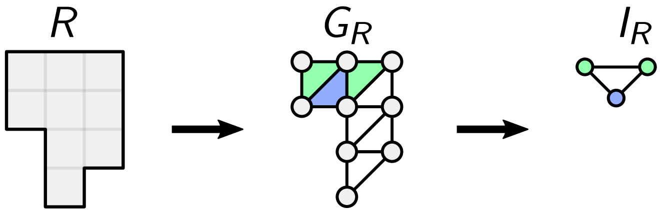

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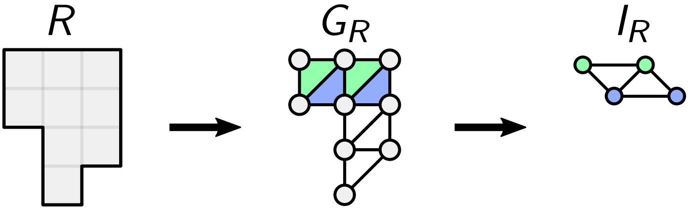

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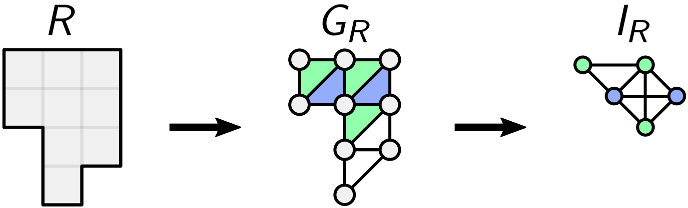

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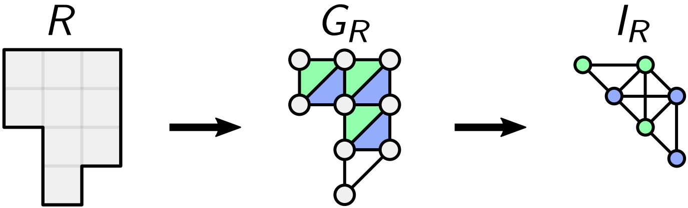

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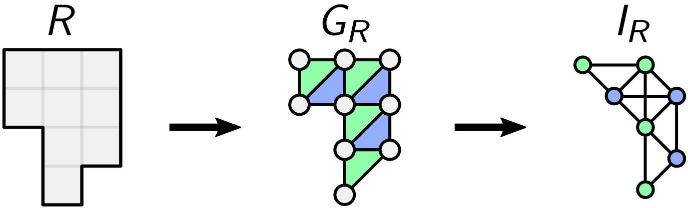

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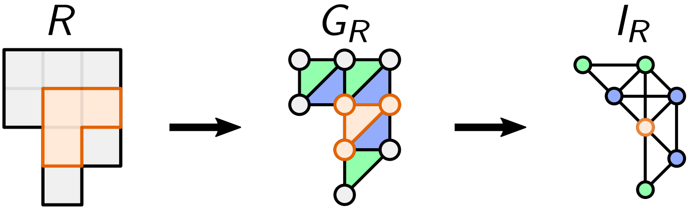

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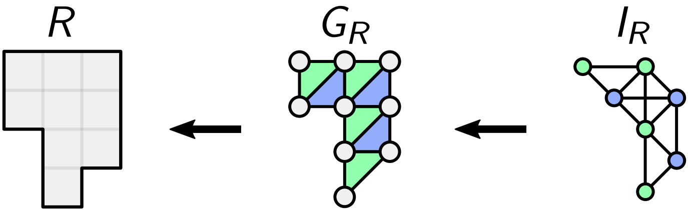
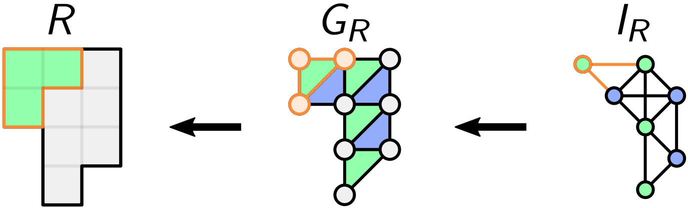


R
G_R
I_R

Forbidden Polyominoes (cont'd)



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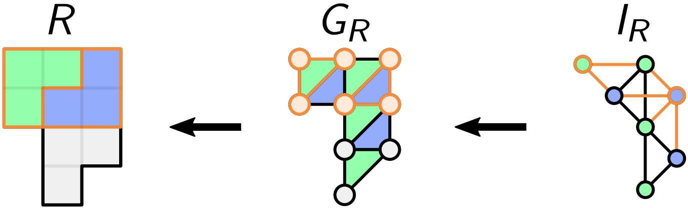

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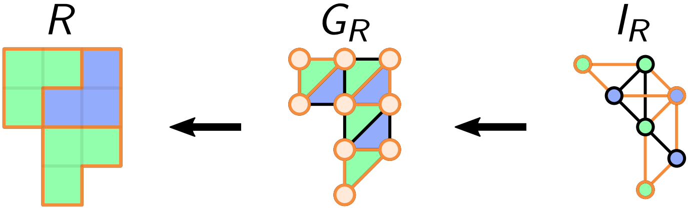

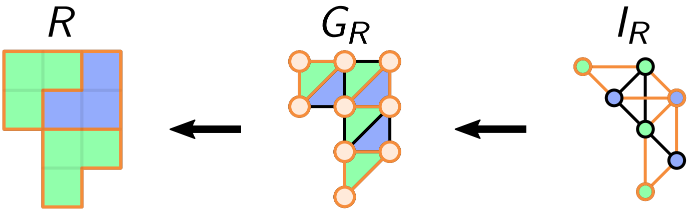

Theorem

Maximum Independent Set of I_R is equal to $\frac{|R|}{3}$
$\iff$ *R has a 180-tromino tiling .*

where $|R|$ the number of cells in a region R.

Forbidden Polyominoes (cont'd)

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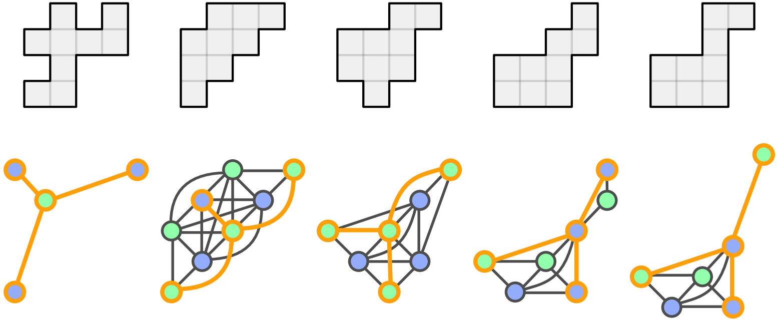
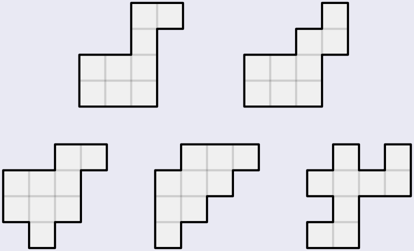


Forbidden Polyominoes (cont'd)

Theorem

*If a region R **doesn't** contains a rotated, reflected or sheared **forbidden polyomino**, then 180-tromino tiling can be computed in a polynomial time.*



Open Problems

1. *Hardness of tiling the Aztec rectangle with a given number of defects.* We saw that an Aztec rectangle with 0 or 1 defects can be covered with L-trominoes in polynomial time, whereas in general the problem is NP-complete when the Aztec rectangle has an unknown number of defects. It is open if there exists a polynomial time algorithm for deciding a tiling for an Aztec rectangle with a given number of defects.

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2. *Tiling of orthogonally-convex regions.* In this work we showed several instances where a tiling can be found in polynomial time. In general, it is open if an orthogonally-convex region with no defects can be covered in polynomial time or if it is NP-complete to decide if a tiling exists.

Enumeration of tilings

- We have not considered the problem of enumerating tromino tilings of the regions described in this talk. In general, there are no such formulas known in the literature for the shapes studied so far.

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If $n = 3k$ for some $k > 0$, then we have

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It appears that these bound can be improved substantially.

Thank you!

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You can try the tetrasected cell tiling program in your phone browser: http://bit.ly/TetrasectedTiling