Cataland: A Romance of Many Bijections

Manjil P. Saikia

27 November, 2024

Lattice Paths Inside a Square

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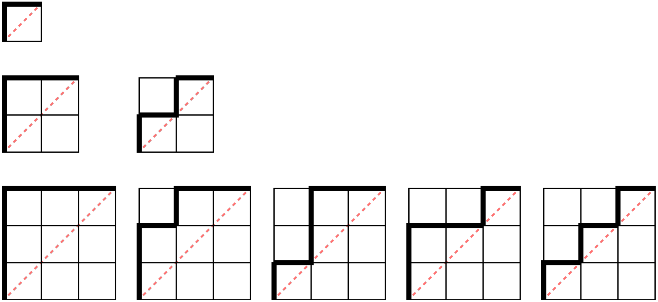

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This bijection clearly shows that $a_1(n)$, the number of lattice paths that do not fall below the diagonal is the same as $a_2(n)$, the number of those that do not rise above it.

Fillings of a Rectangular Grid

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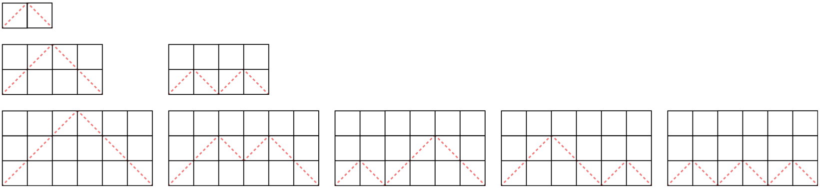

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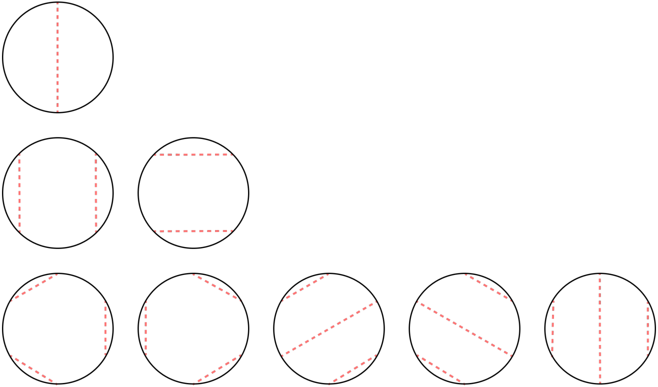

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Count $a_6(n)$, the number of legal sequences of $2n$ parentheses, where, by a legal sequence of parentheses we mean one in which the parentheses can be properly matched, i.e., each opening parenthesis should be matched to a closing one that lies further to its right.

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\begin{aligned}
n = 1 : &\quad () \\
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n = 3 : &\quad ((())) \,, ()\,(()) \,, (())\,() \,, ()\,()\,()
\end{aligned}
$$

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Count $a_7(n)$, the number of non-crossing partitions on the set $[2n] := \{1, 2, 3, \ldots, 2n\}$, where by a non-crossing partition on $[2n]$ we refer to an arrangement of $2n$ points on a line, with n non-intersecting arcs joining them.

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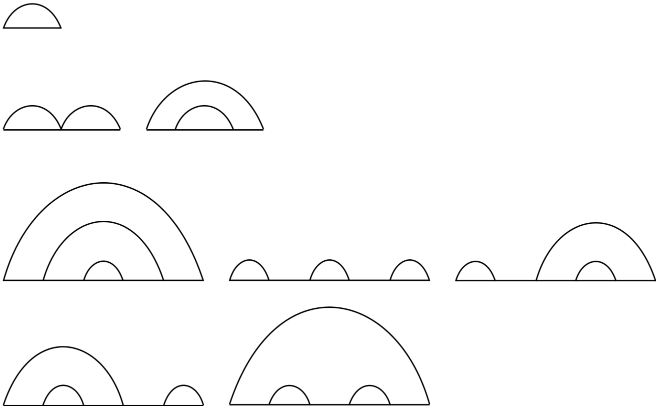

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Not only these 7 combinatorial objects, but more than 200+ other objects are counted by the same numbers.

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or, by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

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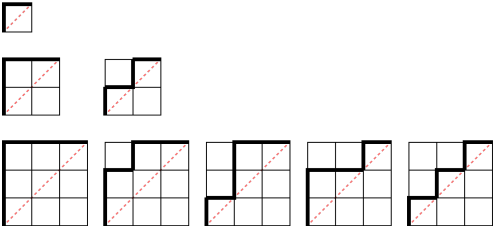

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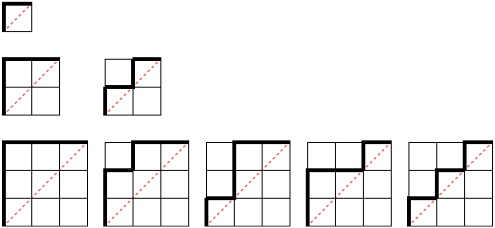

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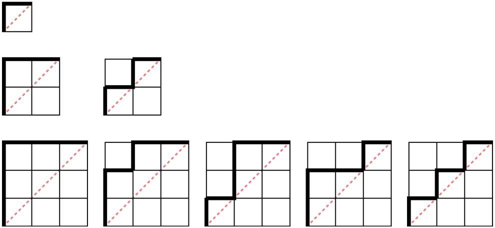

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The first row keeps track of the E steps, while the second row keeps track of the N steps.

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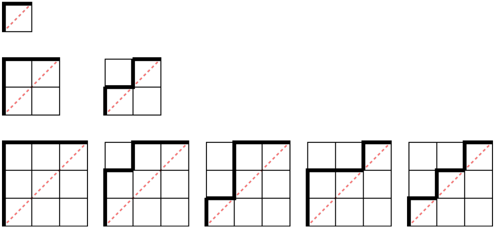

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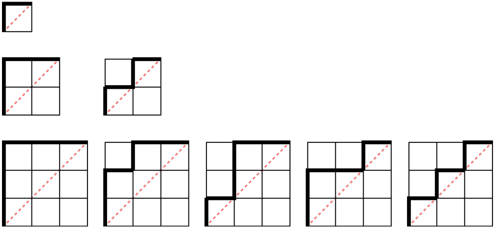

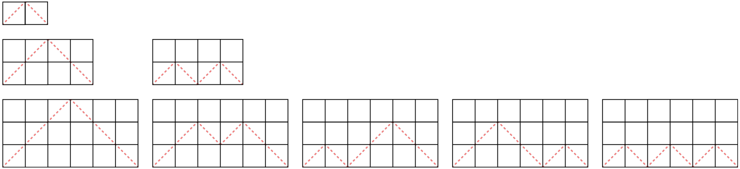

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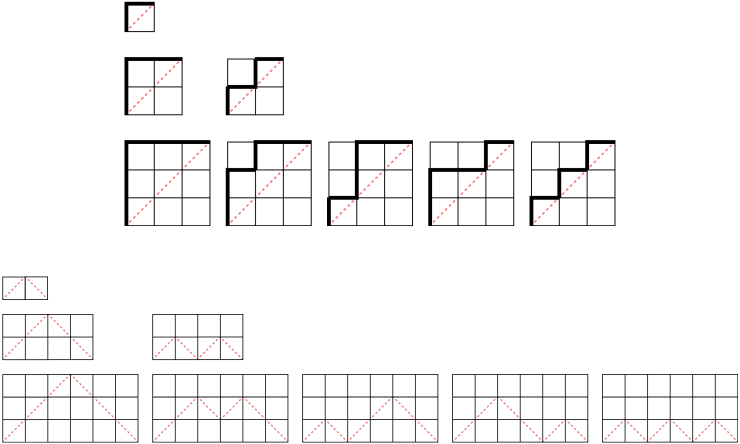

Just rotate!

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We now prove the recurrence

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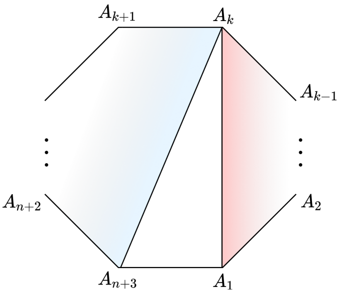

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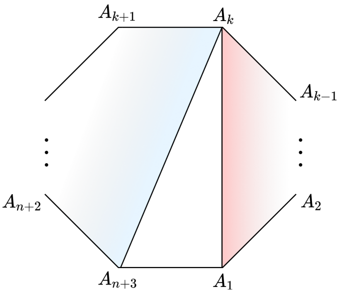

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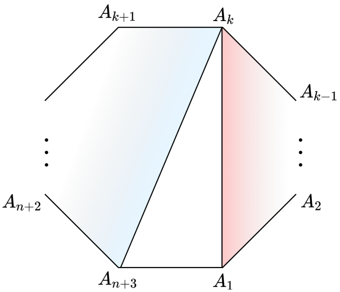

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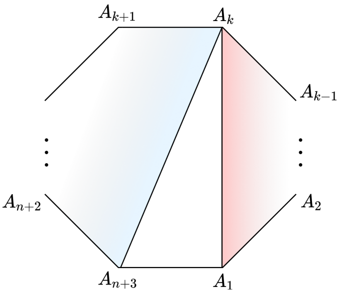

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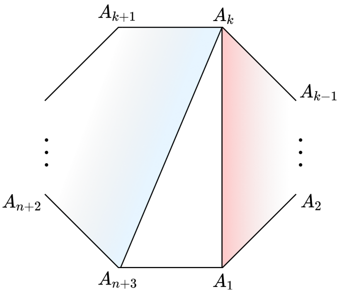

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Solving for $C(x)$ gives

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

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Pictures

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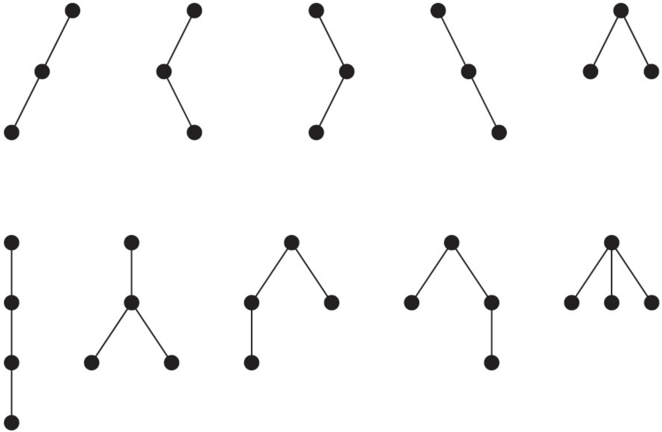

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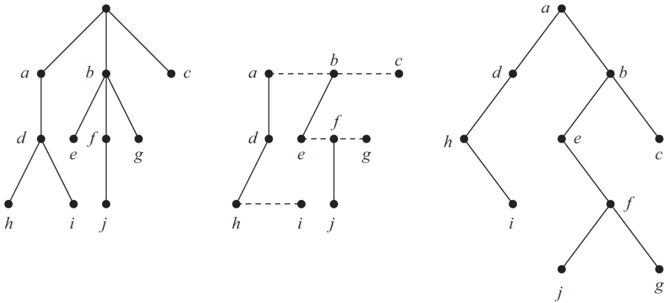

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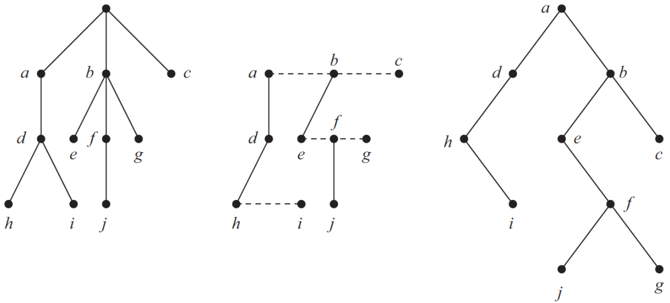

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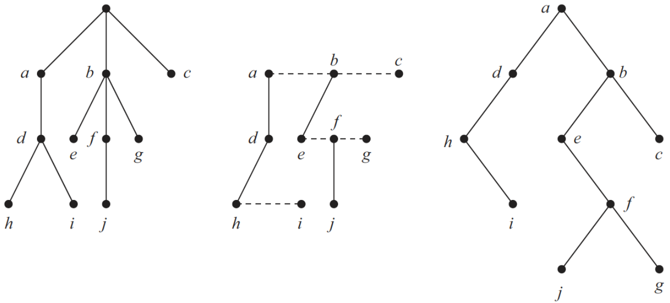

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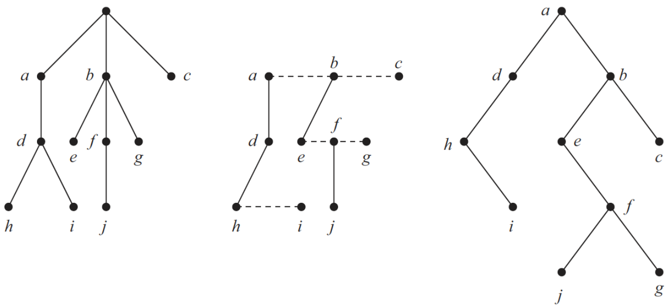

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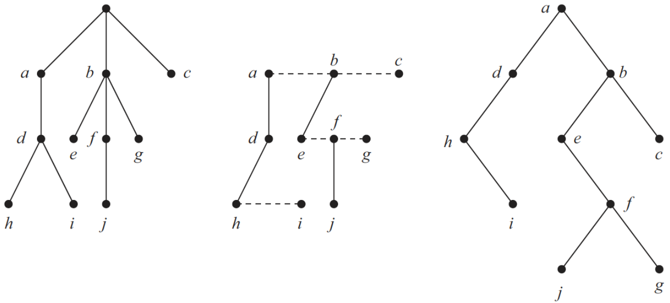

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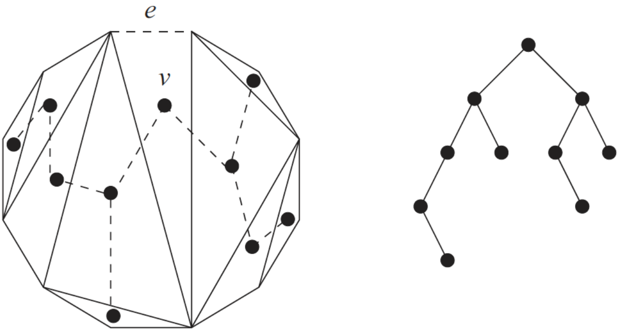

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Let σ be a bijection on $[n]$. An inversion of σ is a tuple $(\sigma(i), \sigma(j))$ such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$.

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(123)	No Inversions	Trivial
(132)	(32)	$2 < 3$ but $3 > 2$
(213)	(21)	$1 < 2$ but $2 > 1$
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If I_n denote the number of inversions on $[n]$, then $I_n = \binom{n}{2} \frac{n!}{2}$.

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1. If $(\sigma(i), \sigma(j))$ is an inversion of σ, then it's not an inversion of σ's mate.
2. Each one of the $\binom{n}{2}$ pairs is an inversion exactly once in each couple.

q-Counting

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In the formal variable q, we define the inversion polynomial on $[n]$ as

$$\sum_{\sigma \in S_n} q^{\mathrm{inv} \sigma}$$

where $\mathrm{inv}(\sigma)$ denotes the number of inversions of σ. As an example, we compute the inversion polynomial on the set $[3]$.

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(4231)	(42), (43), (41), (21), (31)	q^{2+3}
(4312)	(43), (41), (42), (31), (32)	q^{2+3}
(4321)	(43), (42), (41), (32), (31), (21)	q^{3+3}

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q-analogs

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$$\begin{aligned}\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} &= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) \\ &= \frac{1-q}{1-q}\cdot\frac{1-q^2}{1-q}\cdot\frac{1-q^3}{1-q}\cdots\frac{1-q^n}{1-q} \\ &= [1]_q[2]_q\cdots[n]_q \\ &= n!_q\end{aligned}$$

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\end{aligned}
$$

For appropriate choices of n and k, we denote the q-analogue of $\binom{n}{k}$ by

$$
\binom{n}{k}_q = \frac{n!_q}{(n-k)!_q k!_q}.
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$$\sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = n!_q.$$

q-Catalan Recurrence

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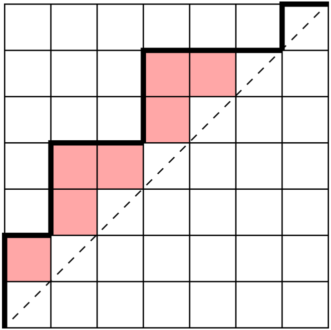

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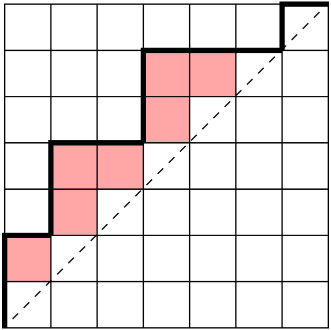

If we set $C_n(q) = \sum_{\pi \in L^+} q^{\text{area}(\pi)}$, then

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_k(q) C_{n-k}(q).$$

Quotable Quotes

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Mathematicians are often amazed that certain mathematical objects (numbers, sequences, etc.) show up so often. For example, in enumerative combinatorics, we encounter the Fibonacci and Catalan sequences in many problems that seem to have nothing to do with each other. [..] The answer, once again, is our human predilection for triviality. [..] The Catalan sequence is the simplest sequence whose generating function is a (genuine) algebraic formal power series. – Doron Zeilberger

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Catalan numbers are even more fascinating [than the Fibonacci numbers]. Like the North Star in the evening sky, they are a beautiful and bright light in the mathematical heavens. They continue to provide a fertile ground for number theorists, especially, Catalan enthusiasts and computer scientists. – Thomas Koshy

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I'd have to say my favorite number sequence is the Catalan numbers. [..] Catalan numbers just come up so many times. It was well-known before me that they had many different combinatorial interpretations. [..] When I started teaching enumerative combinatorics, of course I did the Catalan numbers. When I started doing these very basic interpretations – any enumerative course would have some of this – I just liked collecting more and more of them and I decided to be systematic. – Richard Stanley

References

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Thank you for your attention!