Arithmetic Properties of Overpartition k-Tuples with Odd Parts

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Joint with

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- Arithmetic Properties Modulo Powers of 2 for Overpartition k-Tuples with Odd Parts (with A. Sarma and J. A. Sellers) out on arXiv: 2312.12011

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so $p(4) = 5$.

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For brevity, we set $f_k := (q^k;q^k)_\infty$.

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Formally, first studied by Corteel and Lovejoy (but appeared earlier, as well).

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The arithmetic properties were first studied by Keister, Sellers, and Vary.

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- the cases $k \geq 4$ had not been studied before.

Motivation

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A notable topic in the world of q-series revolves around the arithmetic properties of the coefficients $c(n)$ generated by

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We look for congruences of the form

$$c(An+B) \equiv C \pmod{M},$$

holding for any $n \geq 0$, in which the modulus M and the parameters A, B and C are fixed, with C usually being 0.

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$$p(\ell^\alpha n + \delta_{\alpha,\ell}) \equiv \begin{cases} 0 \pmod{\ell^\alpha} & \ell = 5, 11, \\ 0 \pmod{7^{\lceil \frac{\alpha+1}{2} \rceil}} & \ell = 7, \end{cases}$$

with $0 \leq \delta_{\alpha,\ell} \leq \ell^\alpha - 1$ being such that

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Watson proved the cases of powers of 5 and 7, while Atkin confirmed the case of powers of 11.

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$$\overline{OPT}_3(16n+14) \equiv 0 \pmod{8},$$

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For any integer $n \geq 0$, we have

$$\overline{OPT}_3(3n + i) \equiv 0 \pmod 3,$$
$$\overline{OPT}_3(24(3n + j) + 3) \equiv 0 \pmod 3,$$

where $i = j = 1, 2$.

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A similar result also holds for $k = 3 \pmod 8$, $k = 4 \pmod{16}$, and for odd $k \pmod 4$.

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Using the binomial theorem and a lemma (involving congruences of binomial coefficients), we can conclude

$$\sum_{n\geq 0} \overline{OPT}_k(n)q^n \equiv 1 \pmod{2^{m+1}}.$$

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$$\sum_{n\geq 0} \overline{OPT}_t(n)q^n = \phi(q)^t\phi(q^2)^t\phi(q^4)^{2t}\phi(q^8)^{4t}\cdots,$$

$$= \left(\sum_{j=0}^{7} a_{t,j}q^jF_{t,j}(q^8)\right)\left(\prod_{i\geq 3}\phi(q^{2^i})\right)^{2^{i-1}\cdot t}.$$

The result now follows easily from the lemma. □

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We can now prove some cases of the above.

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- The first case follows immediately from a previous observation.
- The second case uses tricky case by case analysis of divisibility of binomial coefficients.
- The third case follows from the following: Let $k = 2^m r$, $m > 0$ and r be odd, then for all $n \geq 1$ we have

$$\overline{OPT}_{2^m r}(n) \equiv \begin{cases} 2^{m+1} \pmod{2^{m+2}} & \text{if } n \text{ is sq, } 2\times \text{ sq or } 4\times \text{ sq,} \\ 0 \pmod{2^{m+2}} & \text{otherwise.} \end{cases}$$

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We have proofs for several cases, and a conjectured lemma which would give the proof for all cases.

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This is stronger than the conjecture from SSS, and generalizes some results of Adiga and Dasappa (for the case $k = 1, i = 1$) and DSSS (for the case $\alpha = 1$).

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Internal congruences modulo an arbitrary power of a number are not known widely.

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This doesn't seem to generalize as easily.

Thank you for your attention!