Tiling Problems and Perfect Matchings

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Tiling Problems

Combinatorialists study many different kinds of tiling problems. One of the simplest, is to find the number of ways to tile a $2 \times n$ board with dominoes.

Figure: Two types of Dominoes

Figure: A $2 \times n$ board

Tiling Problems

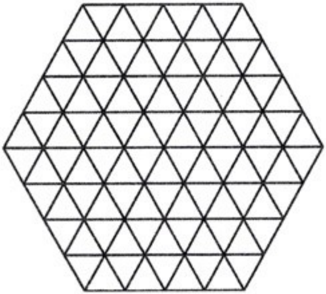
Figure: First choice of the right most corner placement

Figure: Second choice of the right most corner placement

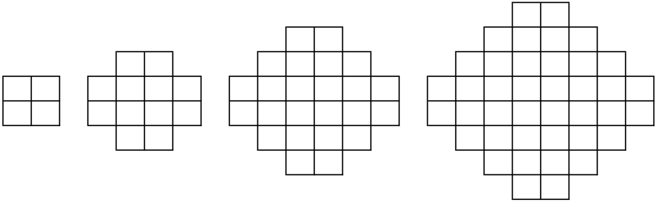
So, these are just the Fibonacci numbers!

But not all tiling problems are easy!

Tiling a hexagon



Tiling a diamond



We will focus on these objects, in this talk.

Aztec Diamonds

- In 1991, Elkies, Kuperberg, Larsen and Propp introduced a new class of object which they called Aztec Diamonds.
- The Aztec Diamond of order n (denoted by AD(n)) is the union of all unit squares inside the contour $|x| + |y| = n + 1$

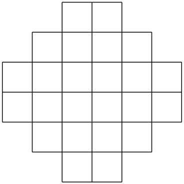


Figure: AD(3), Aztec Diamond of order 3

Aztec Diamonds

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Figure: A Mezoamerican pyramid

Aztec Diamond Theorem

- A domino is the union of any two unit squares sharing an edge, and a domino tiling of a region is a covering of the region by dominoes so that there are no gaps or overlaps.
- They considered the problem of counting the number of domino tiling of the Aztec Diamond and presented four different proofs of the following result.

Theorem (Elkies–Kuperberg–Larsen–Propp)

The number of domino tilings of an Aztec Diamond of order n is $2^{n(n+1)/2}$.

We will prove this result in this talk.

Aztec Rectangle

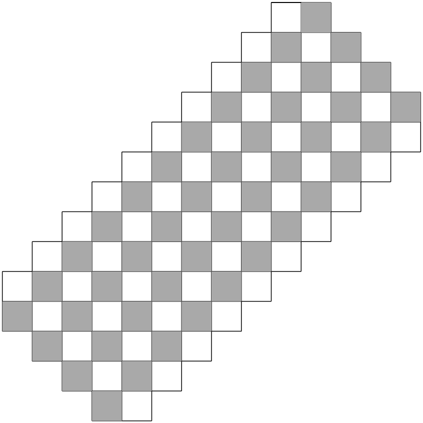

Figure: Checkerboard representation of an Aztec Rectangle

Aztec Rectangles

- We denote by $\mathcal{AR}_{a,b}$ the Aztec rectangle which has a unit squares on the southwestern side and b unit squares on the northwestern side.
- For $a < b$, $\mathcal{AR}_{a,b}$ does not have any tiling by dominoes.
- The non-tileability of the region $\mathcal{AR}_{a,b}$ becomes evident if we look at the checkerboard representation of $\mathcal{AR}_{a,b}$

Aztec Rectangle Theorem

If we remove $b - a$ unit squares from the southeastern side then we have a simple product formula found by Mills, Robbins and Rumsey.

Theorem (Mills-Robbins-Rumsey)

Let $a < b$ be positive integers and $1 \leq s_1 < s_2 < \cdots < s_a \leq b$. Then the number of domino tilings of $\mathcal{AR}_{a,b}$ where all unit squares from the south-eastern side are removed except for those in positions $s_1, s_2, \ldots, s_a$ is

$$2^{a(a+1)/2} \prod_{1 \leq i < j \leq a} \frac{s_j - s_i}{j - i}.$$

But, how does one count such tilings?

We will show one technique in this talk.

Kuo's Graphical Condensation

For a graph G, $\mathrm{M}(G)$ denotes the number of perfect matchings of G.

Theorem (Eric Kuo)

Let G be a planar graph with four vertices w, x, y, z that appear in that cyclic order on a face of G. Then

$$\mathrm{M}(G)\,\mathrm{M}(G-\{w,x,y,z\})+\mathrm{M}(G-\{w,y\})\,\mathrm{M}(G-\{x,z\})$$

$$=\mathrm{M}(G-\{w,x\})\,\mathrm{M}(G-\{y,z\})+\mathrm{M}(G-\{w,z\})\,\mathrm{M}(G-\{x,y\}).$$

Kuo's Graphical Condensation

If $G = (V_1, V_2, E)$ is bipartite, and

- $w, y \in V_1, x, z \in V_2, |V_1| = |V_2|$; second term vanishes

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Kuo's Graphical Condensation

Theorem (Eric Kuo)

Let G be a planar graph with four vertices w, x, y, z that appear in that cyclic order on a face of G. Then

$$\mathrm{M}(G)\,\mathrm{M}(G - \{w,x,y,z\}) + \mathrm{M}(G - \{w,y\})\,\mathrm{M}(G - \{x,z\})$$

$$= \mathrm{M}(G - \{w,x\})\,\mathrm{M}(G - \{y,z\}) + \mathrm{M}(G - \{w,z\})\,\mathrm{M}(G - \{x,y\}).$$

Kuo's Graphical Condensation

If $G = (V_1, V_2, E)$ is bipartite, and

- $w, y \in V_1, x, z \in V_2, |V_1| = |V_2|$; second term vanishes
- $w, x \in V_1, y, z \in V_2, |V_1| = |V_2|$; third term vanishes

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Kuo's Graphical Condensation

If $G = (V_1, V_2, E)$ is bipartite, and

- $w, y \in V_1, x, z \in V_2, |V_1| = |V_2|$; second term vanishes
- $w, x \in V_1, y, z \in V_2, |V_1| = |V_2|$; third term vanishes
- $w, x, y, z \in V_1, |V_1| = |V_2| + 2$; first term vanishes

Kuo's Graphical Condensation

Theorem (Eric Kuo)

Let G be a planar graph with four vertices w, x, y, z that appear in that cyclic order on a face of G. Then

$$\mathrm{M}(G)\,\mathrm{M}(G-\{w,x,y,z\})+\mathrm{M}(G-\{w,y\})\,\mathrm{M}(G-\{x,z\})$$

$$=\mathrm{M}(G-\{w,x\})\,\mathrm{M}(G-\{y,z\})+\mathrm{M}(G-\{w,z\})\,\mathrm{M}(G-\{x,y\}).$$

Sketch Proof

- Superimpose a perfect matching of G (blue) and a perfect matching of $G - \{w, x, y, z\}$ (red) on the same copy of G
- There is a blue-red alternating path from w to one of x, y, z
- Two such paths cannot cross, so w does not connect to y
- Switch the edges in the path of w and get a pair of matchings of $G - \{w, x\}$ and $G - \{y, z\}$ or of $G - \{w, z\}$ and $G - \{x, y\}$

Pfaffians

Let $A = (a_{i,j})$ be a $2n \times 2n$ antisymmetric matrix and Γ_n be the set of all perfect matchings of K_{2n}.Then

$$\mathrm{Pf}(A) = \sum_{\pi=\{(i_1,j_1),\ldots,(i_n,j_n)\}\in\Gamma_n} \operatorname{sgn}\pi \prod_{k=1}^{n} a_{i_k,j_k}$$

where $\operatorname{sgn}\pi = \operatorname{sgn} i_1 j_1 i_2 j_2 \ldots i_n j_n$.

- There are many ways to write π, so to see that $\mathrm{Pf}(A)$ is well-defined we can assume that $i_k < j_k$ and $i_1 < i_2 < \ldots < i_n$.
- Pfaffians have many interesting properties, such as

$$\mathrm{Pf}(A)^2 = \det(A).$$

An Example

Let $n = 2$, then

$$\mathrm{Pf}(A) = \sum_{\pi=\{(i_1,j_1),(i_2,j_2)\}\in\Gamma_2} \operatorname{sgn}\pi \prod_{k=1}^{2} a_{i_k,j_k}$$

$$\mathrm{Pf}(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}.$$

Some notations

- We consider the symmetric difference on the vertices and edges of a graph.
- Let H be a planar graph and G be an induced subgraph of H and let $W \subseteq \mathsf{V}(H)$.
- Then we define $G + W$ as follows: $G + W$ is the induced subgraph of H with vertex set $\mathsf{V}(G + W) = \mathsf{V}(G) \Delta \mathsf{V}(W)$, where Δ denotes the symmetric difference of sets.

Our generalization of Kuo's result

Theorem

Let H be a planar graph and let G be an induced subgraph of H with the vertices $a_1, a_2, \ldots, a_{2k}$ appearing in that cyclic order on a face of H. Consider the skew-symmetric matrix $A = (a_{ij})_{1 \le i,j \le 2k}$ with entries given by

$$a_{ij} := \mathrm{M}(G + \{a_i, a_j\}), \textit{if } i < j. \tag{1.1}$$

Then we have that

$$\mathrm{M}(G + \{a_1, a_2, \ldots, a_{2k}\}) = \frac{\mathrm{Pf}(A)}{[\mathrm{M}(G)]^{k-1}}. \tag{1.2}$$

Idea of the proof

The main ingredients are induction and the following Proposition.

Proposition

Let H be a planar graph and G be an induced subgraph of H with the vertices $a_1, \ldots, a_{2k}$ appearing in that cyclic order among the vertices of some face of H. Then

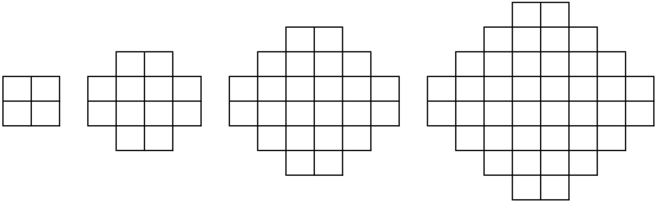
$$\mathrm{M}(G)\,\mathrm{M}(G + \{a_1, \ldots, a_{2k}\}) + \sum_{l=2}^{k} \mathrm{M}(G + \{a_1, a_{2l-1}\})\,\mathrm{M}(G + \overline{\{a_1, a_{2l-1}\}})$$

$$= \sum_{l=1}^{k} \mathrm{M}(G + \{a_1, a_{2l}\})\,\mathrm{M}(G + \overline{\{a_1, a_{2l}\}}),$$

where $\overline{\{a_i, a_j\}}$ stands for the complement of $\{a_i, a_j\}$ in the set $\{a_1, \ldots, a_{2k}\}$.

Back to the beginning

Let's go back to the beginning of the talk.



We are interested in counting the number of domino tilings of these regions.

Proof of Aztec Diamond Theorem

We give a short pictorial proof of the fact that number of domino tilings of an Aztec Diamond of order n is $2^{n(n+1)/2}$.

For that, we recast the tiling problem as a graphical enumeration problem.

It is well known (?) that domino tilings of an Aztec Diamond are in bijection with perfect matchings of the so called dual graph of an Aztec Diamond.

Perfect matchings of a graph are subgraphs where each vertex of the original graph is of degree one.
What is the dual graph?

Dual Graphs

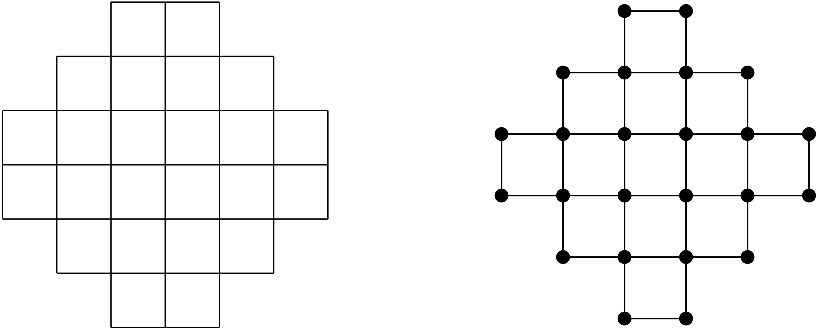


Figure: Aztec Diamond of order 3 and it's dual graph

So, we can use the terms matchings and tilings equivalently.

Example

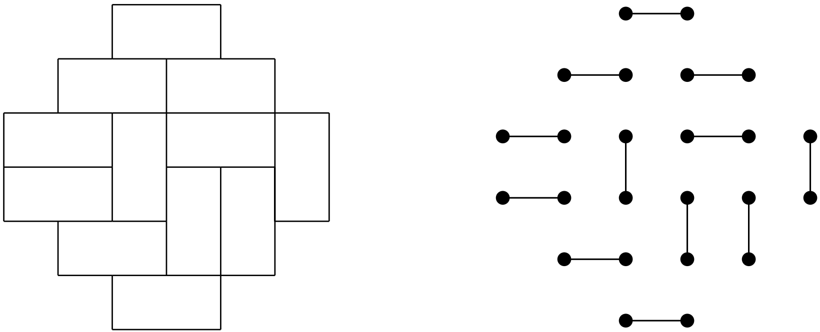
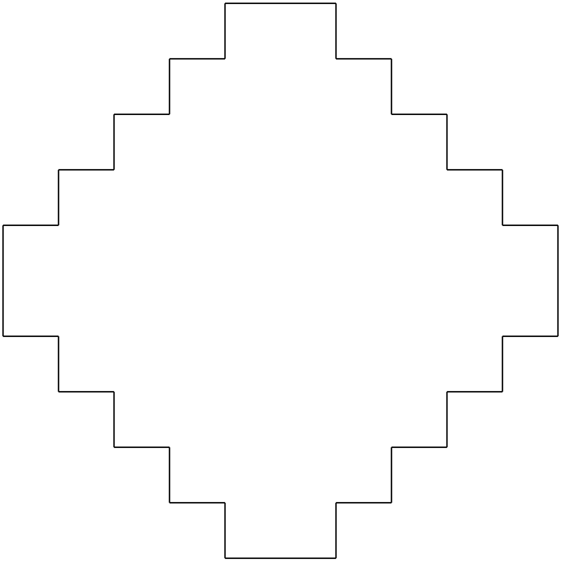
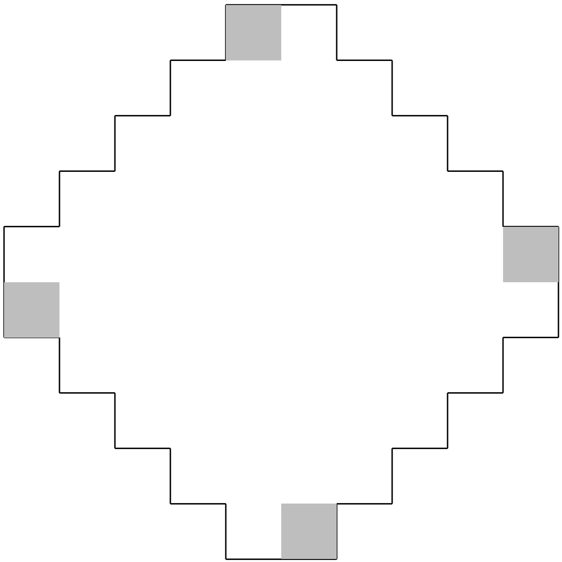


Figure: Equivalence of tilings and matchings

Proof of Aztec Diamond Theorem



Proof of Aztec Diamond Theorem



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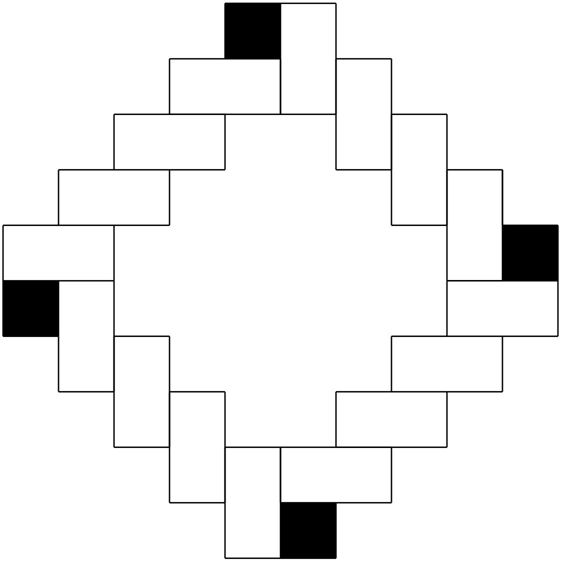

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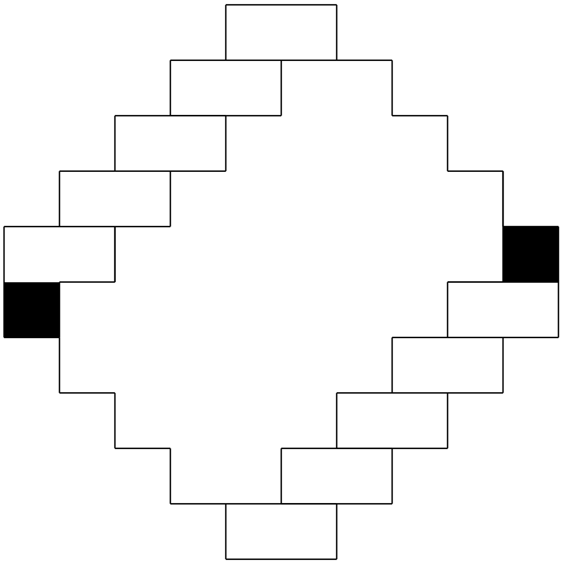

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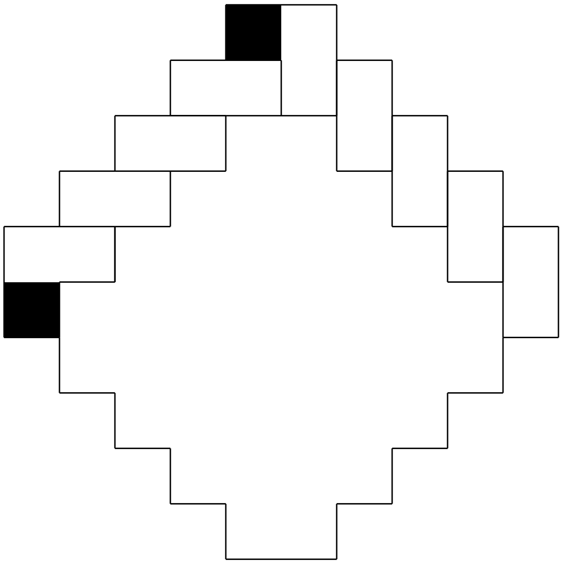

Extensions?

What about other type of regions?

Or, regions with some holes (defects) on them?

Aztec Diamond with defects on adjacent sides

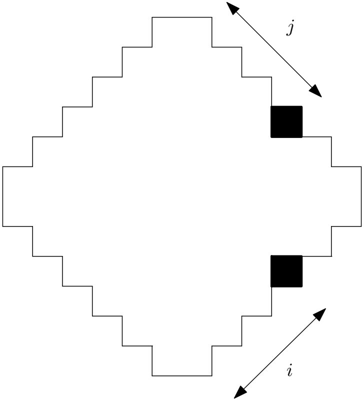

Figure: Aztec Diamond with defects on adjacent sides

Aztec Diamond with defects on adjacent sides

Proposition

Let a, i, j be positive integers such that $1 \leq i, j \leq a$, then the number of domino tilings of $\mathrm{AD}(a)$ with one defect on the southeastern side at the i-th position counted from the south corner and one defect on the northeastern side on the j-th position counted from the north corner is given by

$$2^{a(a-1)/2}\binom{a-1}{i-1}\binom{a-1}{j-1}\,{}_3F_2\left[\begin{matrix}1, 1-i, 1-j\\ 1-a, 1-a\end{matrix}; 2\right].$$

Here

$${}_rF_s\left[\begin{matrix}a_1, a_2, \ldots, a_r\\ b_1, \ldots, b_s\end{matrix}; z\right] = \sum_{k\geq 0} \frac{(a_1)_k(a_2)_k\cdots(a_r)_k}{(b_1)_k(b_2)_k\cdots(b_s)_k}\frac{z^k}{k!}$$

and

$$(p)_n = p(p+1)(p+2)\cdots(p+n-1).$$

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Proof

We use Kuo condensation, with the vertices marked as follows.

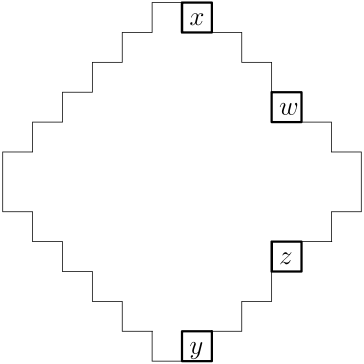

Figure: Aztec Diamond with some marked squares; here $a = 6$

Forced dominoes for different choices of labels

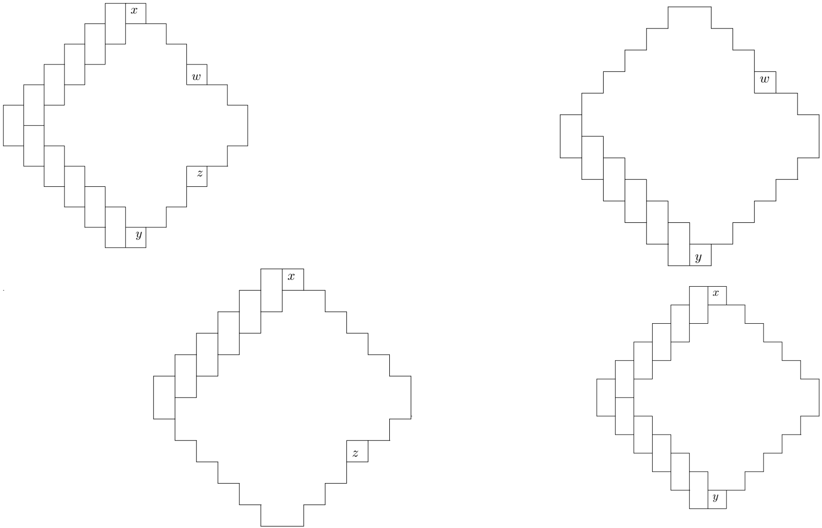

Figure: Forced dominoes, where the vertices we remove are marked

Proof contd.

$$\mathsf{M}(\mathsf{AD}_a(i,j))\,\mathsf{M}(\mathsf{AD}(a-1)) = \mathsf{M}(\mathsf{AD}(a))\,\mathsf{M}(\mathsf{AD}_{a-1}(i-1,j-1)) + \mathsf{M}(\mathcal{AR}_{a-1,a}(j))\,\mathsf{M}(\mathcal{AR}_{a-1,a}(i)). \tag{1.3}$$

$$\mathsf{M}(\mathsf{AD}_a(i,j)) = 2^a\,\mathsf{M}(\mathsf{AD}_{a-1}(i-1,j-1)) + 2^{a(a-1)/2}\binom{a-1}{j-1}\binom{a-1}{i-1} \tag{1.4}$$

Now, we use induction to get the result.

More holes?

But, what about arbitrary holes?

On the boundary?

Regions with defects

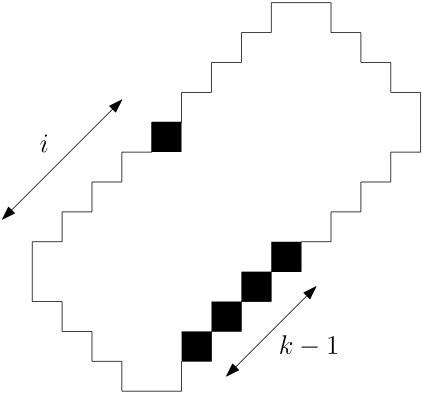

Figure: An $a \times b$ Aztec rectangle with defects marked in black; here $a = 4, b = 9. k = 5, i = 5$

Regions with defects

Proposition

Let $1 \leq a, i \leq b$ be positive integers with $k = b - a > 0$, then the number of domino tilings of $\mathcal{AR}_{a,b}(2,3,\ldots,k)$ with a defect on the northwestern side in the i-th position counted from the west corner is given by

$$2^{a(a+1)/2}\binom{a+k-2}{k-1}\binom{a}{a-i+k}\,{}_3F_2\left[\begin{matrix}1, -k-1, i-a-k\\ i-k+1, 2-a-k\end{matrix}; -1\right].$$

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Preliminaries

We define the region $\mathcal{AR}^k_{a,b}$ to be the region obtained from $\mathcal{AR}_{a.b}$ by adding a string of k unit squares along the boundary of the southeastern side (γ defects) as shown in the figure below.

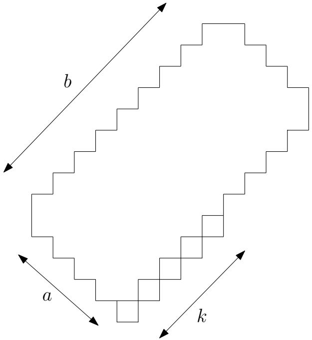

Figure: $\mathcal{AR}^k_{a,b}$ with $a = 4, b = 8, k = 4$

General Result

Theorem

Assume that one of the two shorter sides does not have any defects on it. We assume this to be the southwestern side. Let $\delta_1, \ldots, \delta_{2n+2k}$ be the elements of the set $\{\beta_1, \ldots, \beta_{n+k}\} \cup \{\alpha_1, \ldots, \alpha_n\} \cup \{\gamma_1, \ldots, \gamma_k\}$ listed in a cyclic order, where β_i's are defects on the shorter side, and α_i's are defects on the longer sides.
Then we have

$$\mathrm{M}(\mathcal{AR}_{a,b} \setminus \{\beta_1, \ldots, \beta_{n+k}, \alpha_1, \ldots, \alpha_n\}) =$$
$$\frac{1}{[\mathrm{M}(\mathcal{AR}^k_{a,b})]^{n-k+1}} \operatorname{Pf}[(\mathrm{M}(\mathcal{AR}^k_{a,b} \setminus \{\delta_i, \delta_j\}))_{1 \le i < j \le 2n+2k}],$$

where all the terms on the right hand side are given by explicit formulas.

General Case

Theorem

Let $\beta_1, \ldots, \beta_{n+k}$ *be arbitrary defects of type* β *and* $\alpha_1, \ldots, \alpha_n$ *be arbitrary defects of type* α *along the boundary of* $\mathcal{AR}_{a,b}$*. Then* $\mathrm{M}(\mathcal{AR}_{a,b} \setminus \{\beta_1, \ldots, \beta_{n+k}, \alpha_1, \ldots, \alpha_n\})$ *is equal to the Pfaffian of a* $2n \times 2n$ *matrix whose entries are Pfaffians of* $(2k+2) \times (2k+2)$ *matrices of the type in the statement of main theorem.*

Aztec Rectangles with defects contd.

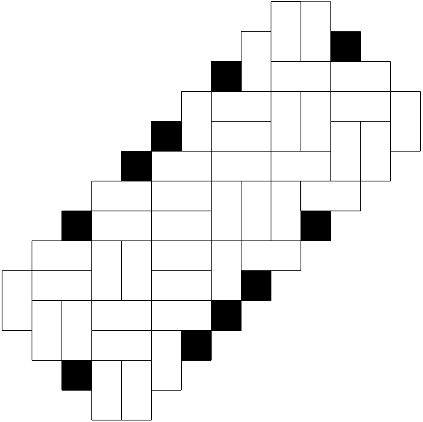

Figure: Tiling with arbitrary defects

Other type of tilings

If instead of dominoes, we had trominoes?

Figure: L-trominoes

The problem becomes quite difficult, and is not solvable using the techniques shown today.

Tromino Cover

However, such tilings (called covers in this case) exists.

Theorem

$\mathcal{AR}_{a,b}$ has a cover if and only if $a(b+1) + b(a+1) \equiv 0 \pmod 3$.

Theorem

A tromino cover for $\mathcal{AR}^k_{a,b}$ can be found in time $O(b^2)$.

Covers with Defects

With defects the problem becomes even harder.

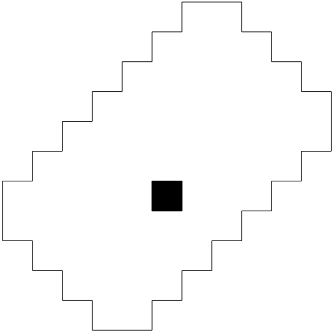

Figure: $\mathcal{AR}_{4,7}$ with defect

Covers with Defects

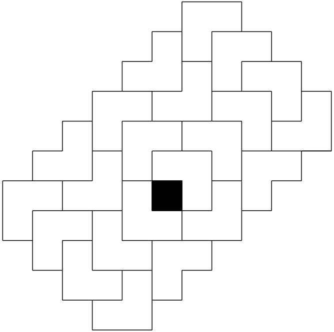

Figure: Covered $\mathcal{AR}_{4,7}$ with defect

Other defects

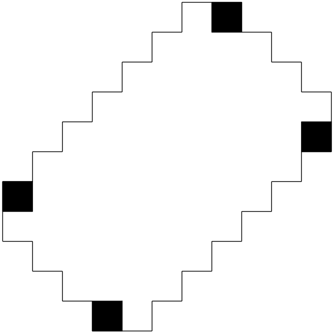

Figure: $\mathcal{AR}_{4,7}$ with defects

Other defects

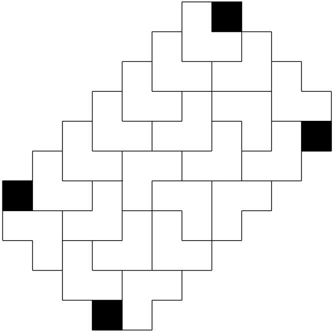

Figure: Covered $\mathcal{AR}_{4,7}$ with defects

Theorem

It is NP-complete to decide if a cover exists for $\mathcal{AR}^k_{a,b}$ with fixed number of defects.

Thank you for your attention.