Partitions with fixed differences between largest and smallest parts with fixed multiplicity of the smallest part

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- Extensions of some results of Vladeta and Dhar (with P. J. Mahanta) out on arXiv

Partitions

We define a partition λ of a non-negative integer n to be an integer sequence $(\lambda_1, \ldots, \lambda_\ell)$ such that

- $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and,
- $\sum_{i=1}^{\ell} \lambda_i = n$.

We say that λ is a partition of n, denoted by $\lambda \vdash n$.

The set of partition of n is denoted by $P(n)$ and $|P(n)| = p(n)$.

For example, there are 5 partitions of 4:

$$4, 3+1, 2+2, 2+1+1, 1+1+1+1,$$

so $p(4) = 5$.

Generating Functions

Let us denote the generating function of $p(n)$ by $P(q)$.

$$P(q) := \sum_{n\geq 0} p(n)q^n.$$

Here, $p(0) = 1$.

Euler proved that

$$P(q) = \prod_{i\geq 1} \frac{1}{1-q^i}.$$

We use the standard notations

$$(a)_n = (a;q)_n := \prod_{i=0}^{n-1}(1-aq^i),$$

and

$$(a)_\infty = (a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

Partition Statistics

- $a_m(n)$: the number of partitions of n where the smallest part occurs at least m times,
- $p(m, n)$: the number of partitions of m with fixed difference between the largest and smallest parts equal to n (studied by Andrews, Beck and Robbins; PAMS 2015)

Dhar (2021) recently proved

- $a_2(n) = 2p(n) - p(n+1)$,
- $a_2(n) = p(2n, n)$.

We generalize these results and statistics.

Results

Theorem (Mahanta - S., 2022)

For all $n \geq 1$ and $m \geq 2$, we have

$$a_m(n) = 2p(n) - p(n+1) - p(n-2) + p(n-m) - \sum_{\ell=2}^{m-1} \sum_{k=3}^{\left\lfloor \frac{n}{\ell} \right\rfloor + 1} \mathcal{Q}_{\ell,k}(n),$$

where $\mathcal{Q}_{\ell,k}(n)$ is the number of partitions of $n - \ell(k-1)$ with smallest part k.

Dhar's result follows as a corollary when $m = 2$.

Generating Function

Theorem (Mahanta - S., 2022)

For $n, m > 0$*, we have*

$$\sum_{n=1}^{\infty} a_m(n)q^n = \frac{1}{(q)_\infty}\left(1 + \sum_{k=1}^{m-1}(-1)^k \prod_{i=0}^{k-1}\left(\frac{1}{q^{m-1-i}} - 1\right)\right) - \mathcal{D}_m,$$

where $\mathcal{D}_m$ *is the sum of the terms with power of* q *less than or equal to 0 in the expansion of*

$$1 + \sum_{k=1}^{m-1}(-1)^k \prod_{i=0}^{k-1}\left(\frac{1}{q^{m-1-i}} - 1\right).$$

Easy Cases

Two easy cases of the previous result are:

Theorem (Mahanta - S., 2022)

We have

$$a_3(n) = 3p(n) - p(n+1) - 2p(n+2) + p(n+3),$$

for all natural numbers n.

Theorem (Mahanta - S., 2022)

We have

$$\begin{aligned} a_4(n) = 4p(n) - p(n+1) - 2p(n+2) - 2p(n+3) \\ + p(n+4) + 2p(n+5) - p(n+6), \end{aligned}$$

for all natural numbers n.

More Statistics

Let $a_m(n, k)$ be the total number of partitions of n where the smallest part occurs at least m times and the difference between the largest and smallest parts is k.

For all natural number n and $m \geq 2$ we have,

$$a_m(n) = a_{m-1}(2n, n).$$

Proof.

If the smallest part is k and the largest part is $n + k$ for a partition counted by $a_{m-1}(2n, n)$, then just remove the part $n + k$ and add a part of size k to get a partition counted by $a_m(n)$. $\square$

Clearly $a_1(2n, n) = p(2n, n)$, so Dhar's result follows immediately from the above.

Generating Function

Theorem (Mahanta - S., 2022)

For $\ell > 1$ we have

$$\sum_{n=1}^{\infty} a_m(n,\ell) q^n = \frac{q^{\ell+m+1}(q)_m (q)_{\ell-m+1}}{((q)_\ell)^2} (-1)^{-m-1} q^{(-m^2-3m-2)/2} \times \left((q)_\ell - \sum_{j=0}^{m} \binom{\ell}{j}_q (-1)^j q^{j+j(j-1)/2} \right).$$

Overpartitions

Overpartitions of n are the partitions of n in which the first occurrence (equivalently, the last occurrence) of a part may be overlined. The number of overpartitions of n are denoted by $\bar{p}(n)$. For example, $\bar{p}(3) = 8$, and the overpartitions of 3 are

$$3, \bar{3}, 2+1, \bar{2}+1, 2+\bar{1}, \bar{2}+\bar{1}, 1+1+1, \bar{1}+1+1.$$

We have the generating function

$$\sum_{n=0}^{\infty} \bar{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots .$$

We define $\bar{a}_m(n)$ to be the number of overpartitions of n where the smallest part occurs at least m times.

Results

Theorem (Mahanta - S., 2022)

For all $n \geq 1$, we have

$$\bar{a}_2(n) = 2\bar{p}(n) - \bar{p}(n+1) + \bar{u}(n+1),$$

where $\bar{u}(n)$ is the number of overpartitions of n with the following conditions:

1. 1 *is not a part.*
2. *Smallest part must be overlined, and no other part of that value is present. (For example, $7 + 7 + \bar{2}$ is included, but $7 + 7 + \bar{2} + 2$ is not.)*
3. *The value of the first and second greatest parts are equal or consecutive, but if they are consecutive then the second greatest part must be overlined. (For example, $7 + \bar{6} + \bar{2}$ is included, but $7 + 6 + \bar{2}$ is not.)*

Overpartitions contd.

Theorem (Mahanta - S., 2022)

Define $\bar{p}(n, t)$ to be the number of overpartitions of n where the difference between the largest and smallest parts equal t. For all $n \geq 1$, we have

$$2\bar{a}_2(n) = \bar{p}(2n, n).$$

We add n to the rightmost smallest part of each overpartition in the set counted by $\bar{a}_2(n)$. For example, if $\bar{k} + k$ is a part in a partition in this set, then we add n to k.

Then the new overpartition belongs to the set counted by $\bar{p}(2n, n)$ and its largest part is not overlined, which is greater than all other parts.

So corresponding to this overpartition there is another unique overpatition in the second set, all of whose parts are same except now the largest part is overlined.

Concluding Remarks

- We have similar results for ℓ-regular partitions (no parts divisible by ℓ).
- Is it possible to find generating functions of analogues of $a_m(n)$ (and other statistics defined here) for other partition functions such as (ℓ, m)-regular partitions, t-core partitions, partition with designated summands, k-colored partitions, etc.?
- Andrews, Beck and Robbins also give a generalization to partitions with a set of specified distances. It would be interesting to explore this direction with some of the statistics defined in this paper.
- Breuer and Kronholm extended the result of Andrews, Beck and Robbins to partitions where the fixed difference between the largest and smallest parts is at most a fixed integer. Chapman gave a combinatorial proof of this result. It would be interesting to extend this setting for the statistics defined in this paper.

Thank you for your attention!