On the parity of Andrews' Singular overpartition

Ajit Singh

Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati-781039

December 11, 2020

Outline

Outline

Partition

A partition of a positive integer n is any nonincreasing sequence of positive integers whose sum is n. The positive integers in the partition are called parts. The number of partitions of n is denoted by $p(n)$.

Partition

A partition of a positive integer n is any nonincreasing sequence of positive integers whose sum is n. The positive integers in the partition are called parts. The number of partitions of n is denoted by $p(n)$.

For example, the partitions of $n = 5$ are

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$$

and so $p(5) = 7$. Conventionally, we set $p(0) = 1$.

Partition

A partition of a positive integer n is any nonincreasing sequence of positive integers whose sum is n. The positive integers in the partition are called parts. The number of partitions of n is denoted by $p(n)$.

For example, the partitions of $n = 5$ are

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$$

and so $p(5) = 7$. Conventionally, we set $p(0) = 1$.

The generating function for the partition function $p(n)$ is given by, for $|q| < 1$,

Partition

A partition of a positive integer n is any nonincreasing sequence of positive integers whose sum is n. The positive integers in the partition are called parts. The number of partitions of n is denoted by $p(n)$.

For example, the partitions of $n = 5$ are

$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$

and so $p(5) = 7$. Conventionally, we set $p(0) = 1$.

The generating function for the partition function $p(n)$ is given by, for $|q| < 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= (1 + q + q^2 + \cdots)(1 + q^2 + q^4 + \cdots)\cdots \\ &= (1 - q)^{-1}(1 - q^2)^{-1}\cdots \\ &= \prod_{k=1}^{\infty} \frac{1}{1 - q^k}.\end{aligned}$$

Ramanujan's congruences

In 1919, Ramanujan announced that he had found three simple congruences satisfied by $p(n)$, namely,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Ramanujan's congruences

In 1919, Ramanujan announced that he had found three simple congruences satisfied by $p(n)$, namely,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

He gave the proofs of first two congruences in [S. Ramanujan, Proc. Cambridge Philos. Soc. (1919)] and derived the following q-series identities:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6} = 5 + 30q + 135q^2 + \ldots,$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q^7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^7}{(1-q^n)^8} = 7 + 77q + \cdots.$$

Ramanujan's congruences

Later in a short one page note [S. Ramanujan, Proc. Cambridge Philos. Soc. (1920)], he announced that he had also found a proof of third congruence.

Ramanujan's congruences

Later in a short one page note [S. Ramanujan, Proc. Cambridge Philos. Soc. (1920)], he announced that he had also found a proof of third congruence. In 1919, Ramanujan offered a more general conjecture after studying a table of values of $p(n)$, $0 \leq n \leq 200$, made by MacMahon. Let $\delta = 5^a 7^b 11^c$ and let λ be an integer such that $24\lambda \equiv 1 \pmod{\delta}$. Then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta}. \tag{1}$$

Ramanujan's congruences

Later in a short one page note [S. Ramanujan, Proc. Cambridge Philos. Soc. (1920)], he announced that he had also found a proof of third congruence. In 1919, Ramanujan offered a more general conjecture after studying a table of values of $p(n)$, $0 \leq n \leq 200$, made by MacMahon. Let $\delta = 5^a 7^b 11^c$ and let λ be an integer such that $24\lambda \equiv 1 \pmod{\delta}$. Then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta}. \tag{1}$$

However, the conjecture is not quite correct. In 1930 Chowla and Gupta discovered a counterexample for the modulus 7^3

Ramanujan's type congruences

In 2000, Ken Ono developed aspects of the p-adic theory of half-integral weight modular forms and used this to prove the existence of infinite families of partition congruences modulo every prime $\ell \geq 5$.

Theorem (K. Ono- Ann. of Math. (2000))

For any prime $\ell \geq 5$, there are infinitely many distinct arithmetic progression $An + B$ such that for all n,

$$p(An + B) \equiv 0 \pmod{\ell}.$$

Ramanujan's type congruences

In 2000, Ken Ono developed aspects of the p-adic theory of half-integral weight modular forms and used this to prove the existence of infinite families of partition congruences modulo every prime $\ell \geq 5$.

Theorem (K. Ono- Ann. of Math. (2000))

For any prime $\ell \geq 5$, there are infinitely many distinct arithmetic progression $An + B$ such that for all n,

$$p(An + B) \equiv 0 \pmod{\ell}.$$

Unlike the original Ramanujan congruences, these results are typically quite complicated, as exhibited by the example

$$p(1977147619n + 815655) \equiv 0 \pmod{19}.$$

Distribution of the partition function

In addition to the study of Ramanujan-type congruences, it is natural to consider the general distribution of the partition function modulo positive integers M.

Distribution of the partition function

In addition to the study of Ramanujan-type congruences, it is natural to consider the general distribution of the partition function modulo positive integers M.

If M is a positive integer and $0 \leq r < M$, then define $\delta_r(M;X)$ by

$$\delta_r(M;X) := \frac{\#\{0 \leq n < X : p(n) \equiv r \pmod{M}\}}{X}.$$

Distribution of the partition function

In addition to the study of Ramanujan-type congruences, it is natural to consider the general distribution of the partition function modulo positive integers M.

If M is a positive integer and $0 \le r < M$, then define $\delta_r(M;X)$ by

$$\delta_r(M;X) := \frac{\#\{0 \le n < X : p(n) \equiv r \pmod{M}\}}{X}.$$

It is a difficult problem to study the density

$$\lim_{X\to\infty} \delta_r(M;X).$$

Distribution of the partition function: $M = 2$ case

The well-known parity conjecture of Parkin and Shanks predicts that the values of $p(n)$ are evenly distributed modulo 2.

Distribution of the partition function: $M = 2$ case

The well-known parity conjecture of Parkin and Shanks predicts that the values of $p(n)$ are evenly distributed modulo 2.

Conjecture (Parkin and Shanks- Math. Comp. (1967))

If $r \in \{0, 1\}$, *then*

$$\lim_{X\to\infty} \delta_r(2; X) = \frac{1}{2}.$$

Distribution of the partition function: $M = 2$ case

The well-known parity conjecture of Parkin and Shanks predicts that the values of $p(n)$ are evenly distributed modulo 2.

Conjecture (Parkin and Shanks- Math. Comp. (1967))

If $r \in \{0, 1\}$, *then*

$$\lim_{X \to \infty} \delta_r(2; X) = \frac{1}{2}.$$

This conjecture is supported by the following numerical evidence.

Distribution of the partition function: $M = 2$ case

The well-known parity conjecture of Parkin and Shanks predicts that the values of $p(n)$ are evenly distributed modulo 2.

Conjecture (Parkin and Shanks- Math. Comp. (1967))

If $r \in \{0, 1\}$, *then*

$$\lim_{X \to \infty} \delta_r(2; X) = \frac{1}{2}.$$

This conjecture is supported by the following numerical evidence.

X	$\delta_0(2;X)$	$\delta_1(2;X)$
$200,000$	0.5012	0.4988
$400,000$	0.5000	0.5000
$600,000$	0.5000	0.5000
$800,000$	0.5006	0.4994
$1,000,000$	0.5004	0.4996

Distribution of the partition function: $M = 2$ case

The well-known parity conjecture of Parkin and Shanks predicts that the values of $p(n)$ are evenly distributed modulo 2.

Conjecture (Parkin and Shanks- Math. Comp. (1967))

If $r \in \{0, 1\}$, *then*

$$\lim_{X\to\infty} \delta_r(2; X) = \frac{1}{2}.$$

This conjecture is supported by the following numerical evidence.

X	$\delta_0(2; X)$	$\delta_1(2; X)$
$200,000$	0.5012	0.4988
$400,000$	0.5000	0.5000
$600,000$	0.5000	0.5000
$800,000$	0.5006	0.4994
$1,000,000$	0.5004	0.4996

Little is known regarding this conjecture.

Parity conjecture

From the works of Scott Ahlgren and J. P. Serre, we know that $p(n)$ is even for infinitely many values of n. But, it is not known if the set of such n has a positive density.

Parity conjecture

From the works of Scott Ahlgren and J. P. Serre, we know that $p(n)$ is even for infinitely many values of n. But, it is not known if the set of such n has a positive density.

For the odd case, we have the same piece of information known till date.

Parity conjecture

From the works of Scott Ahlgren and J. P. Serre, we know that $p(n)$ is even for infinitely many values of n. But, it is not known if the set of such n has a positive density.

For the odd case, we have the same piece of information known till date. More precisely, the works of Scott Ahlgren and J. P. Serre can be stated as follows: For large X,

$$\#\{n \leq X | \ p(n) \text{ is even}\} \gg \sqrt{X};$$

$$\#\{n \leq X | \ p(n) \text{ is odd}\} \gg \frac{\sqrt{X}}{log(X)}.$$

Parity conjecture

From the works of Scott Ahlgren and J. P. Serre, we know that $p(n)$ is even for infinitely many values of n. But, it is not known if the set of such n has a positive density.

For the odd case, we have the same piece of information known till date. More precisely, the works of Scott Ahlgren and J. P. Serre can be stated as follows: For large X,

$$\#\{n \leq X \mid p(n) \text{ is even}\} \gg \sqrt{X};$$

$$\#\{n \leq X \mid p(n) \text{ is odd}\} \gg \frac{\sqrt{X}}{log(X)}.$$

There are various ways to prove parity results of partition functions. One of the useful modern approach is to use the theory of modular forms. We will review some of the useful facts of modular forms and will show how they are used in partition theory.

Outline

Modular Group

Definition

The set

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

forms a group under matrix multiplication and is called modular group.

Modular Group

Definition

The set

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

forms a group under matrix multiplication and is called modular group.

Theorem

The full modular group $\mathrm{SL}_2(\mathbb{Z})$ *is generated by the elements*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Congruence Subgroups

Definition

A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subset \Gamma$, for some N. The level of Γ is the smallest N such that $\Gamma(N) \subset \Gamma$.

Congruence Subgroups

Definition

A subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subset \Gamma$, for some N. The level of Γ is the smallest N such that $\Gamma(N) \subset \Gamma$.

Some important congruence subgroups of $SL_2(\mathbb{Z})$:

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\};$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\};$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod N, \ d \equiv 1 \pmod N \right\}$$

where N is a positive integer.

Action of $SL_2(\mathbb{Z})$

Let $\mathbb{H}$ denote the upper half plane,

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\} \subseteq \mathbb{C}.$$

Action of $\mathrm{SL}_2(\mathbb{Z})$

Let $\mathbb{H}$ denote the upper half plane,

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\} \subseteq \mathbb{C}.$$

The group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - cb > 0 \right\}$$

on $\mathbb{H}$ via the folmula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

Action of $SL_2(\mathbb{Z})$

Let $\mathbb{H}$ denote the upper half plane,

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\} \subseteq \mathbb{C}.$$

The group

$$GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - cb > 0 \right\}$$

on $\mathbb{H}$ via the folmula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

Definition

The group $GL_2^+(\mathbb{R})$ acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and ℓ is an integer, then define the slash operator $|_\ell$ by

$$(f|_\ell \gamma)(z) := (\det \gamma)^{\ell/2}(cz + d)^{-\ell} f(\gamma z).$$

Modular forms of level one

Consider a function $f(z) : \mathbb{H} \to \mathbb{C}$ for which

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2}$$

Modular forms of level one

Consider a function $f(z) : \mathbb{H} \to \mathbb{C}$ for which

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2}$$

To define the holomorphicity of f at $i\infty$, we use the following transformation. Let $q = e^{2i\pi z}$, then q transform $\mathbb{H}$ to punctured unit disc. If $f(z)$ is holomorphic on $\mathbb{H}$ then $f(q)$ will be holomorphic on the punctured unit disc. Hence, we have a *Laurent series* expansion centered at $q = 0$, as

$$f(q) = \sum_{n=-\infty}^{\infty} a_n q^n$$

Modular forms of level one

Consider a function $f(z) : \mathbb{H} \to \mathbb{C}$ for which

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2}$$

To define the holomorphicity of f at $i\infty$, we use the following transformation. Let $q = e^{2i\pi z}$, then q transform $\mathbb{H}$ to punctured unit disc. If $f(z)$ is holomorphic on $\mathbb{H}$ then $f(q)$ will be holomorphic on the punctured unit disc. Hence, we have a *Laurent series* expansion centered at $q = 0$, as

$$f(q) = \sum_{n=-\infty}^{\infty} a_n q^n$$

which is also called the q-series associated with f.Hence, we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi izn}$$

as the *Fourier series* of f at $i\infty$. We call f is holomorphic at $i\infty$ if $a_n = 0$ for all negative integer n.

Modular forms of level one

Definition

A modular form of weight $k \in \mathbb{Z}$ for the full modular group is a holomorphic function $f(z) : \mathbb{H} \longrightarrow \mathbb{C}$ which satisfies (2) and is holomorphic at $i\infty$.
A modular form f is called a cusp form if $a_0 = 0$, that is, f vanishes at $i\infty$.

Modular forms of level one

Definition

A modular form of weight $k \in \mathbb{Z}$ for the full modular group is a holomorphic function $f(z) : \mathbb{H} \longrightarrow \mathbb{C}$ which satisfies (2) and is holomorphic at $i\infty$.
A modular form f is called a cusp form if $a_0 = 0$, that is, f vanishes at $i\infty$.

An example of Modular forms :

Modular forms of level one

Definition

A modular form of weight $k \in \mathbb{Z}$ for the full modular group is a holomorphic function $f(z) : \mathbb{H} \longrightarrow \mathbb{C}$ which satisfies (2) and is holomorphic at $i\infty$.
A modular form f is called a cusp form if $a_0 = 0$, that is, f vanishes at $i\infty$.

An example of Modular forms :
Let $k \geq 4$ be an even integer. The Eisenstein series of weight k is defined as

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$$

We have

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

Where $\sigma_{k-1}(n)$ is the arithmetic function

$$\sigma_{k-1}(n) = \sum_{d|n, d>0} d^{k-1}.$$

Modular forms of higher level

Definition

Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ on Γ if the following hold:

1. We have
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\ell} f(z)$$
for all $z \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
2. If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_{\ell}\gamma)(z)$ has a Fourier expansion of the form
$$(f|_{\ell}\gamma)(z) = \sum_{n \geq 0} a_{\gamma}(n) q_N^n.$$

Modular forms of higher level

Definition

Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ on Γ if the following hold:

1. We have
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\ell} f(z)$$
for all $z \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
2. If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_{\ell}\gamma)(z)$ has a Fourier expansion of the form
$$(f|_{\ell}\gamma)(z) = \sum_{n\geq 0} a_{\gamma}(n) q_N^n.$$

For a positive integer ℓ, the complex vector space of modular forms of weight ℓ with respect to a congruence subgroup Γ is denoted by $M_{\ell}(\Gamma)$. The space of cusp forms is denoted by $S_{\ell}(\Gamma)$.

Modular forms of higher level

Definition

If χ is a Dirichlet character modulo N, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$ has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. .

Modular forms of higher level

Definition

If χ is a Dirichlet character modulo N, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$ has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. .

The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$.

Modularity of eta-quotients

Definition

The Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$.

Modularity of eta-quotients

Definition

The Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$.

Definition

A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer.

Modularity of eta-quotients

We now recall two theorems on modularity of eta-quotients.

Modularity of eta-quotients

We now recall two theorems on modularity of eta-quotients.

Theorem (Theorem 1.64, The web of modularity (2004))

If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ *is an eta-quotient such that* $\ell = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ *satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *Here the character* χ *is defined by* $\chi(d) := \left(\frac{(-1)^\ell s}{d}\right)$, *where* $s := \prod_{\delta|N} \delta^{r_\delta}$.

Modularity of eta-quotients

The following theorem gives the necessary criterion for determining orders of an eta-quotient at cusps.

Modularity of eta-quotients

The following theorem gives the necessary criterion for determining orders of an eta-quotient at cusps.

Theorem (Theorem 1.65, The web of modularity (2004))

Let c, d *and* N *be positive integers with* $d \mid N$ *and* $\gcd(c, d) = 1$. *If* f *is an eta-quotient satisfying the conditions of previous Theorem for* N, *then the order of vanishing of* $f(z)$ *at the cusp* $\frac{c}{d}$ *is*

$$\frac{N}{24} \sum_{\delta \mid N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}.$$

A density result of Serre

Theorem (Theorem 2.65, The web of modularity (2004))

If $f(z)$ is an integral weight modular form in $M_k(\Gamma_0(N), \chi)$ which has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathbb{Z}[[q]],$$

then there is a constant $\alpha > 0$ such that

$$\#\{n \leq X : a(n) \not\equiv 0 \pmod{m}\} = O\left(\frac{X}{(\log X)^{\alpha}}\right).$$

A density result of Serre

Theorem (Theorem 2.65, The web of modularity (2004))

If $f(z)$ is an integral weight modular form in $M_k(\Gamma_0(N), \chi)$ which has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \in \mathbb{Z}[[q]],$$

then there is a constant $\alpha > 0$ such that

$$\#\{n \leq X : a(n) \not\equiv 0 \pmod{m}\} = \mathcal{O}\left(\frac{X}{(\log X)^{\alpha}}\right).$$

This yields

$$\lim_{X \to \infty} \frac{\#\{0 < n \leq X : a(n) \equiv 0 \pmod{m}\}}{X} = 1.$$

Outline

Divisibility of certain partition function

Let $b_k(n)$ be the number of partitions of n into parts none of which are multiples of k.

Divisibility of certain partition function

Let $b_k(n)$ be the number of partitions of n into parts none of which are multiples of k.

For example, $b_2(5) = 3$ with $5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. Note that $p(5) = 7$.

Divisibility of certain partition function

Let $b_k(n)$ be the number of partitions of n into parts none of which are multiples of k.

For example, $b_2(5) = 3$ with $5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. Note that $p(5) = 7$.

For any fixed positive integer $k = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, B. Gordon and K. Ono (Ramanujan J. (1997)) proved that for any t, $b_k(n)$ is divisible by p_i^t for almost all n.

Divisibility of certain partition function

Let $b_k(n)$ be the number of partitions of n into parts none of which are multiples of k.

For example, $b_2(5) = 3$ with $5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. Note that $p(5) = 7$.

For any fixed positive integer $k = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, B. Gordon and K. Ono (Ramanujan J. (1997)) proved that for any t, $b_k(n)$ is divisible by p_i^t for almost all n.

That is,

$$\lim_{X \to \infty} \frac{\#\left\{0 < n \leq X : b_k(n) \equiv 0 \pmod{p_i^t}\right\}}{X} = 1.$$

Outline

Andrews' Singular overpartition

Overpartition:

In (Trans. AMS (2004)), Corteel and Lovejoy introduced the notion of overpartitions.

Andrews' Singular overpartition

Overpartition:

In (Trans. AMS (2004)), Corteel and Lovejoy introduced the notion of overpartitions.
An overpartition of a nonnegative integer n is a partition of n in which the first occurrence of a part may be over-lined and $\overline{p}(n)$ counts the number of overpartitions of n.

Andrews' Singular overpartition

Overpartition:

In (Trans. AMS (2004)), Corteel and Lovejoy introduced the notion of overpartitions.
An overpartition of a nonnegative integer n is a partition of n in which the first occurrence of a part may be over-lined and $\overline{p}(n)$ counts the number of overpartitions of n.

For example, the eight overpartitions of 3 are
$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1$.

Andrews' Singular overpartition

Overpartition:

In (Trans. AMS (2004)), Corteel and Lovejoy introduced the notion of overpartitions.
An overpartition of a nonnegative integer n is a partition of n in which the first occurrence of a part may be over-lined and $\overline{p}(n)$ counts the number of overpartitions of n.

For example, the eight overpartitions of 3 are
$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1$.

Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have the generating function for overpartitions,

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

Andrews' Singular overpartition

Andrews (Int. J. Number Theory (2015)) defined the so-called singular overpartitions.

Andrews' Singular overpartition

Andrews (Int. J. Number Theory (2015)) defined the so-called singular overpartitions.

Andrews singular overpartition function $\overline{C}_{k,i}(n)$ counts the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined.

Andrews' Singular overpartition

Andrews (Int. J. Number Theory (2015)) defined the so-called singular overpartitions.

Andrews singular overpartition function $\overline{C}_{k,i}(n)$ counts the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined.

For example, $\overline{C}_{3,1}(4) = 10$ with the relevant partitions being $4, \overline{4}, 2+2, \overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1$

For $k \geq 3$ and $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty}, \tag{3}$$

Andrews' Singular overpartition

In a recent paper (Congruences for Andrews' (k, i)-singular overpartitions, Ramanujan J. 43 (2017)), Aricheta has studied the parity of $\overline{C}_{3k,k}(n)$.

Andrews' Singular overpartition

In a recent paper (Congruences for Andrews' (k, i)-singular overpartitions, Ramanujan J. 43 (2017)), Aricheta has studied the parity of $\overline{C}_{3k,k}(n)$.

To be specific, represent any positive integer k as $k = 2^a m$ where the integer $a \geq 0$ and m is positive odd. Assume further that $2^a \geq m$. He proved that $\overline{C}_{3\ell,\ell}(n) \equiv b_\ell(n) \pmod 2$, where $b_\ell(n)$ denotes the number of partitions of n into parts none of which are multiples of ℓ. He then used the density result of Gordon and Ono [Ramanujan J. (1997)] regarding $b_\ell(n)$ to prove a density result about $\overline{C}_{3\ell,\ell}(n)$ modulo 2 for an infinite family of ℓ. Then Aricheta proved that $\overline{C}_{3k,k}(n)$ is almost always even, that is

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3k,k}(n) \equiv 0 \pmod 2\right\}}{X} = 1.$$

Andrews' Singular overpartition

Further, recently Barman & Ray (Research in Number Theory (2019)), has studied the parity of $\overline{C}_{3,1}(n)$.

Andrews' Singular overpartition

Further, recently Barman & Ray (Research in Number Theory (2019)), has studied the parity of $\overline{C}_{3,1}(n)$.

More precisely, Let k be a fixed positive integer. Then $\overline{C}_{3,1}(n)$ is almost always divisible by 2^k, namely,

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3,1}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

Andrews' Singular overpartition

Further, recently Barman & Ray (Research in Number Theory (2019)), has studied the parity of $\overline{C}_{3,1}(n)$.

More precisely, Let k be a fixed positive integer. Then $\overline{C}_{3,1}(n)$ is almost always divisible by 2^k, namely,

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \le X : \overline{C}_{3,1}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

They, further proved that the partition function $\overline{C}_{3,1}(n)$ is also divisible by 3^k for almost all n.

Arithmatic density of Andrews' Singular overpartition

Theorem (Barman & Singh, Journal of Number Theory (To be appeared))

Let k be a fixed positive integer. Then for all $\alpha \geq 1$ and all prime p satisfying $2^\alpha \geq p$, $\overline{C}_{3p\cdot 2^\alpha, p\cdot 2^\alpha}(n)$ is almost always divisible by 2^k, namely,

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3p\cdot 2^\alpha, p\cdot 2^\alpha}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

Arithmatic density of Andrews' Singular overpartition

Theorem (Barman & Singh, Journal of Number Theory (To be appeared))

Let k *be a fixed positive integer. Then for all* $\alpha \geq 1$ *and all prime* p *satisfying* $2^{\alpha} \geq p$, $\overline{C}_{3p\cdot 2^{\alpha}, p\cdot 2^{\alpha}}(n)$ *is almost always divisible by* 2^k*, namely,*

$$\lim_{X \to \infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3p\cdot 2^{\alpha}, p\cdot 2^{\alpha}}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

We further prove that the partition function $\overline{C}_{3\cdot 3\cdot 2^{\alpha}, 3\cdot 2^{\alpha}}(n)$ is also divisible by 3^k for almost all n.

Arithmatic density of Andrews' Singular overpartition

Theorem (Barman & Singh, Journal of Number Theory (To be appeared))

Let k be a fixed positive integer. Then for all $\alpha \geq 1$ and all prime p satisfying $2^{\alpha} \geq p$, $\overline{C}_{3p\cdot 2^{\alpha}, p\cdot 2^{\alpha}}(n)$ is almost always divisible by 2^k, namely,

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3p\cdot 2^{\alpha}, p\cdot 2^{\alpha}}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

We further prove that the partition function $\overline{C}_{3\cdot 3\cdot 2^{\alpha}, 3\cdot 2^{\alpha}}(n)$ is also divisible by 3^k for almost all n.

Theorem (Barman & Singh, Journal of Number Theory (To be appeared))

Let k be a fixed positive integer. Then for each α, $0 \leq \alpha \leq 3$, $\overline{C}_{3\cdot 3\cdot 2^{\alpha}, 3\cdot 2^{\alpha}}(n)$ is almost always divisible by 3^k, namely,

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3\cdot 3\cdot 2^{\alpha}, 3\cdot 2^{\alpha}}(n) \equiv 0 \pmod{3^k}\right\}}{X} = 1.$$

Arithmatic density of Andrews' Singular overpartition

We also proved that the eta-quotient which arises naturally as generating function for $\overline{C}_{3\cdot 3\cdot 2^{\alpha}, 3\cdot 2^{\alpha}}(n)$ is not a modular form if $\alpha \geq 4$.

Arithmatic density of Andrews' Singular overpartition

We also proved that the eta-quotient which arises naturally as generating function for $\overline{C}_{3\cdot 3\cdot 2^{\alpha},3\cdot 2^{\alpha}}(n)$ is not a modular form if $\alpha \geq 4$.

Proof of Theorem 1: The generating function for $\overline{C}_{3p\cdot 2^{\alpha},p\cdot 2^{\alpha}}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{3p\cdot 2^{\alpha},p\cdot 2^{\alpha}}(n)q^n = \frac{(q^{2p\cdot 2^{\alpha}};q^{2p\cdot 2^{\alpha}})_\infty (q^{3p\cdot 2^{\alpha}};q^{3p\cdot 2^{\alpha}})_\infty^2}{(q;q)_\infty (q^{p\cdot 2^{\alpha}};q^{p\cdot 2^{\alpha}})_\infty (q^{6p\cdot 2^{\alpha}};q^{6p\cdot 2^{\alpha}})_\infty}.$$

Arithmatic density of Andrews' Singular overpartition

We also proved that the eta-quotient which arises naturally as generating function for $\overline{C}_{3\cdot 3\cdot 2^{\alpha},3\cdot 2^{\alpha}}(n)$ is not a modular form if $\alpha \geq 4$.

Proof of Theorem 1: The generating function for $\overline{C}_{3p\cdot 2^{\alpha},p\cdot 2^{\alpha}}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{3p\cdot 2^{\alpha},p\cdot 2^{\alpha}}(n)q^n = \frac{(q^{2p\cdot 2^{\alpha}};q^{2p\cdot 2^{\alpha}})_{\infty}(q^{3p\cdot 2^{\alpha}};q^{3p\cdot 2^{\alpha}})_{\infty}^2}{(q;q)_{\infty}(q^{p\cdot 2^{\alpha}};q^{p\cdot 2^{\alpha}})_{\infty}(q^{6p\cdot 2^{\alpha}};q^{6p\cdot 2^{\alpha}})_{\infty}}.$$

We note that $\eta(3p\cdot 2^{\alpha+3}z) = q^{p\cdot 2^{\alpha}}\prod_{n=1}^{\infty}(1-q^{(3p\cdot 2^{\alpha+3})n})$ is a power series of q. Given prime p, Let

$$A_{\alpha,p}(z) = \prod_{n=1}^{\infty}\frac{(1-q^{(3p\cdot 2^{\alpha+3})n})^2}{(1-q^{(3p\cdot 2^{\alpha+4})n})} = \frac{\eta^2(3p\cdot 2^{\alpha+3}z)}{\eta(3p\cdot 2^{\alpha+4}z)}.$$

Proof continued...

Then using binomial theorem we have

$$A_{\alpha,p}^{2^k}(z) = \frac{\eta^{2^{k+1}}(3p \cdot 2^{\alpha+3}z)}{\eta^{2^k}(3p \cdot 2^{\alpha+4}z)} \equiv 1 \pmod{2^{k+1}}.$$

Proof continued...

Then using binomial theorem we have

$$A_{\alpha,p}^{2^k}(z) = \frac{\eta^{2^{k+1}}(3p \cdot 2^{\alpha+3}z)}{\eta^{2^k}(3p \cdot 2^{\alpha+4}z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $B_{\alpha,p,k}(z)$ by

$$B_{\alpha,p,k}(z) = \left(\frac{\eta(3p \cdot 2^{\alpha+4}z)\eta(9p \cdot 2^{\alpha+3}z)^2}{\eta(24z)\eta(3p \cdot 2^{\alpha+3}z)\eta(9p \cdot 2^{\alpha+4}z)}\right)A_{\alpha,p}^{2^k}(z).$$

Proof continued...

Then using binomial theorem we have

$$A_{\alpha,p}^{2^k}(z) = \frac{\eta^{2^{k+1}}(3p \cdot 2^{\alpha+3}z)}{\eta^{2^k}(3p \cdot 2^{\alpha+4}z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $B_{\alpha,p,k}(z)$ by

$$B_{\alpha,p,k}(z) = \left(\frac{\eta(3p \cdot 2^{\alpha+4}z)\eta(9p \cdot 2^{\alpha+3}z)^2}{\eta(24z)\eta(3p \cdot 2^{\alpha+3}z)\eta(9p \cdot 2^{\alpha+4}z)}\right)A_{\alpha,p}^{2^k}(z).$$

Modulo 2^{k+1}, we have

$$\begin{aligned} B_{\alpha,p,k}(z) &\equiv \frac{\eta(3p \cdot 2^{\alpha+4}z)\eta(9p \cdot 2^{\alpha+3}z)^2}{\eta(24z)\eta(3p \cdot 2^{\alpha+3}z)\eta(9p \cdot 2^{\alpha+4}z)} \\ &= q^{p \cdot 2^{\alpha}-1}\left(\frac{(q^{3p\cdot 2^{\alpha+4}}; q^{3p\cdot 2^{\alpha+4}})_\infty (q^{9p\cdot 2^{\alpha+3}}; q^{9p\cdot 2^{\alpha+3}})_\infty^2}{(q^{24}; q^{24})_\infty (q^{3p\cdot 2^{\alpha+3}}; q^{3p\cdot 2^{\alpha+3}})_\infty (q^{9p\cdot 2^{\alpha+4}}; q^{9p\cdot 2^{\alpha+4}})_\infty}\right). \end{aligned}$$

Proof continued...

Combining the above identities, we obtain

$$B_{\alpha,p,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{3p\cdot 2^{\alpha},p\cdot 2^{\alpha}}(n)q^{24n+p\cdot 2^{\alpha}-1} \pmod{2^{k+1}}.$$

Proof continued...

Combining the above identities, we obtain

$$B_{\alpha,p,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{3p\cdot 2^{\alpha},p\cdot 2^{\alpha}}(n) q^{24n+p\cdot 2^{\alpha}-1} \pmod{2^{k+1}}.$$

Now, $B_{\alpha,p,k}$ is an eta-quotient with $N = 9p \cdot 2^{\alpha+5}$. The cusps of $\Gamma_0(9p \cdot 2^{\alpha+5})$ are represented by fractions $\frac{c}{d}$ where $d \mid 9p \cdot 2^{\alpha+5}$ and $\gcd(c, d) = 1$.

Proof continued...

Combining the above identities, we obtain

$$B_{\alpha,p,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{3p\cdot 2^{\alpha},p\cdot 2^{\alpha}}(n) q^{24n+p\cdot 2^{\alpha}-1} \pmod{2^{k+1}}.$$

Now, $B_{\alpha,p,k}$ is an eta-quotient with $N = 9p \cdot 2^{\alpha+5}$. The cusps of $\Gamma_0(9p \cdot 2^{\alpha+5})$ are represented by fractions $\frac{c}{d}$ where $d \mid 9p \cdot 2^{\alpha+5}$ and $\gcd(c,d) = 1$.
we find that $B_{\alpha,p,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d, 3p\cdot 2^{\alpha+3})^2}{3p\cdot 2^{\alpha+3}}(2^{k+1}-1) + \frac{\gcd(d, 3p\cdot 2^{\alpha+4})^2}{3p\cdot 2^{\alpha+4}}(1-2^{k})$$
$$+ 2\frac{\gcd(d, 9p\cdot 2^{\alpha+3})^2}{9p\cdot 2^{\alpha+3}} - \frac{\gcd(d,24)^2}{24} - \frac{\gcd(d, 9p\cdot 2^{\alpha+4})^2}{9p\cdot 2^{\alpha+4}} \geq 0.$$

Proof continued...

Equivalently, if and only if

$$L := 6G_1^2 \cdot (2^{k+1} - 1) + 3G_2^2 \cdot (1 - 2^k) + 4G_3^2 - 3p \cdot 2^{\alpha+1} G_4^2 - 1 \geq 0,$$

Proof continued...

Equivalently, if and only if

$$L := 6G_1^2 \cdot (2^{k+1} - 1) + 3G_2^2 \cdot (1 - 2^k) + 4G_3^2 - 3p \cdot 2^{\alpha+1} G_4^2 - 1 \geq 0,$$

where
$G_1 = \frac{\gcd(d, 3p \cdot 2^{\alpha+3})}{\gcd(d, 9p \cdot 2^{\alpha+4})}$, $G_2 = \frac{\gcd(d, 3p \cdot 2^{\alpha+4})}{\gcd(d, 9p \cdot 2^{\alpha+4})}$, $G_3 = \frac{\gcd(d, 9p \cdot 2^{\alpha+3})}{\gcd(d, 9p \cdot 2^{\alpha+4})}$, and
$G_4 = \frac{\gcd(d, 24)}{\gcd(d, 9p \cdot 2^{\alpha+4})}$, respectively.

Proof continued...

In the following table, we find all the possible values of L.

$d \mid 9p \cdot 2^{\alpha+5}$	G_1	G_2	G_3	G_4	L
$1, 2, 3, 4, 6, 8, 12, 24$	1	1	1	1	$9 \cdot 2^k - 3p \cdot 2^{\alpha+1}$
$p, 2p, 3p, 4p, 6p, 8p, 12p$	1	1	1	$1/p$	$9 \cdot 2^k - 3 \cdot 2^{\alpha+1}/p$
$9, 18, 36, 72$	$1/3$	$1/3$	1	$1/3$	$2^k + 8/3 - 2^{\alpha+1}p/3$
$9p, 18p, 36p, 72p$	$1/3$	$1/3$	1	$1/3p$	$2^k + 8/3 - 2^{\alpha+1}/3p$
$2^{\alpha+4}, 2^{\alpha+5}, 3 \cdot 2^{\alpha+4}$	$1/2$	1	$1/2$	$2^{-1-\alpha}$	$1.5 - 3p \cdot 2^{-1-\alpha}$
$p \cdot 2^{\alpha+4}, p \cdot 2^{\alpha+5}$	$1/2$	1	$1/2$	$2^{-1-\alpha}/p$	$1.5 - 3 \cdot 2^{-1-\alpha}/p$
$2^r, 3 \cdot 2^r : 4 \leq r \leq \alpha+3$	1	1	1	2^{3-r}	$9 \cdot 2^k - 3p \cdot 2^{7+\alpha-2r}$
$p \cdot 2^r, 3p \cdot 2^r : 4 \leq r \leq \alpha + 3$	1	1	1	$2^{3-r}/p$	$9 \cdot 2^k - 3 \cdot 2^{7+\alpha-2r}/p$
$9 \cdot 2^r : 4 \leq r \leq \alpha + 3$	$1/3$	$1/3$	1	$2^{3-r}/3$	$2^k + 8/3 - p \cdot 2^{7+\alpha-2r}/3$
$9p \cdot 2^r : 4 \leq r \leq \alpha + 3$	$1/3$	$1/3$	1	$2^{3-r}/3p$	$2^k + 8/3 - 2^{7+\alpha-2r}/3p$
$9 \cdot 2^{\alpha+4}, 9 \cdot 2^{\alpha+5}$	$1/6$	$1/3$	$1/2$	$2^{-1-\alpha}/3$	$1/6 - p \cdot 2^{-1-\alpha}/3$
$9p \cdot 2^{\alpha+4}, 9p \cdot 2^{\alpha+5}$	$1/6$	$1/3$	$1/2$	$2^{-1-\alpha}/3p$	$1/6 - 2^{-1-\alpha}/3p$

Proof continued...

For $2^\alpha \geq p$, we now find that $L \geq 0$ for all $d \mid 9p \cdot 2^{\alpha+5}$ and for all $k \geq 2\alpha$. Hence, $B_{\alpha,p,k}(z)$ is holomorphic at every cusp $\frac{c}{d}$. We find that the weight of $B_{\alpha,p,k}(z)$ is $\ell = 2^{k-1}$. Also, the associated character for $B_{\alpha,p,k}(z)$ is given by $\chi = \left(\frac{2^{\alpha+2^k\cdot(\alpha+2)}3^{2^k+1}p^{2^k+1}}{\bullet}\right)$.

Thus, $B_{\alpha,p,k}(z) \in M_{2^{k-1}}\left(\Gamma_0(9p \cdot 2^{\alpha+5}), \chi\right)$ for all $k \geq 2\alpha$.

Proof continued...

For $2^{\alpha} \geq p$, we now find that $L \geq 0$ for all $d \mid 9p \cdot 2^{\alpha+5}$ and for all $k \geq 2\alpha$. Hence, $B_{\alpha,p,k}(z)$ is holomorphic at every cusp $\frac{c}{d}$. We find that the weight of $B_{\alpha,p,k}(z)$ is $\ell = 2^{k-1}$. Also, the associated character for $B_{\alpha,p,k}(z)$ is given by $\chi = (\frac{2^{\alpha+2^k\cdot(\alpha+2)}3^{2^k+1}p^{2^k+1}}{\bullet})$.

Thus, $B_{\alpha,p,k}(z) \in M_{2^{k-1}}\left(\Gamma_0(9p \cdot 2^{\alpha+5}), \chi\right)$ for all $k \geq 2\alpha$.

Hence, the result follows from Serre's theorem.

Recent progress

Very recently, Barman & Singh (Divisibility of certain singular overpartitions by powers of 2 and 3, Bull. Aust. Math. Soc. (To be appeared)), has extended our work.

Recent progress

Very recently, Barman & Singh (Divisibility of certain singular overpartitions by powers of 2 and 3, Bull. Aust. Math. Soc. (To be appeared)), has extended our work.

To be specific,

Theorem (Barman & Singh, Bull. Aust. Math. Soc. (To be appeared))

Let k be a fixed positive integer. Then the set

$$\{n \in \mathbb{N} : \overline{C}_{3\ell,\ell}(n) \equiv 0 \pmod{3^k}\}$$

has arithmetic density 1 for $\ell = 2, 4$.

Recent progress

We also proved some Ramanujan's type congruences using a result of Ono and Taguchi on nilpotency of Hecke operators.

Recent progress

We also proved some Ramanujan's type congruences using a result of Ono and Taguchi on nilpotency of Hecke operators. More precisely,

Theorem (Barman & Singh, Bull. Aust. Math. Soc. (To be appeared))

Let n be a non-negative integer. Then there is an integer $s \geq 0$ such that for every $t \geq 1$ and distinct primes $p_1, \ldots, p_{s+t}$ coprime to 6, we have

$$\overline{C}_{6,2}\left(\frac{p_1 \cdots p_{s+t} \cdot n - 1}{24}\right) \equiv 0 \pmod{2^t}$$

whenever n is coprime to $p_1, \ldots, p_{s+t}$.

Thank You