Seminars

Tessellation
by Equilateral Pentagons

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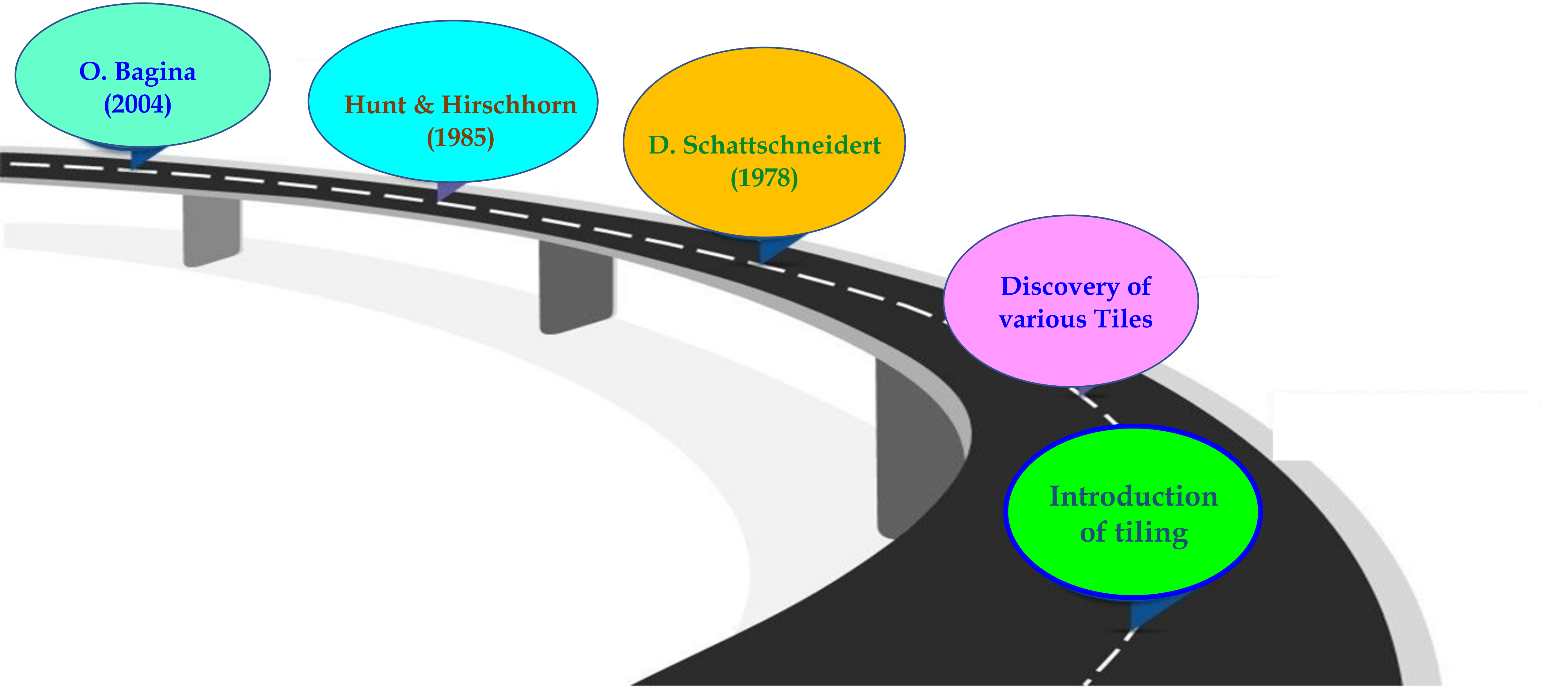
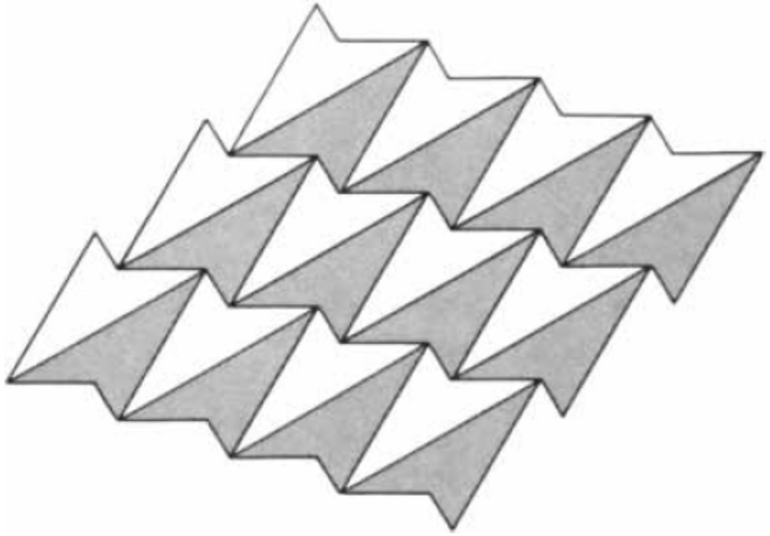
January 29, 2021

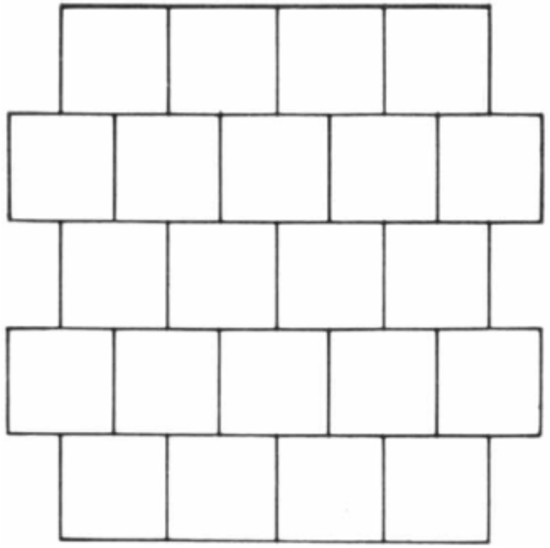
CHRIST (Deemed to be University),
Bangalore

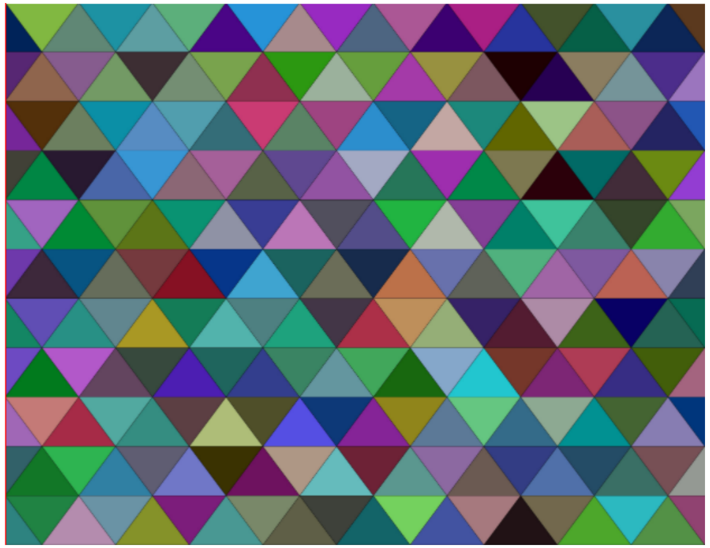
Outline

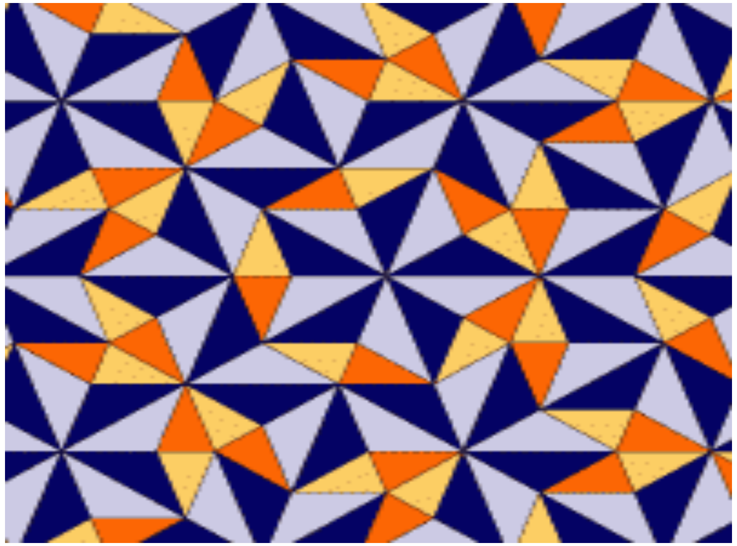

O. Bagina
(2004)
Hunt & Hirschhorn
(1985)
D. Schattschneidert
(1978)
Discovery of
various Tiles
Introduction
of tiling

A Tiling or Tessellation of a flat surface is the covering of a plane by polygons without overlapping and with no gaps. This tessellation can be done by regular as well irregular polygons.



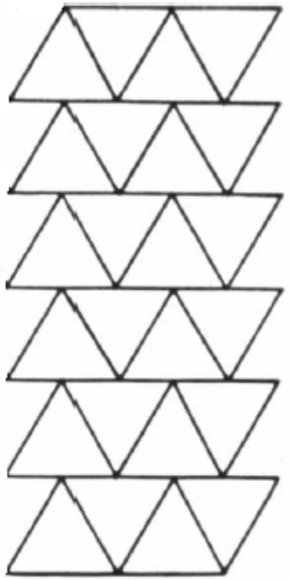


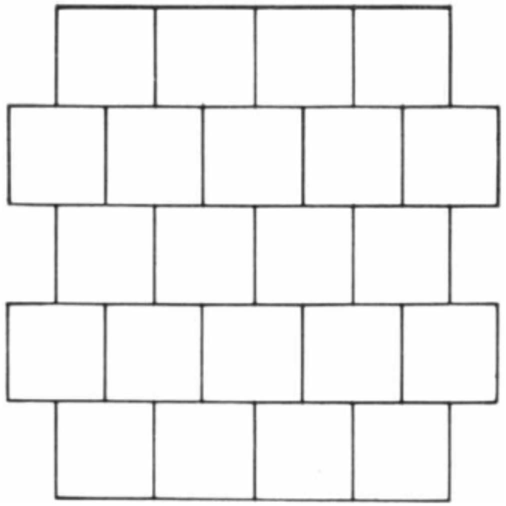


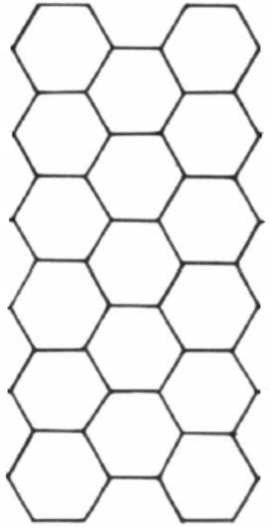


Monohedral tiling

If all the tiles in a tessellation are of the same size and shape, then the tiling is called monohedral.

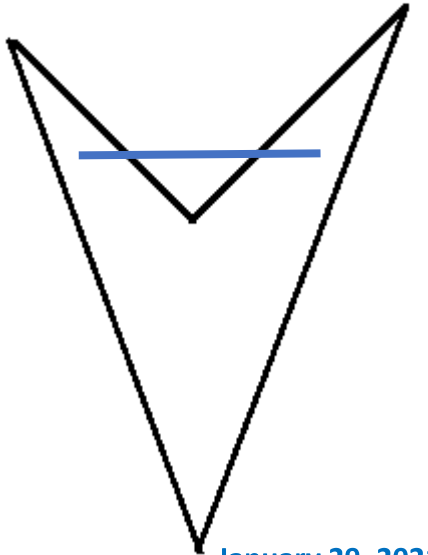






Edge-to-edge tiling

If any two polygons in a polygonal tiling are either disjoint or share one vertex or an entire edge in common, then the tiling is called edge – to – edge tiling.



Convex

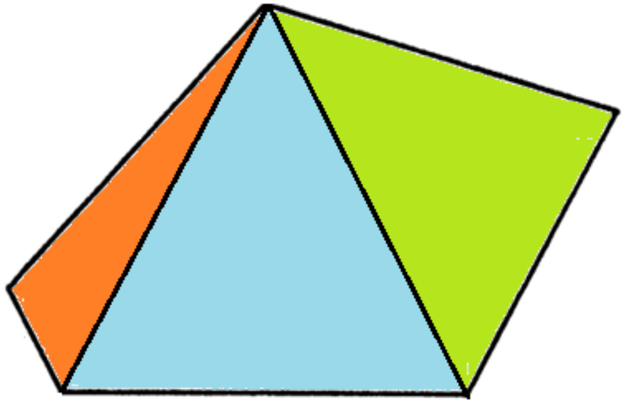
Polygons whose interior angles are each less than 180 degrees, is called convex polygon.

$$x,\ y \in A => t\,x + (1-t)y \in A$$

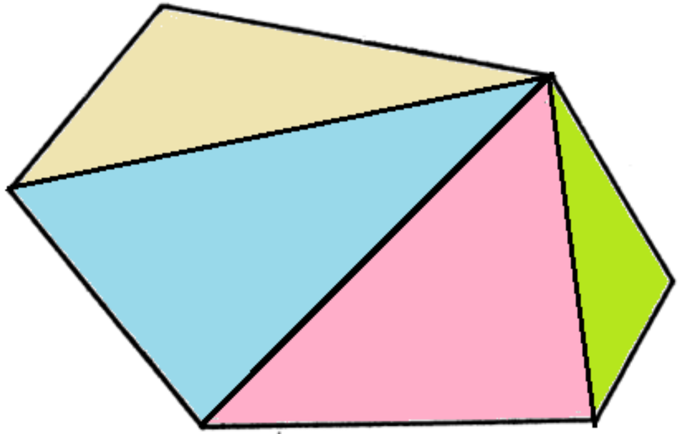
Tiling the plane is an ancient subject in our civilization.

From the ancient Greeks it is known that, among the regular polygons, only the **triangle**, the **square**, and the **hexagon** can tile the plane.

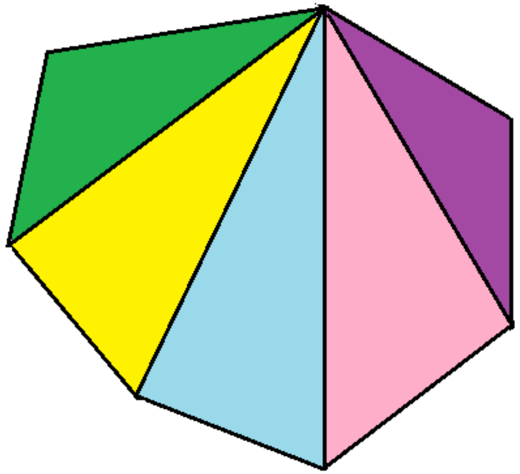
Because any *n*-sided polygon can be divided into $(n-2)$ triangles



A pentagon, 5 – sided polygon divided into three triangles



A hexagon, 6 – sided polygon divided into four triangles



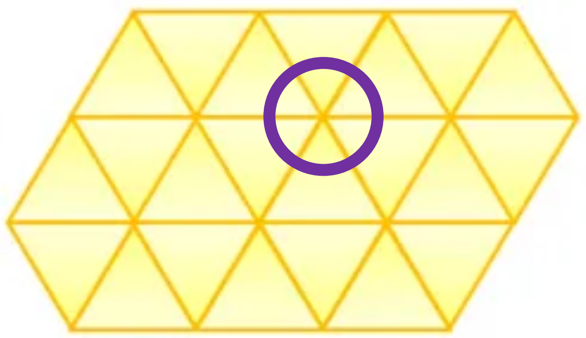
A heptagon, 7 – sided polygon divided into five triangles

Therefore, the sum of the interior angles of *n* – sided polygon is given by

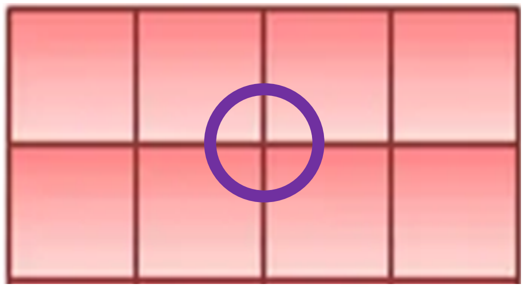
$$S_n = (n-2)\pi, \quad n \geq 3$$

Now the interior angles of a regular polygon being all equal, an **internal angle of a regular polygon of side *n* is given by**

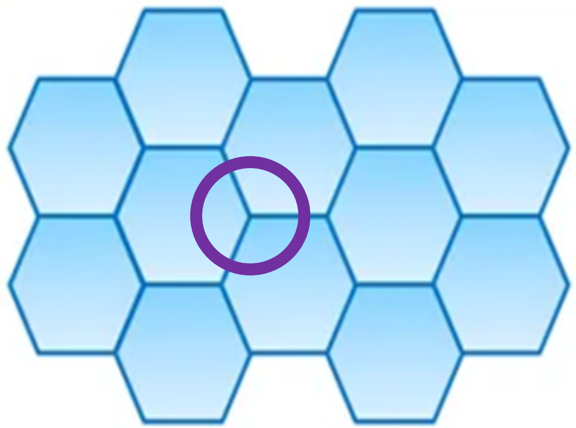
$$\theta_n = \left(\frac{n-2}{n}\right)\pi$$



Each internal angle of an equilateral triangle is 60° and six equilateral triangles around a vertex constitute 360°.



Each internal angle of a square is 90° and four squares around a vertex constitute 360°.



Each internal angle of a hexagon is 120° and three hexagons around a vertex constitute 360°.

These three examples are the only regular, edge-to-edge, convex, monohedral tilings of the plane.

It took almost 100 years, multiple contributors and in the end an exhaustive computer search to find all types of polygons that tile a plane.

In 1910, when **K. Reinhardt** started his doctoral thesis, his supervisor **Bieberbach** suggested that he ***determine all the convex domains which can tile the whole plane*** and later in 1918, Reinhardt, received his doctoral degree with a thesis titled "***On Partitioning the Plane into Polygons***". This is the first formal approach in characterizing all the convex domains that can tile the whole plane. He also obtained that:

A convex m - gon can tile the whole plane $\mathbb{E}^2$ by identical copies only if $m \leq 6$.

So the problem reduces to the determination of those convex hexagons and pentagons which can pave the plane.

Martin Gardner, a famous scientific writer, in the "***Mathematical Games***" column of the ***Scientific American*** magazine, wrote an article in 1975, giving a detailed evolution of all types of polygons that tile a plane. That publication popularized the concept of tiling. Since then, the tiling problem has stimulated many amateurs who went on to make significant contributions to this problem.

The 15 types of convex pentagons that admit tilings (not all edge-to-edge) of the plane were discovered by:

Karl Reinhardt in 1918

Richard Kershner in 1968

Richard James in 1975

Marjorie Rice in 1977

Rolf Stein in 1985

Casey Mann, Jennifer McLoud-Mann and David Von Derau in 2015

Finally in 2017, Michaël Rao verified computationally that 15 types of pentagons that can tessellate is an exhaustive list.

Tessellation by Equilateral Pentagons

1918, Reinhardt gave **three types of hexagon** and **five types of pentagon** that will tile a whole plane.

↓

1968, Kershner gave **three more types of pentagon** that will tile a whole plane.

↓

1975, Richard James gave **one more type of pentagon** that will tile a whole plane.

↓

1980, Marjorie Rice gave **four more types of pentagon** that will tile a whole plane.

↓

1985, Rolf Stein gave **one more type of pentagon** that will tile a whole plane.

↓

2015, Casey Mann, Jennifer McLoud and **David Von Derau** gave **one more type of pentagon** that will tile a whole plane.

↓

2017, Michaël Rao proved no other polygonal tiling is possible except these fifteen type.

3 hexagons
5 pentagons

3 hexagons
8 pentagons

3 hexagons
9 pentagons

3 hexagons
13 pentagons

3 hexagons
14 pentagons

3 hexagons
15 pentagons

Pentagon of Type 1: $A + B + C = 2\pi$

Pentagon of Type 2: $A + B + D = 2\pi, \quad a = d$

Pentagon of Type 3: $A = C = D = \frac{2}{3}\pi, \quad a = b, \quad d = c + e$

Pentagon of Type 4: $A = C = \frac{1}{2}\pi, \quad a = b, \quad c = d$

Pentagon of Type 5: $A = \frac{1}{3}\pi, \quad C = \frac{2}{3}\pi, \quad a = b, \quad c = d$

Pentagon of Type 6: $A + B + D = 2\pi, \quad A = 2C, \quad a = b = e, \quad c = d$

Pentagon of Type 7: $2B + C = 2D + A = 2\pi, \quad a = b = c = d$

Pentagon of Type 8: $2A + B = 2D + C = 2\pi, \quad a = b = c = d$

Pentagon of Type 9: $B + 2E = 2\pi, \quad C + 2D = 2\pi, \quad a = b = c = d$

Pentagon of Type 10: $E = \frac{\pi}{2}, \quad A + D = \pi, \quad 2B - D = \pi, \quad 2C + D = 2\pi, \quad a = e = b + d$

Pentagon of Type 11: $A = \frac{\pi}{2}, \quad C + E = \pi, \quad 2B + C = 2\pi, \quad 2a + c = d = e$

Pentagon of Type 12: $A = \frac{\pi}{2}, \quad C + E = \pi, \quad 2B + C = 2\pi, \quad 2a = c + e = d$

Pentagon of Type 13: $A = C = \frac{\pi}{2}, \quad 2B + D = 2E + D = 2\pi, \quad c = d, \quad 2c = e$

Pentagon of Type 14: $A = \frac{\pi}{2}, \quad 2B + C = 2\pi, \quad C + E = \pi, \quad 2a = 2c = d = e$

Pentagon of Type 15: $A = \frac{\pi}{2}, B = \frac{2\pi}{3}, C = \frac{7\pi}{12}, D = \frac{\pi}{2}, E = \frac{5\pi}{6}, \quad a = 2b = 2d = 2e$

Hexagon of Type 1: $A + B + C = 2\pi, \quad a = d$

Hexagon of Type 2: $A + B + D = 2\pi, \quad a = d, \quad c = e$

Hexagon of Type 3: $A = C = E = \frac{2}{3}\pi, \quad a = b, \quad c = d, \quad e = f$

We now search for tiling by all those convex polygons which are equilateral.

Quite obviously it is observed that Pentagonal tiling of the following types can not be equilateral.

Pentagon of Type 3: $A = C = D = \frac{2}{3}\pi, \quad a = b, \quad d = c + e$

Pentagon of Type 10:

$E = \frac{\pi}{2}, A + D = \pi, 2B - D = \pi, C + 2D = 2\pi, a = e = b + d$

Pentagon of Type 11: $A = \frac{\pi}{2}, C + E = \pi, 2B + C = 2\pi, d = e = 2a + c$

Pentagon of Type 12:

$A = \frac{\pi}{2}, C + E = \pi, 2B + C = 2\pi, \quad 2a = c + e = d$

Pentagon of Type 13:

$A = C = \frac{\pi}{2}, 2B + D = 2E + D = 2\pi, c = d, 2c = e$

Pentagon of Type 14: $A = \frac{\pi}{2}, 2B + C = 2\pi, C + E = \pi, \quad 2a = 2c = d = e$

Pentagon of Type 15:

$A = \frac{\pi}{3}, B = \frac{2\pi}{3}, C = \frac{7\pi}{14}, D = \frac{\pi}{2}, E = \frac{5\pi}{6}, \quad a = 2b = 2d = 2e$

Therefore, the search for equilateral pentagonal tiling, filters down into following 8 types from the exhaustive list of 15 types of pentagonal tiling.

Pentagon of Type 1: $A + B + C = 2\pi$

Pentagon of Type 2: $A + B + D = 2\pi, \quad a = d$

Pentagon of Type 4: $A = C = \frac{1}{2}\pi, \quad a = b, \quad c = d$

Pentagon of Type 5: $A = \frac{1}{3}\pi, \quad C = \frac{2}{3}\pi, \quad a = b, \quad c = d$

Pentagon of Type 6: $A + B + D = 2\pi, \quad A = 2C, \quad a = b = e, \quad c = d$

Pentagon of Type 7: $2B + C = 2D + A = 2\pi, \quad a = b = c = d$

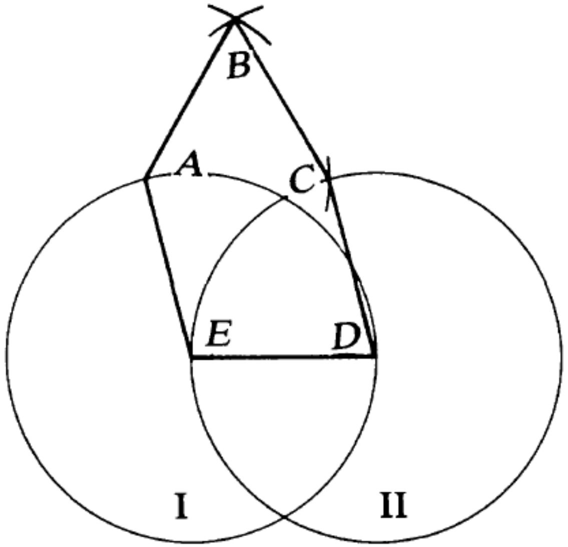
Pentagon of Type 8: $2A + B = 2D + C = 2\pi, \quad a = b = c = d$

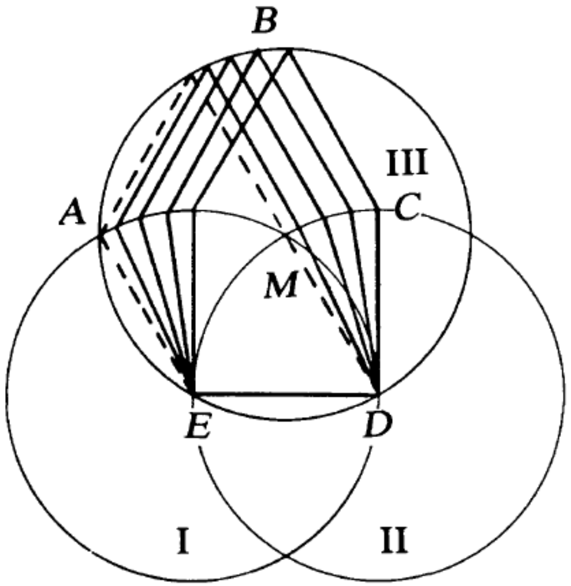
Pentagon of Type 9: $B + 2E = 2\pi, \quad C + 2D = 2\pi, \quad a = b = c = d$

Doris Schattschneidert in 1978 determined all possible equilateral convex pentagons from 13 types pentagonal tiles. He used the techniques of geometrical construction and trigonometrical results to list out all three types equilateral convex pentagons.

•D. Schattschneidert, Tiling the plane with congruent pentagons, Math. Mag. **51** (1), pp. 29-44, 1978.

Pentagon of Type 1: $A + B + C = 2\pi$





Take angle E and draw circle I with E as the center. Find vertex A and D on circle I so that two sides EA and ED been constructed.

Now draw another circle II with D as the center and DE as the radius. Find a vertex C on the circle II so that $DC \parallel EA$. Join C & A.

Next draw third circle III with M, the point of intersection of circle I & II, as the center and DE as the radius. Finally, we construct sides $CB = AB$.

Then for each pentagon of the type I, vertex A lies on circle I, vertex C lies on circle II and vertex B lies on circle III.

Thus, an equilateral pentagon $ABCDE$ of type I is obtained.

From the construction it is observed that $\frac{\pi}{2} \leq E < \frac{2\pi}{3}$

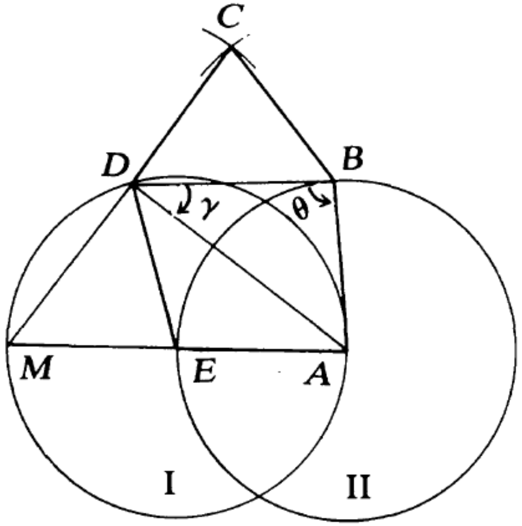
$$\therefore \quad \frac{\pi}{3} < E < \frac{2\pi}{3}, \quad A = \frac{4\pi}{3} - E, \quad B = \frac{\pi}{3}, \quad C = \frac{\pi}{3} + E, \quad D = \pi - E$$

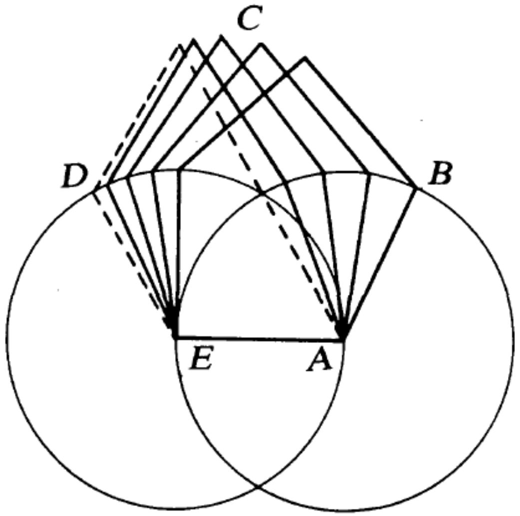
D. Schattschneidert, Tiling the plane with congruent pentagons, Math. Mag. **51** (1), pp. 29-44, 1978.

Pentagon of Type 2:

$A + B + D = 2\pi,$

$a = d$





Take angle ***E*** and draw circle I with ***E*** as the center. Find vertex ***A*** and ***D*** on circle I so that two sides ***EA*** and ***ED*** been constructed. Extend ***AE*** to ***M*** on circle I.

Now draw another circle II with ***A*** as the center and ***AE*** as the radius. Find a vertex ***B*** on the circle II so that ***DB* = *DM***.

Next take a point C such that $\Delta\ DCB \cong \Delta\ DME$ and finally join ***A*** & ***B*** to form the equilateral pentagon ***ABCDE*** of type II.

From the construction it is observed that $\frac{\pi}{2} \leq E < \frac{2\pi}{3}$

$$\therefore \quad \frac{\pi}{3} < E < \frac{2\pi}{3}, \quad \theta = \cos^{-1}\left(\frac{1 + 4\cos E}{4\cos\frac{E}{2}}\right), \quad \gamma = \cos^{-1}\left(\frac{3}{4\sin E}\right)$$

$$A = \frac{3\pi}{2} - \frac{E}{2} - \theta - \gamma, \quad B = \frac{E}{2} + \theta, \quad C = \pi - E, \quad D = \frac{\pi}{2} + \gamma$$

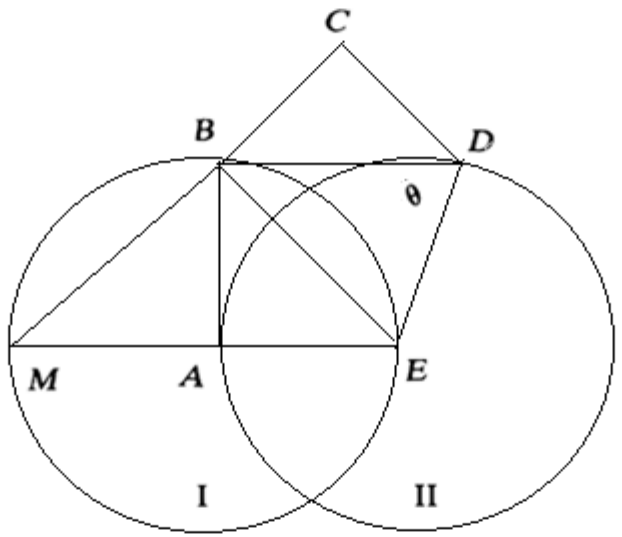
D. Schattschneidert, Tiling the plane with congruent pentagons, Math. Mag. **51** (1), pp. 29-44, 1978.

Pentagon of Type 4:

$A = C = \frac{1}{2}\pi,$

$a = b, \quad c = d$

Take angle A and draw circle I with A as the center. Find vertex B and E on circle I so that two sides AB and AE been constructed. Extend EA to M on circle I.



Now draw another circle II with E as the center and EA as the radius. Find a vertex D on the circle II so that $BD = BM$.

Next take a point C such that $\Delta\ BCD \cong \Delta\ BMA$ and finally join E & D to form the equilateral pentagon $ABCDE$ of type IV.

From the construction it is observed that

$$A = \frac{\pi}{2}, \quad C = \frac{\pi}{2}, \quad \theta = \cos^{-1}\left(\frac{1}{2\sqrt{2}}\right),$$

$$B = 3\pi - (A + C) - (B + D) = 2\pi - 2\cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

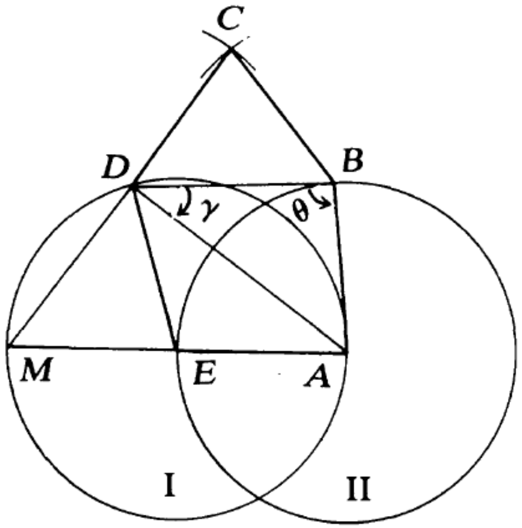
$$D = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

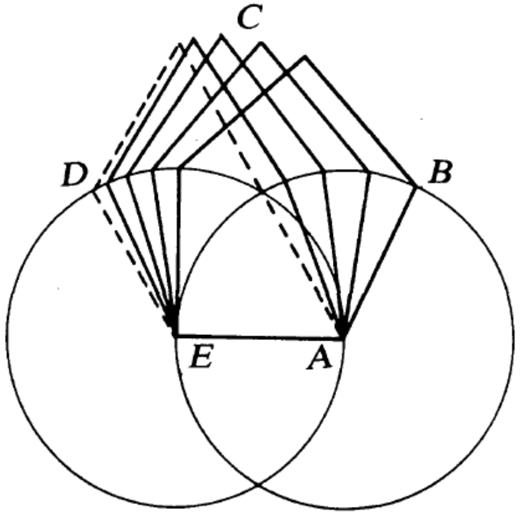
$$E = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

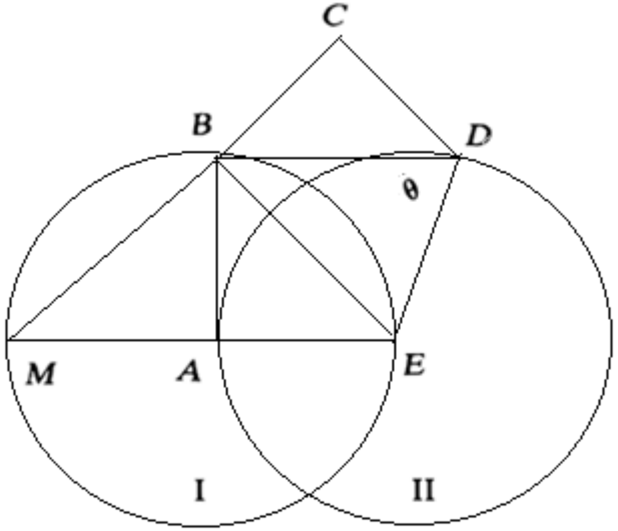
D. Schattschneidert, Tiling the plane with congruent pentagons, Math. Mag. 51 (1), pp. 29-44, 1978.

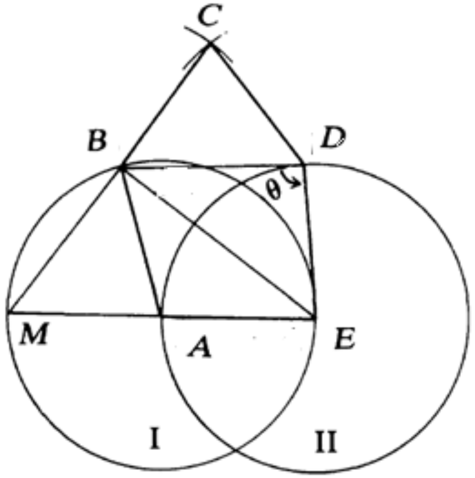
Pentagon of Type 2: $A + B + D = 2\pi, \quad a = d$

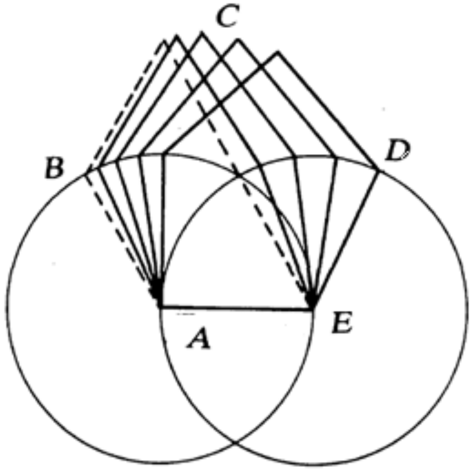
Pentagon of Type 4: $A = C = \frac{1}{2}\pi, \quad a = b, \quad c = d$











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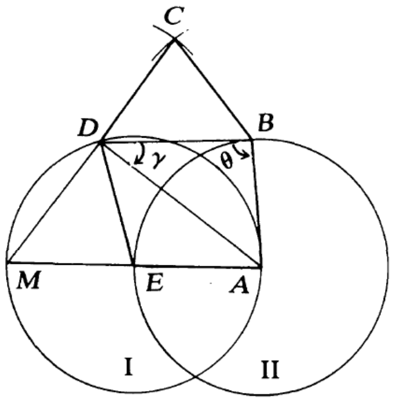
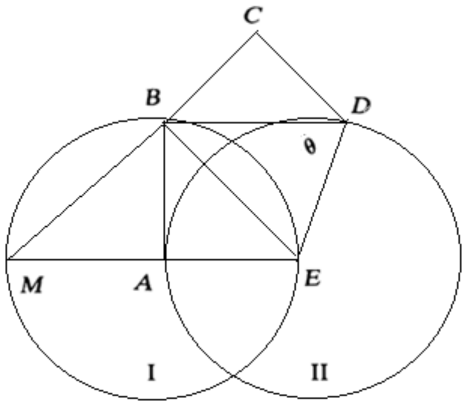


From the construction it is observed that $\frac{\pi}{2} \leq E < \frac{2\pi}{3}$

$$\therefore \quad \frac{\pi}{3} < E < \frac{2\pi}{3}, \quad \theta = \cos^{-1}\left(\frac{1+4\cos E}{4\cos\frac{E}{2}}\right), \quad \gamma = \cos^{-1}\left(\frac{3}{4\sin E}\right)$$

$$A = \frac{3\pi}{2} - \frac{E}{2} - \theta - \gamma, \quad B = \frac{E}{2} + \theta, \quad C = \pi - E, \quad D = \frac{\pi}{2} + \gamma$$



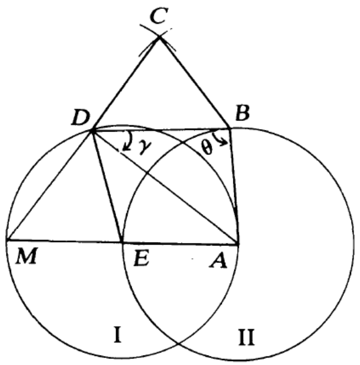
D. Schattschneidert, Tiling the plane with congruent pentagons, Math. Mag. **51** (1), pp. 29-44, 1978.

Pentagon of Type 5:

$$A = \frac{1}{3}\pi, \quad C = \frac{2}{3}\pi, \quad a = b, \quad c = d$$

$$A = \frac{\pi}{3}, C = \frac{2\pi}{3} \implies A + C = \pi \quad \text{and thus} \quad B + D + E = 2\pi$$

This leads to the construction of a limiting equilateral pentagon of type II, obtained by relabelling the vertices while replacing the angle ***E*** by ***A*** and ***D*** by ***B***.



Pentagon of Type 2:

$$A + B + D = 2\pi, \quad a = d$$

$$\therefore \quad \frac{\pi}{3} < E < \frac{2\pi}{3}, \quad \theta = \cos^{-1}\left(\frac{1 + 4\cos E}{4\cos\frac{E}{2}}\right), \quad \gamma = \cos^{-1}\left(\frac{3}{4\sin E}\right)$$

$$A = \frac{3\pi}{2} - \frac{E}{2} - \theta - \gamma, \quad B = \frac{E}{2} + \theta, \quad C = \pi - E, \quad D = \frac{\pi}{2} + \gamma$$

Pentagon of Type 5:

$$B + D + E = 2\pi, \quad a = b, \quad c = d$$

$$\frac{\pi}{3} < A < \frac{2\pi}{3}, \quad \theta = \cos^{-1}\left(\frac{1 + 4\cos A}{4\cos\frac{A}{2}}\right) = \frac{\pi}{6}, \quad \gamma = \cos^{-1}\left(\frac{3}{4\sin A}\right) = \frac{\pi}{6}$$

$$E = \frac{3\pi}{2} - \frac{A}{2} - \theta - \gamma = \frac{3\pi}{2} - \frac{\pi}{6} - \frac{\pi}{6} - \frac{\pi}{6} = \pi$$

This is IMPOSSIBLE.

D. Schattschneidert, Tiling the plane with congruent pentagons, Math. Mag. **51** (1), pp. 29-44, 1978.

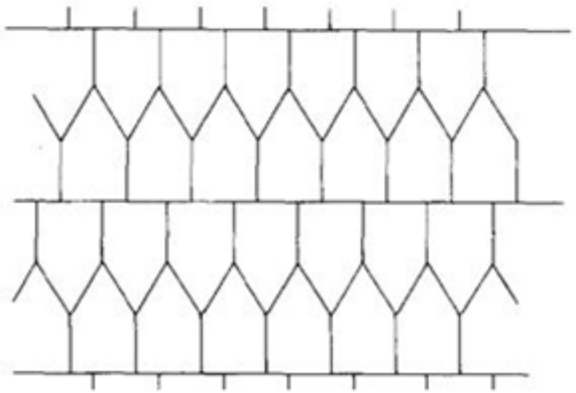
M.D. Hirschhorn and D. C. Hunt in 1985 have developed a theorem for the problem of finding all equilateral convex pentagons which tile the plane.

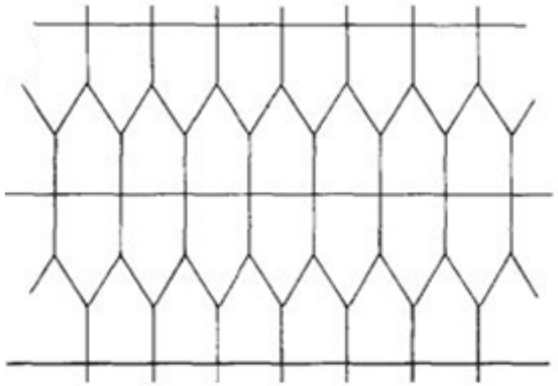
THEOREM: An equilateral convex pentagon tiles the plane if and only if it has two angles adding to 180°, or it is the unique equilateral convex pentagon X with angles A, B, C, D, E satisfying $A + 2B = 360°$, $C + 2E = 360°$, $A + C + 2D = 360°$

For equilateral convex pentagon of edge –to-edge tiling, they proposed

$$m_A A + m_B B + m_C C + m_D D + m_E E = 360^{o}$$

where m_A, m_B, m_C, m_D, m_E are nonnegative integers.

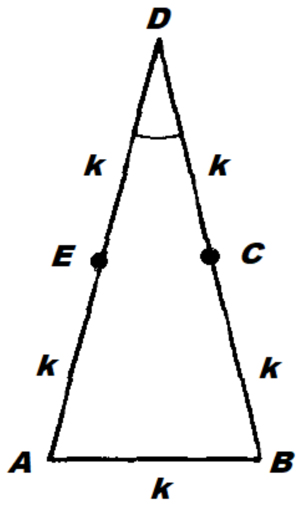




M.D. Hirschhorn, D.C. Hunt, Equilateral Convex Pentagons Which Tile the Plane, J. Combin. Theory. Ser. A 39 (1985) 1–18.

Each angle in an equilateral convex pentagon is greater than $\cos^{-1}\left(\frac{7}{8}\right) = 28^{o}$ since if any angle was less than or equal to this, the polygon would fail to be a convex pentagon.

$$28^{o} < A, B, C, D, E < 180^{o}$$



$$\therefore\ m_A A + m_B B + m_C C + m_D D + m_E E = 360^{o} \Longrightarrow m_A + m_B + m_C + m_D + m_E \le 12$$

$$m_A A + m_B B + m_C C + m_D D + m_E E = 360^{o} \Longrightarrow m_A + m_B + m_C + m_D + m_E \ge 3$$

Thus, they have proved the existence of a finite set of relations that any equilateral convex pentagon can satisfy.

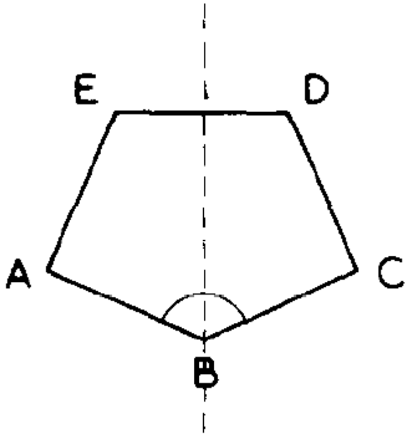
Then, they proved a series of Lemmas in order to reduce the possible cases of feasible solutions, in order to prove the proposed theorem.

M.D. Hirschhorn, D.C. Hunt, Equilateral Convex Pentagons Which Tile the Plane, J. Combin. Theory. Ser. A 39 (1985) 1–18.

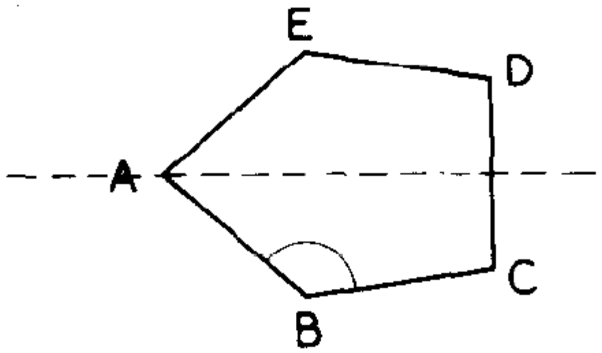
Lemma 1 $A \leq C \leq D \leq E \leq B$

$$A \leq C \leq \lim C = \lim D \leq D \leq E \leq B$$

The greatest value of A occurs when $A = C$, $D = E$.



The least value of A occurs when $B = E$, $C = D$.



M.D. Hirschhorn, D.C. Hunt, Equilateral Convex Pentagons Which Tile the Plane, J. Combin. Theory. Ser. A 39 (1985) 1–18.

Lemma 2

$$108^o \leq B < 180^o$$

$$180^o - \frac{B}{2} - \sin^{-1}\left(\sin\frac{B}{2} - \frac{1}{2}\right) \geq A \geq 180^o - B + 2\sin^{-1}\left(\frac{1}{4\sin\frac{B}{2}}\right)$$

$$D = \cos^{-1}\left(\cos A + \cos B - \cos(A+B) - \frac{1}{2}\right)$$

$$C = 270^o - B - \frac{D}{2} + \theta$$

$$E = 270^o - A - \frac{D}{2} - \theta \qquad \text{where} \qquad \theta = \tan^{-1}\left(\frac{\sin A - \sin B}{1 - \cos A - \cos B}\right)$$

In an equilateral convex pentagon, the angles ***C***, ***D*** and ***E*** are uniquely determined by the angles ***A*** and ***B***.

$$\cos^{-1}\left(\frac{7}{8}\right) < A \leq 108^o,\ 108^o \leq B < 180^o,\ 60^o < C \leq 108^o, \cos^{-1}\left(\frac{1}{4}\right) < D < 120^o,\ 108^o \leq E < 180^o$$

These constraints allow just **220** solutions to the equation

$$m_A A + m_B B + m_C C + m_D D + m_E E = 360^o$$

M.D. Hirschhorn, D.C. Hunt, Equilateral Convex Pentagons Which Tile the Plane, J. Combin. Theory. Ser. A 39 (1985) 1–18.

For each set $(m_A, m_B, m_C, m_D, m_E)$, they considered a function of A , B defined over the region

$$\wp\left(= (28^{o},\ 108^{o}] \times [108^{o},\ 180^{o}) \right) \text{ by}$$

$$m_A m_B m_C m_D m_E = m_A A + m_B B + m_C C + m_D D + m_E E$$

They further defined a partial order

$$m_A m_B m_C m_D m_E < m'_A m'_B m'_C m'_D m'_E$$

on the set of functions by

$$m_A A + m_B B + m_C C + m_D D + m_E E \leq m'_A A + m'_B B + m'_C C + m'_D D + m'_E E$$

This reduces the feasible solutions from 220 to **100**, eliminating 120 relations.

M.D. Hirschhorn, D.C. Hunt, Equilateral Convex Pentagons Which Tile the Plane, J. Combin. Theory. Ser. A 39 (1985) 1–18.

They have listed out all the 100 relations satisfied by some equilateral convex pentagons.

1. $A+2E=360^\circ$
2. $A+B+E=360^\circ$
3. $A+2B=360^\circ$
4. $C+2E=360^\circ$
5. $B+C+E=360^\circ$
6. $2B+C=360^\circ$
7. $B+2D=360^\circ$
8. $D+2E=360^\circ$
9. $B+D+E=360^\circ$
10. $2B+D=360^\circ$
11. $3E=360^\circ$
12. $B+2E=360^\circ$
13. $2B+E=360^\circ$
14. $3B=360^\circ$
15. $4A=360^\circ$
16. $3A+C=360^\circ$
17. $3A+D=360^\circ$
18. $3A+E=360^\circ$
19. $3A+B=360^\circ$
20. $2A+2C=360^\circ$
21. $2A+C+D=360^\circ$
22. $2A+C+E=360^\circ$
23. $2A+B+C=360^\circ$
24. $2A+2D=360^\circ$
25. $2A+D+E=360^\circ$
26. $2A+B+D=360^\circ$
27. $A+3C=360^\circ$
28. $A+2C+D=360^\circ$
29. $A+2C+E=360^\circ$
30. $A+C+2D=360^\circ$
31. $A+3D=360^\circ$
32. $4C=360^\circ$
33. $3C+D=360^\circ$
34. $3C+E=360^\circ$
35. $2C+2D=360^\circ$
36. $C+3D=360^\circ$
37. $4D=360^\circ$
38. $5A=360^\circ$
39. $4A+C=360^\circ$
40. $4A+D=360^\circ$
41. $4A+E=360^\circ$
42. $4A+B=360^\circ$
43. $3A+2C=360^\circ$
44. $3A+C+D=360^\circ$
45. $3A+C+E=360^\circ$
46. $3A+B+C=360^\circ$
47. $3A+2D=360^\circ$
48. $3A+D+E=360^\circ$
49. $3A+B+D=360^\circ$
50. $2A+3C=360^\circ$
51. $2A+2C+D=360^\circ$
52. $2A+C+2D=360^\circ$
53. $2A+3D=360^\circ$
54. $A+4C=360^\circ$
55. $A+3C+D=360^\circ$
56. $A+2C+2D=360^\circ$
57. $A+C+3D=360^\circ$
58. $A+4D=360^\circ$
59. $5C=360^\circ$
60. $6A=360^\circ$
61. $5A+C=360^\circ$
62. $5A+D=360^\circ$
63. $5A+E=360^\circ$
64. $5A+B=360^\circ$
65. $4A+2C=360^\circ$
66. $4A+C+D=360^\circ$
67. $4A+2D=360^\circ$
68. $3A+3C=360^\circ$
69. $3A+2C+D=360^\circ$
70. $3A+C+2D=360^\circ$
71. $3A+3D=360^\circ$
72. $7A=360^\circ$
73. $6A+C=360^\circ$
74. $6A+D=360^\circ$
75. $6A+E=360^\circ$
76. $6A+B=360^\circ$
77. $5A+2C=360^\circ$
78. $5A+C+D=360^\circ$
79. $5A+2D=360^\circ$
80. $4A+3C=360^\circ$
81. $4A+2C+D=360^\circ$
82. $4A+C+2D=360^\circ$
83. $4A+3D=360^\circ$
84. $8A=360^\circ$
85. $7A+C=360^\circ$
86. $7A+D=360^\circ$
87. $6A+2C=360^\circ$
88. $6A+C+D=360^\circ$
89. $6A+2D=360^\circ$
90. $9A=360^\circ$
91. $8A+C=360^\circ$
92. $8A+D=360^\circ$
93. $7A+2C=360^\circ$
94. $7A+C+D=360^\circ$
95. $7A+2D=360^\circ$
96. $10A=360^\circ$
97. $9A+C=360^\circ$
98. $9A+D=360^\circ$
99. $11A=360^\circ$
100. $12A=360^\circ$

M.D. Hirschhorn, D.C. Hunt, Equilateral Convex Pentagons Which Tile the Plane, J. Combin. Theory. Ser. A 39 (1985) 1–18.

Finally, they observed that if an equilateral convex pentagon tiles the plane, it simultaneously satisfies at least two of the 100 relations.

So, considering the intersection of the 100 relations, 10 relations identified which are equivalent.

(2) $A + B + E = 360^o$ (5) $B + C + E = 360^o$ (9) $B + D + E = 360^o$

(23) $2A + B + C = 360^o$ (24) $2A + 2D = 360^o$ (20) $2A + 2C = 360^o$

(25) $2A + D + E = 360^o$

(35) $2C + 2D = 360^o$

(44) $3A + C + D = 360^o$

(60) $6A = 360^o$

They concluded that an equilateral pentagonal tiling consists of two types. Type 1 consist of all equilateral convex pentagons with two adjacent angles adding to 180^o and Type 2 consists of all equilateral convex pentagons with two non - adjacent angles adding to 180^o.

Further elimination from the remaining 90 relations, they obtained **54** and then finally the only **three** possible relations

(2) $A + 2B = 360^o$ (4) $C + 2E = 360^o$ (30) $A + C + 2D = 360^o$

M.D. Hirschhorn, D.C. Hunt, Equilateral Convex Pentagons Which Tile the Plane, J. Combin. Theory. Ser. A 39 (1985) 1–18.

O. Bagina in 2004 has given an alternative demonstration, based on Euler's theorem on plane for the result proved by Hunt and Hirschhorn.

Bagina proved that

In each edge – to – edge tiling of the plane by uniformly bounded pentagons, there exists a tile with at least three vertices of valence three.

Bagina first proved a lemma given by B.N. Delone, N.P. Dolbilin, M.I. Shtogrinthat in 1978

Lemma

Let $\boldsymbol{T}$ be a tiling of the plane by polygons that are uniformly bounded. Then there exists an infinite sequence $\boldsymbol{U_i}$, $\boldsymbol{i = 1, 2, \dots}$ of finite unions of polygons from $\boldsymbol{T}$ such that

$$\Gamma(U_i) \to \infty, \quad \text{as} \quad i \to \infty \qquad \frac{\gamma(U_i)}{\Gamma(U_i)} \to \infty, \quad \text{as} \quad i \to \infty \qquad \chi(U_i) = 1, \quad \text{for all} \quad i.$$

Bagina O, Tiling the plane with congruent equilateral convex pentagons. J. Comb. Theory Ser. A **105**, 221–232, 2004.

Bagina considered a tile in the tiling with at least three vertices of valence 3 so that the vertices of valence 3 satisfy one of the following relations:

$$x_i \qquad + \qquad 2$$

$$x_j = 360^o, \quad i, j = 0, 1, 2, 3, 4$$

This leads two cases:

(1) Two adjacent angles are involved in the above relation $\quad x_0 + 2x_1 = 360^o$

(2) Only non - adjacent angles are involved in the above relation $\quad x_0 + 2x_2 = 360^o$

Finally using sorting techniques, Bagina proved the theorem.

Bagina O, Tiling the plane with congruent equilateral convex pentagons. J. Comb. Theory Ser. A **105**, 221–232, 2004.

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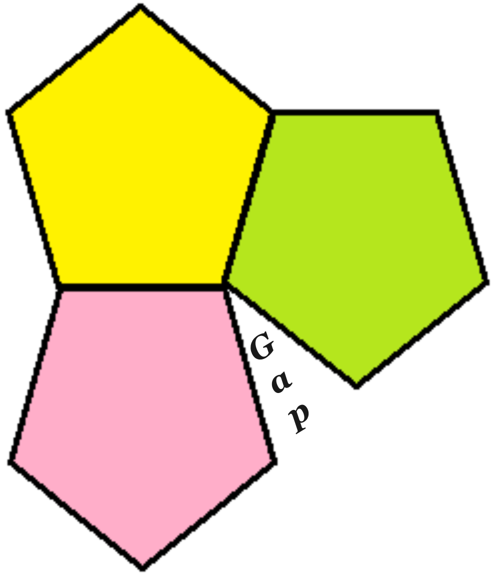
Thank You

These three examples are the only regular, edge-to-edge, monohedral tilings of the plane.

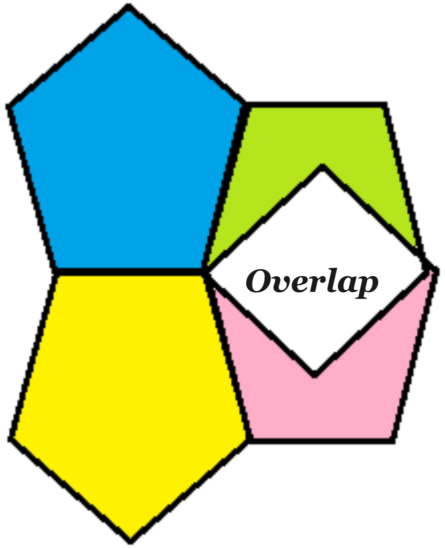
Why??

Each internal angle of a regular pentagon is 108°.

Three pentagons at a vertex gives us $(\mathbf{3\times 108^\circ}) = \mathbf{324^\circ}$, which leaves a gap of 36° that is too small to fill with another pentagon.

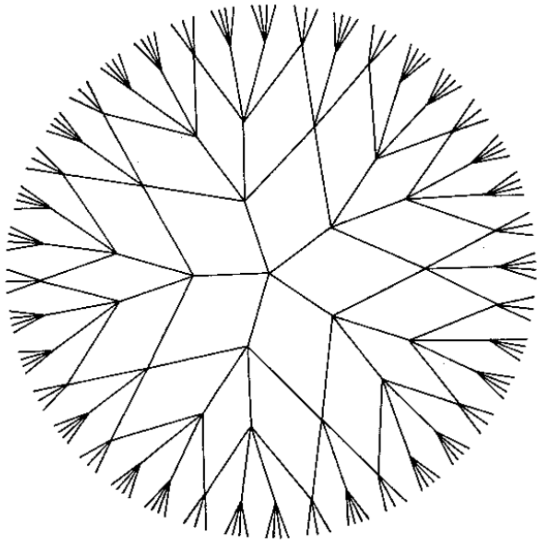


Four pentagons at a vertex gives us $(\mathbf{4\times 108^\circ}) = \mathbf{432^\circ}$, which produces unwanted overlap at a point.



No matter how they are arranged, regular pentagons will never match up around a vertex with no gap and no overlap. **This indicates regular pentagon does not tile a plane.**

The ***valence*** of a vertex in the tiling is the number of tiles meeting at the vertex.



The tiling ***T*** of the plane by polygons are said to be ***uniformly bounded*** when there exist real number ***s*** > 0 and ***r*** > 0 such that each tile in ***T*** contains a disc of radius ***s*** and is contained in a disk of radius ***r***.

In this tiling, each tile is a closed topological disk, these are 5- valanced convex quadrangles