Some aspects of Γ_2 graph over some of the finite commutative rings

Anurag Baruah

Research Scholar
Department of Mathematics
Gauhati University

14-08-2020

Online Seminar of Assamese Mathematicians

Introduction

• In 1988, I. Beck first introduced the concept of *zero divisor graph*, where he was mainly interested in colorings.
He conjectured that, $\chi(R) = \omega(R)$.

Introduction

• In 1988, I. Beck first introduced the concept of *zero divisor graph*, where he was mainly interested in colorings.
He conjectured that, $\chi(R) = \omega(R)$.
• In 1993, D. D. Anderson and M. Naseer came up with a counter example.

Introduction

• In 1988, I. Beck first introduced the concept of *zero divisor graph*, where he was mainly interested in colorings.
He conjectured that, $\chi(R) = \omega(R)$.
• In 1993, D. D. Anderson and M. Naseer came up with a counter example.
• In 1999, D.F. Anderson and P. Livingston modified this definition, which is now considered as the standard definition of *zero divisor graph*.

Introduction

• In 1988, I. Beck first introduced the concept of *zero divisor graph*, where he was mainly interested in colorings.
He conjectured that, $\chi(R) = \omega(R)$.

• In 1993, D. D. Anderson and M. Naseer came up with a counter example.

• In 1999, D.F. Anderson and P. Livingston modified this definition, which is now considered as the standard definition of *zero divisor graph*. They denoted this graph as $\Gamma(R)$ whose vertices are the nonzero zero-divisors of a commutative ring R, and in which two vertices a and b are connected by an edge if and only if $ab = 0$.

Introduction

• In 1988, I. Beck first introduced the concept of *zero divisor graph*, where he was mainly interested in colorings.
He conjectured that, $\chi(R) = \omega(R)$.

• In 1993, D. D. Anderson and M. Naseer came up with a counter example.

• In 1999, D.F. Anderson and P. Livingston modified this definition, which is now considered as the standard definition of *zero divisor graph.* They denoted this graph as $\Gamma(R)$ whose vertices are the nonzero zero-divisors of a commutative ring R, and in which two vertices a and b are connected by an edge if and only if $ab = 0$.

Consider the ring $\mathbb{Z}_3 \times \mathbb{Z}_3$
Here the non-zero zero divisors are $(0, 1), (1, 0), (0, 2), (2, 0)$

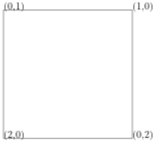

Figure 1. $\Gamma(\mathbf{Z}_3 \times \mathbf{Z}_3)$

Introduction

- In 2008, D.F.Anderson and A. Badawi, constructed a new type of graph called the *Total graph* of a commutative ring where the addition operation involving the zero divisors has been used.

Introduction

• In 2008, D.F.Anderson and A. Badawi, constructed a new type of graph called the *Total graph* of a commutative ring where the addition operation involving the zero divisors has been used.
They took all the elements of the ring R as the vertices of the graph, and for two distinct $x, y \in R$, x and y are adjacent if and only if $x + y$ is a zero divisor.

Ring theoretic definitions and elementary results

Definition

A subring S of a ring R is an *ideal* of R if $a \in S, r \in R \implies ra \in S$ and $ar \in S$.

Ring theoretic definitions and elementary results

Definition

A subring S of a ring R is an *ideal* of R if $a \in S, r \in R \implies ra \in S$ and $ar \in S$.

Definition

For a ring R, an ideal $M \neq R$ is *maximal* in R if for any ideal U of R satisfying $M \subset U \subset R$, either $U = M$ or $U = R$.

Ring theoretic definitions and elementary results

Definition

A subring S of a ring R is an *ideal* of R if $a \in S, r \in R \implies ra \in S$ and $ar \in S$.

Definition

For a ring R, an ideal $M \neq R$ is *maximal* in R if for any ideal U of R satisfying $M \subset U \subset R$, either $U = M$ or $U = R$.

Definition

Let R be a ring and $a \in R$. The smallest ideal of a ring R containing a is said to be the *principal ideal* generated by a, denoted as $< a >$.

Ring theoretic definitions and elementary results

• If $< n >$ be the principal ideal generated by n in $\mathbb{Z}[i]$, and let $\mathbb{Z}_n$ be the ring of integers modulo n. Then $\mathbb{Z}[i]/ < n >$ is isomorphic to $\mathbb{Z}_n[i] = \{\bar{a} + i\bar{b} : \bar{a}, \bar{b} \in \mathbb{Z}_n\}$, which implies that $\mathbb{Z}_n[i]$ is a principal ideal ring. This ring is called the ring of Gaussian integers modulo n.

Ring theoretic definitions and elementary results

• If $< n >$ be the principal ideal generated by n in $\mathbb{Z}[i]$, and let $\mathbb{Z}_n$ be the ring of integers modulo n. Then $\mathbb{Z}[i]/ < n >$ is isomorphic to $\mathbb{Z}_n[i] = \{\bar{a} + i\bar{b} : \bar{a}, \bar{b} \in \mathbb{Z}_n\}$, which implies that $\mathbb{Z}_n[i]$ is a principal ideal ring. This ring is called the ring of Gaussian integers modulo n.

Definition

A *local ring* is a ring R that contains a single maximal ideal.

Ring theoretic definitions and elementary results

• If $< n >$ be the principal ideal generated by n in $\mathbb{Z}[i]$, and let $\mathbb{Z}_n$ be the ring of integers modulo n. Then $\mathbb{Z}[i]/ < n >$ is isomorphic to $\mathbb{Z}_n[i] = \{\bar{a} + i\bar{b} : \bar{a}, \bar{b} \in \mathbb{Z}_n\}$, which implies that $\mathbb{Z}_n[i]$ is a principal ideal ring. This ring is called the ring of Gaussian integers modulo n.

Definition

A *local ring* is a ring R that contains a single maximal ideal.

Theorem

If $m = t^k$ for some prime t and positive integer k, then $\mathbb{Z}_m[i]$ is a local ring if and only if $t = 2$ or $t \equiv 3 (mod 4)$.

Ring theoretic definitions and elementary results

Theorem

$\bar{a}+i\bar{b}$ is an unit in $\mathbb{Z}_n[i]$ if and only if a^2+b^2 is an unit in $\mathbb{Z}_n$.

Theorem

If $n=\prod_{j=1}^{s} {a_j}^{k_j}$ is the prime power decomposition of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{{a_j}^{k_j}}[i]$.

These results can be found in 'The maximal regular ideal of some commutative rings' by E.A.Osba et al., Comment.Math.Univ.Carolin. 47,1 (2006)1–10.

Some background from graph theory

Definition

A graph G is said to be *connected* if any two distinct vertices of G are joined by a path. A graph G is said to be *disconnected* if G is not connected. A *maximal connected subgraph* of G is called a connected component of G.

Example :

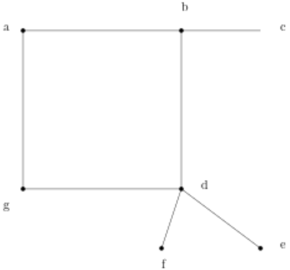

Figure 2.

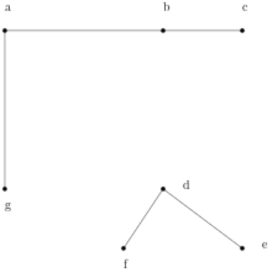

Figure 3.

Some background from graph theory

Definition

The *diameter* of a graph G is $diam(G) = sup\{d(x,y) \| x$ and y are vertices of $G\}$.

Definition

The *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in G.

Definition

The chromatic number $\chi(G)$ is the minimum k such that G can be colored using k different colors such that no two adjacent vertices have the same color.

Some background from graph theory

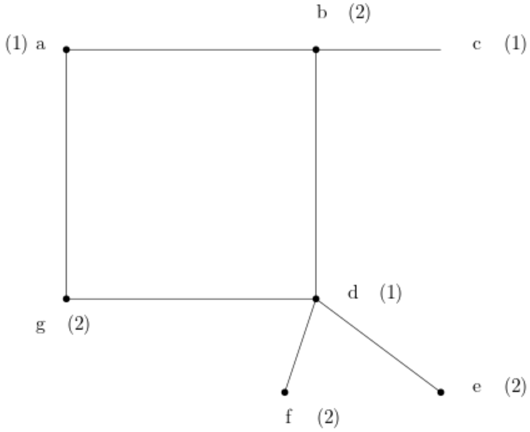

Figure 4. G

$diam(G) = 3.$

$gr(G) = 4$

$\chi(G) = 2$

Definition

For any vertex v in a graph, $nbd(v)$ be the set of vertices adjacent to v.

Definition

A graph is called a *complete graph* if every two distinct vertices are adjacent.

K_n : complete graph on n vertices.

Definition

If the vertex set of a graph G can be splitted into two parts A and B so that each edge of G joins a vertex of A to a vertex of B, then G is called a *bipartite graph*. If each vertex in A is joined to each vertex in B by an edge, then G is *complete bipartite*.

$K_{m,n}$: complete bipartite graph with having m and n number of vertices in the two disjoint sets.

Some background from graph theory

Definition

A graph is planar if it can be drawn in the plane so that its edges intersect only at their ends.

Theorem

A graph is planar if and only if it contains no subgraph homeomorphic to K_5 *or* $K_{3,3}$.

Some background from graph theory

Definition

A graph is planar if it can be drawn in the plane so that its edges intersect only at their ends.

Theorem

A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

Definition

An eulerian graph is a graph containing an eulerian cycle.

Theorem

A connected graph is eulerian if and only if the degree of each vertex is even.

Notations

- p will denote prime integers such that $p \equiv 1 (\text{mod } 4)$.
- q will denote prime integers such that $q \equiv 3 (\text{mod } 4)$.
- $Reg(\Gamma_2(R))$: the (induced) subgraph of $\Gamma_2(R)$ with vertices $Reg(R)$ i.e. the units of the ring R.
- $Z(\Gamma_2(R))$: the (induced) subgraph of $\Gamma_2(R)$ with vertices $Z(R)$ i.e. the zero divisors of the ring R.

Γ_2 graph

Let R be a ring then Γ_2 is an undirected graph (V, E) in which $V = R \setminus \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a.b = 0$ or $b.a = 0$ or $a + b$ is a zero-divisor (including 0).

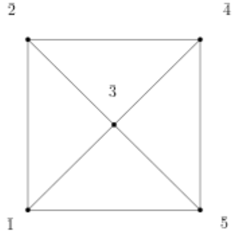

Figure 5. $\Gamma_2(\mathbf{Z_6})$

Here, we will maily focus on the rings $\mathbb{Z}_n$ and $\mathbb{Z}_n[i]$.

Γ_2 graph over $\mathbb{Z}_n$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_n)$ *is never complete for* $n > 3$.

Γ_2 graph over $\mathbb{Z}_n$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_n)$ *is never complete for* $n > 3$.

For $n > 3$, atleast two units 1 and -1. So, 1 is not adjacent to -2.

Γ_2 graph over $\mathbb{Z}_n$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is never complete for $n > 3$.

For $n > 3$, atleast two units 1 and -1. So, 1 is not adjacent to -2.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is connected if and only if either $n \leq 3$ or the prime-power factorization of n has more than one prime factor and in the latter case diam$(\Gamma_2(\mathbb{Z}_n)) = 2$.

Γ_2 graph over $\mathbb{Z}_n$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is never complete for $n > 3$.

For $n > 3$, atleast two units 1 and -1. So, 1 is not adjacent to -2.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is connected if and only if either $n \leq 3$ or the prime-power factorization of n has more than one prime factor and in the latter case diam$(\Gamma_2(\mathbb{Z}_n)) = 2$.

If n has only one prime factor say t. Zero divisors are precisely the multiples of t. Hence sum of a zero divisor and an unit can not be a zero divisor.

Γ_2 graph over $\mathbb{Z}_n$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is never complete for $n > 3$.

For $n > 3$, atleast two units 1 and -1. So, 1 is not adjacent to -2.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is connected if and only if either $n \leq 3$ or the prime-power factorization of n has more than one prime factor and in the latter case $diam(\Gamma_2(\mathbb{Z}_n)) = 2$.

If n has only one prime factor say t. Zero divisors are precisely the multiples of t. Hence sum of a zero divisor and an unit can not be a zero divisor.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is disjoint union of $(n-1)/2$ copies of K_2 when n is an odd prime.

Γ_2 graph over $\mathbb{Z}_n$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_{2^r})$,where $r \in \mathbb{N}$, has two components consisting of zero-divisors and units of $\mathbb{Z}_{2^r}$ respectively. The first is a $K_{2^{r-1}-1}$ and the other is a $K_{2^{r-1}}$.

Γ_2 graph over $\mathbb{Z}_n$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_{2^r})$,where $r \in \mathbb{N}$, has two components consisting of zero-divisors and units of $\mathbb{Z}_{2^r}$ respectively. The first is a $K_{2^{r-1}-1}$ and the other is a $K_{2^{r-1}}$.

Theorem

Let $n = t^r$, where t is an odd prime and $r \in \mathbb{N}$. Then $\Gamma_2(\mathbb{Z}_n)$ has $(t+1)/2$ components, one $K_{t^{r-1}-1}$ consisting of the zero-divisors, and $(t-1)/2$ copies of $K_{t^{r-1},t^{r-1}}$ for the units.

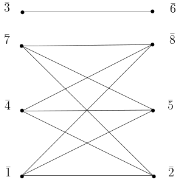

Figure 6. $\Gamma_2(\mathbf{Z}_9)$

Γ_2 graph over $\mathbb{Z}_n$: other properties

Corollary

$$gr(\Gamma_2(\mathbb{Z}_n))) = \begin{cases} 4, & \textit{if } n = 9 \\ \infty, & \textit{if } n = 2,3,4 \textit{ or } n \textit{ is a prime} \\ 3, & \textit{otherwise} \end{cases}$$

Corollary

$$\chi(\Gamma_2(\mathbb{Z}_{2^r})) = 2^{r-1}.$$

$$\chi(\Gamma_2(\mathbb{Z}_{t^r})) = t^{r-1} - 1.$$

Γ_2 graph over $\mathbb{Z}_n$: other properties

Theorem

For $n > 4$, $\Gamma_2(\mathbb{Z}_n)$ has no vertex of degree 0. A vertex can have maximum degree if and only if the prime power factorization of n has more than one prime and there is an idempotent x such that 2x=0.

Γ_2 graph over $\mathbb{Z}_n$: other properties

Theorem

For $n > 4$, $\Gamma_2(\mathbb{Z}_n)$ has no vertex of degree 0. A vertex can have maximum degree if and only if the prime power factorization of n has more than one prime and there is an idempotent x such that 2x=0.

Theorem

For an unit u in $\mathbb{Z}_n$, $deg(u) = (n - \phi(n) - 1)$ or $(n - \phi(n))$ according as n is even or odd.

Γ_2 graph over $\mathbb{Z}_n$: other properties

Theorem

For $n > 4$, $\Gamma_2(\mathbb{Z}_n)$ has no vertex of degree 0. A vertex can have maximum degree if and only if the prime power factorization of n has more than one prime and there is an idempotent x such that 2x=0.

Theorem

For an unit u in $\mathbb{Z}_n$, $deg(u) = (n - \phi(n) - 1)$ or $(n - \phi(n))$ according as n is even or odd.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is not Eulerian for any positive integer n.

Γ_2 graph over $\mathbb{Z}_n$: other properties

Theorem

For $n > 4$, $\Gamma_2(\mathbb{Z}_n)$ has no vertex of degree 0. A vertex can have maximum degree if and only if the prime power factorization of n has more than one prime and there is an idempotent x such that 2x=0.

Theorem

For an unit u in $\mathbb{Z}_n$, $deg(u) = (n - \phi(n) - 1)$ or $(n - \phi(n))$ according as n is even or odd.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is not Eulerian for any positive integer n.

Degree of the vertex 1 is always odd.

Γ_2 graph over $\mathbb{Z}_n$: other properties

Theorem

For $n > 4$, $\Gamma_2(\mathbb{Z}_n)$ has no vertex of degree 0. A vertex can have maximum degree if and only if the prime power factorization of n has more than one prime and there is an idempotent x such that 2x=0.

Theorem

For an unit u in $\mathbb{Z}_n$, $deg(u) = (n - \phi(n) - 1)$ or $(n - \phi(n))$ according as n is even or odd.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is not Eulerian for any positive integer n.

Degree of the vertex 1 is always odd.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is planar if and only if n=4,6,8 or n is a prime.

Γ_2 graph over $\mathbb{Z}_n$: other properties

Theorem

For $n > 4$, $\Gamma_2(\mathbb{Z}_n)$ has no vertex of degree 0. A vertex can have maximum degree if and only if the prime power factorization of n has more than one prime and there is an idempotent x such that 2x=0.

Theorem

For an unit u in $\mathbb{Z}_n$, $deg(u) = (n - \phi(n) - 1)$ or $(n - \phi(n))$ according as n is even or odd.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is not Eulerian for any positive integer n.

Degree of the vertex 1 is always odd.

Theorem

$\Gamma_2(\mathbb{Z}_n)$ is planar if and only if n=4,6,8 or n is a prime.

Γ_2 graph over $\mathbb{Z}_{2^n}[i]$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_{2^n}[i])$ is not connected. $\Gamma_2(\mathbb{Z}_{2^n}[i])$ has exactly two components consisting of the zero divisors and units of $\mathbb{Z}_{2^n}[i]$ respectively. First is $K_{2^{2n-1}-1}$, second is $K_{2^{2n-1}}$.

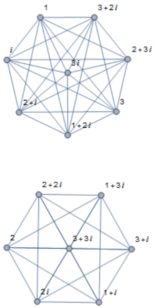

Figure 7. $\Gamma_2(\mathbf{Z}_4[i])$

Γ_2 graph over $\mathbb{Z}_{2^n}[i]$: other properties

Corollary

From the previous theorem these results follow directly:

1. *As* $\Gamma_2(\mathbb{Z}_{2^n}[i])$ *is disconnected, so* $diam(\Gamma_2(\mathbb{Z}_{2^n}[i])) = \infty$.
2. $diam(Z(\Gamma_2(\mathbb{Z}_{2^n}[i]))) = 1$.
3. $diam(Reg(\Gamma_2(\mathbb{Z}_{2^n}[i]))) = 1$.

Corollary

The chromatic number and girth for the graph $\Gamma_2(\mathbb{Z}_{2^n}[i])$ *are as follows:*

1. $\chi(\Gamma_2(\mathbb{Z}_{2^n}[i]) = 2^{2n-1}$.
2. $gr(\Gamma_2(\mathbb{Z}_{2^n}[i]) = 3, \; n \geq 2$.

Γ_2 graph over $\mathbb{Z}_{q^n}[i]$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_q[i])$ *is always disconnected having* $(q^2 - 1)/2$ *components,each being equal to* K_2*.*

Γ_2 graph over $\mathbb{Z}_{q^n}[i]$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_q[i])$ is always disconnected having $(q^2 - 1)/2$ components,each being equal to K_2.

There are no non zero zero divisors. Only additive inverses will be adjacent to each other.

Γ_2 graph over $\mathbb{Z}_{q^n}[i]$: connectedness

Theorem

$\Gamma_2(\mathbb{Z}_q[i])$ is always disconnected having $(q^2-1)/2$ components,each being equal to K_2.

There are no non zero zero divisors. Only additive inverses will be adjacent to each other.

Theorem

$\Gamma_2(\mathbb{Z}_{q^n}[i])$, $n \geq 2$ is never connected. Zero divisors and units are not connected. The component of the graph $\Gamma_2(\mathbb{Z}_{q^n}[i])$; having the zero divisors as the vertices is complete and it is $K_{q^{2n-2}-1}$. The set of units form $(q^2-1)/2$ number of complete bipartite graphs each $K_{q^{2n-2},q^{2n-2}}$.

Γ_2 graph over $\mathbb{Z}_{q^n}[i]$: other properties

Corollary

1. *As* $\Gamma_2(\mathbb{Z}_{q^n}[i])$ *is disconnected, so* $diam(\Gamma_2(\mathbb{Z}_{q^n}[i])) = \infty$.
2. $diam(Z(\Gamma_2(\mathbb{Z}_{q^n}[i]))) = 1$.
3. $diam(Reg(\Gamma_2(\mathbb{Z}_{q^n}[i]))) = \infty$.

Corollary

The chromatic number and girth for the graph $\Gamma_2(\mathbb{Z}_{q^n}[i])$ *are as follows:*

1. $\chi(Reg(\Gamma_2(\mathbb{Z}_{q^n}[i]))) = 2$.
2. $\chi(\Gamma_2(\mathbb{Z}_{q^n}[i]) = q^{2n-2} - 1$.
3. $gr(\Gamma_2(\mathbb{Z}_{q^n}[i]) = 3, \ n \geq 2$.

Γ_2 graph over $\mathbb{Z}_{p^n}[i]$: connectedness

Γ_2 graph over $\mathbb{Z}_{p^n}[i]$: connectedness

Theorem

For every unit u, $\exists$ a pair $x, y \in Z(\mathbb{Z}_{p^n}[i])$ such that, $u = x + y$. In $\mathbb{Z}_p[i]$, this pair $x, y \in Z(\mathbb{Z}_p[i])$ is unique.

Γ_2 graph over $\mathbb{Z}_{p^n}[i]$: connectedness

Theorem

For every unit u, $\exists$ a pair $x, y \in Z(\mathbb{Z}_{p^n}[i])$ such that, $u = x + y$. In $\mathbb{Z}_p[i]$, this pair $x, y \in Z(\mathbb{Z}_p[i])$ is unique.

Theorem

$\Gamma_2(\mathbb{Z}_p[i])$ is connected. The induced subgraph generated by the zero divisors is complete i.e. K_{2p-2}.

Γ_2 graph over $\mathbb{Z}_{p^n}[i]$: connectedness

Theorem

For every unit u, $\exists$ a pair $x, y \in Z(\mathbb{Z}_{p^n}[i])$ such that, $u = x + y$. In $\mathbb{Z}_p[i]$, this pair $x, y \in Z(\mathbb{Z}_p[i])$ is unique.

Theorem

$\Gamma_2(\mathbb{Z}_p[i])$ is connected. The induced subgraph generated by the zero divisors is complete i.e. K_{2p-2}.

Theorem

In $\Gamma_2(\mathbb{Z}_{p^n}[i])$, for an unit t,
$nbd(u) = [-x+ < a - ib >] \cup [-y+ < a + ib >]$, where
$x \in < a + ib > \setminus W$ and $y \in < a - ib > \setminus W$ such that $x + y = u$,
where $p = a^2 + b^2$.

Γ_2 graph over $\mathbb{Z}_{p^n}[i]$: connectedness

Theorem

For every unit u, $\exists$ a pair $x, y \in Z(\mathbb{Z}_{p^n}[i])$ such that, $u = x + y$. In $\mathbb{Z}_p[i]$, this pair $x, y \in Z(\mathbb{Z}_p[i])$ is unique.

Theorem

$\Gamma_2(\mathbb{Z}_p[i])$ is connected. The induced subgraph generated by the zero divisors is complete i.e. K_{2p-2}.

Theorem

In $\Gamma_2(\mathbb{Z}_{p^n}[i])$, for an unit t,
$nbd(u) = [-x + < a - ib >] \cup [-y + < a + ib >]$, where
$x \in < a + ib > \setminus W$ and $y \in < a - ib > \setminus W$ such that $x + y = u$,
where $p = a^2 + b^2$.

Theorem

$Z(\Gamma_2(\mathbb{Z}_{p^n}[i]))$ is p^{2n-2}- connected. $\Gamma_2(\mathbb{Z}_{p^n}[i]))$ is connected.

Γ_2 graph over $\mathbb{Z}_{p^n}[i]$: other properties

Corollary

1. $diam(\Gamma_2(\mathbb{Z}_{p^n}[i])) = 3.$
2. $gr(\Gamma_2(\mathbb{Z}_{p^n}[i])) = 3.$
3. $\chi(\Gamma_2(\mathbb{Z}_p[i])) = 2p - 2.$
4. $\chi(\Gamma_2(\mathbb{Z}_{p^n}[i])) = p^{2n-1} - 1$, *for* $n \geq 2.$

Γ_2 graph over $\mathbb{Z}_n[i]$: general case

Theorem

$\Gamma_2(\mathbb{Z}_n[i])$ is always connected if and only if $n = p^r$ or the prime power factorization of n has more than one prime factor.

Theorem

$\Gamma_2(\mathbb{Z}_n[i])$ is planar if and only if n=2 or n is a prime $q \equiv 3 (mod\ 4)$.

Theorem

$\Gamma_2(\mathbb{Z}_n[i])$ is not eulerian for any n.

References

- D.F. Anderson and P.S. Livingston; *The Zero-Divisor Graph of a Commutative Ring*, Journal of Algebra 217, 434-447(1999).
- E.A. Osba, S. Al-Addasi, N.A. Jaradeh; *Zero Divisor Graph for the Ring of Gaussian Integers Modulo n*, Communications in Algebra,36:10,3865-3877,DOI : 10.1080/00927870802160859.
- T. Gaohua, S. Huadong, Y. Zhong; *Unit group structure of* $\mathbb{Z}_n[i]$, Journal of Guangxi Normal University : Natural Science Edition, Vol.28 , No.2 , 2010.
- R. Sen Gupta; *The Graph* $\Gamma_2(R)$ *over a ring R*, International Journal of Pure and Applied Mathematics, Volume 86, No.6 ,2013,893-904.
- James T. Cross, *The Euler phi-function in the Gaussian integers.* American Mathematical Monthly. 90. 10.2307/2322785,1983.

References

Khalida Nazzal and Manal Ghanem, *On the Line Graph of the Zero Divisor Graph for the Ring of Gaussian Integers Modulo n.* International Journal of Combinatorics. 2012. 10.1155/2012/957284.

D. Anderson and Ayman Badawi, *The total graph of a commutative ring*, Journal of Algebra. 320. 2706-2719. 10.1016/j.jalgebra.2008.06.028.

THANK YOU