HYPERGEOMETRIC SERIES OVER FINITE FIELDS

Research Talk

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We start with certain examples of certain examples of series having closed forms. Denote the rising factorial as $(a)_k$, such that .

$$\sum_{k=0}^{\infty}(4k+1)\binom{-\frac{1}{2}}{k}^5 = \frac{2}{\Gamma(\frac{3}{4})^4}; \quad \sum_{k=0}^{\infty}(6k+1)\binom{-\frac{1}{3}}{k}^3 = \frac{3}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}$$

Using $\binom{z}{k} = (-1)^k \frac{(-z)_k}{(1)_k}$, where $(a)_k := a(a+1)\cdots(a+k-1)$ denotes the rising factorial, we have $\frac{(\frac{1}{2})_k}{k!} = (-1)^k\binom{-\frac{1}{2}}{k}$. The above series can be rewritten as

In terms of rising factorial

$$\sum_{k=0}^{\infty}\frac{\left(\frac{5}{4}\right)_k\left(\frac{1}{2}\right)_k^5}{\left(\frac{1}{4}\right)_k(1)_k^5} = \frac{2}{\Gamma(\frac{3}{4})^4}; \quad \sum_{k=0}^{\infty}\frac{\left(\frac{7}{6}\right)_k\left(\frac{1}{3}\right)_k^3}{\left(\frac{1}{6}\right)_k(1)_k^3} = \frac{2}{\Gamma(\frac{3}{4})^4}$$

This type of series are examples of hypergeometric series. We give a formal definition of this series.

For a complex number a and a non-negative integer n, let $(a)_n$ denote the rising factorial defined by

$$(a)_0 := 1 \qquad \text{and} \qquad (a)_k := a(a+1)\cdots(a+k-1) \quad \text{for } k \geq 1.$$

Thus, for complex numbers a_i, b_j and λ, with none of the b_j being negative integers or zero, the classical hypergeometric series ${}_{r+1}F_r$ is defined as

$$ {}_{r+1}F_r\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_r \end{matrix} \,\middle|\, \lambda\right] := \sum_{k=0}^{\infty} \frac{(a_0)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_r)_k} \frac{\lambda^k}{k!}.$$

Similarly, the truncated hypergeometric series is given by

$$ {}_{r+1}F_r\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_r \end{matrix} \,\middle|\, \lambda\right]_n := \sum_{k=0}^{n} \frac{(a_0)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_r)_k} \frac{\lambda^k}{k!}.$$

For a detailed study of this series, see the book **Special functions** by G. Andrews, R. Askey, and R. Roy and **Generalized hypergeometric series** by W. Bailey.

Characters are homomorphisms from a group to the group of complex numbers.

In a finite field $\mathbb{F}_q$ there are two finite abelian groups that are of significance– the additive group and the multiplicative group.

Additive characters

Characters of the additive group $(\mathbb{F}_{p^n}, +)$.

Multiplicative characters

Characters of the multiplicative group $(\mathbb{F}^*_{p^n}, \times)$, consisting of all non-zero elements.

For the analogues of the hypergeometric series over finite fields we shall mostly refer to multiplicative characters.

Notations

- Capital letters $A, B, C, \cdots$ and Greek letters $\chi, \psi, \cdots$ will denote multipicative characters.
- Trivial character and quadratic characters are denoted by ϵ and ϕ respectively.
- Extend the multiplicative character to all of $\mathbb{F}_q$ by setting $\chi(0) = 0$.
- $\overline{\chi}$ is the inverse of χ, i.e., $\chi\overline{\chi} = \epsilon$, $\sum_{\chi}$ denotes the summation over all multiplicative characters of $\mathbb{F}_q$.
- $\delta(A) = \begin{cases} 1 & \text{if } A = \epsilon \\ 0 & \text{otherwise.} \end{cases}$
- $q = p^n$, where p is a prime.

Elementary definitions

Before moving to the formal definition of hypergeometric series over finite fields, we recall elementary definitions of Gauss sums and Jacobi sums.

Trace

$$Tr(\alpha) := \alpha + \alpha^q + \cdots + \alpha^{q^{m-1}}$$

Gauss Sum

Set $\zeta = e^{2\pi i/p}$, Gauss sum is defined as

$$G(\chi) = \sum_{t \in \mathbb{F}_q} \chi(t) \zeta^{Tr(t)}$$

Jacobi Sum

Set $\zeta = e^{2\pi i/p}$, Jacobi sum is defined as

$$J(A, B) = \sum_{t \in \mathbb{F}_q} A(t) B(1 - t).$$

We start with certain examples.

Example 1

The Gauss sum $G(\chi) = \sum_{t \in \mathbb{F}_q} \chi(t)\zeta^{Tr(t)}$ is the analogue of the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Example 2

The Jacobi sum $J(A, B) = \sum_{t \in \mathbb{F}_q} A(t)B(1-t)$ is the analogue of the beta function

$$B(x, y) = \int_0^\infty t^{x-1}(1-t)^{y-1} dt.$$

This type of analogy, essentially expressed as $x^k \leftrightarrow \chi(x)$ dates back to Jacobi.

Following this analogy, Helversen-Pasotto gave the following result.

$$\frac{1}{q-1}\sum_{\chi} G(A\chi)G(B\overline{\chi})G(C\chi)G(D\overline{\chi}) = \frac{G(AB)G(AD)G(BC)G(CD)}{G(ABCD)} + q(q-1)AC(-1)\delta(ABCD)$$

which is a finite field analogue of Barnes' lemma,

Barnes' Lemma

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(a+z)\Gamma(b-z)\Gamma(c+z)\Gamma(d-z)dz = \frac{\Gamma(a+b)\Gamma(a+d)\Gamma(b+c)\Gamma(c+d)}{\Gamma(a+b+c+d)}$$

Using similar type of analogue, John Greene [Trans. Amer. Math. Soc. (1987)] developed a finite field analogue for the hypergeometric series.

We shall discuss the hypergeometric series over finite field as defined by John Greene [(1987)] which is based on the following observation. This expression represents a finite field analogue for the power series expansion of a function.

Observation

Any function $f : \mathbb{F}_q \to \mathbb{C}$ has a unique representation

$$f(x) = f_\delta \delta(x) + \sum_{\chi} f_\chi \chi(x),$$

where the sum ranges over all multiplicative characters of $\mathbb{F}_q$ and δ is given by

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Also, f_δ and f_χ are given by

$$f_\delta = f(0), \quad f_\chi = \frac{1}{q-1} \sum_{x \in \mathbb{F}_q} f(x) \overline{\chi}(x).$$

Binomial coefficients and the binomial theorem

The binomial theorem states

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k,$$

where $\binom{a}{k}$ is the binomial co-efficient defined as $\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}$.

The observation discussed previously gives the character sum analogue for the binomial theorem as follows.

Theorem

For any character $A \in \widehat{\mathbb{F}_q}$ and $x \in \mathbb{F}_q$,

$$A(1+x) = \delta(x) + \frac{1}{q-1} \sum_{\chi} J(A, \overline{\chi}) \chi(-x).$$

It follows from the previous theorem that the finite field analogue for the binomial coefficients is the Jacobi sum. Thus Greene defined the binomial coefficient over finite fields as follows.

Definition

For characters A and B of $\mathbb{F}_q$, define

$$\binom{A}{B} = \frac{B(-1)}{q} J(A, \overline{B}).$$

In terms of binomial coefficients over finite fields, the binomial theorem over finite field can therefore rewritten as

$$A(1+x) = \delta(x) + \frac{q}{q-1} \sum_{\chi} \binom{A}{\chi} \chi(x).$$

Properties of binomial coefficients

From the properties of Jacobi sums, certain properties of the binomial coefficients follows easily.

$$\binom{A}{B} = \binom{A}{A\overline{B}}$$

$$\binom{A}{B} = \binom{B\overline{A}}{B} B(-1)$$

$$\binom{A}{B} = \binom{\overline{B}}{\overline{A}} AB(-1)$$

$$\binom{A}{A} = -\frac{1}{q} + \frac{q-1}{q}\delta(A)$$

$$\binom{\epsilon}{A} = -\frac{A(-1)}{q} + \frac{q-1}{q}\delta(A)$$

We now express the hypergeometric series in terms of binomial coefficients over finite fields.

Recall that the hypergeometric series is defined by

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| x\right) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} x^k, $$

where $(a)_k = a(a+1)\cdots(a+k-1) = \frac{(a+k-1)!}{(a-1)!} = \frac{\Gamma(a+k)}{\Gamma(a)}$. The hypergeometric series can thereby written in the form of Gamma function as

$$ \begin{aligned} {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| x\right) &= \frac{(c-1)!}{(a-1)!(b-1)!} \sum_{k=0}^{\infty} \frac{(a+k-1)!(b+k-1)!}{k!(c+k-1)!} x^k \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(k-1)\Gamma(c+k)}. \end{aligned} $$

Since the analogue of the Gamma function is the Gauss sum, the natural approach to define a character sum analogue would be

$$ \frac{G(C)}{G(A)G(B)} \sum_{\chi} \frac{G(A\chi)G(B\chi)}{G(\chi)G(C\chi)} \chi(x). $$

But the character sum analogue in terms of Gauss sums leads to poor results and hence as an alternative Greene developed an analogue using the integral representation for the hypergeometric series.

Integral formula for the hypergeometric series

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \mid x \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^b (1-t)^{c-b} (1-tx)^{-a} \frac{dt}{t(1-t)}. $$

Greene defined the finite field analogue as follows.

Definition

For characters A, B and C of $\mathbb{F}_q$, and $x \in \mathbb{F}_q$,

$$ {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \mid x \right) := \epsilon(x) \frac{BC(-1)}{q} \sum_y B(y)\overline{B}C(1-y)\overline{A}(1-xy). $$

Using this definition and the binomial theorem over finite fields, the hypergeometric series can be easily expressed in terms of binomial coefficients as

Theorem

For characters A, B and C of $\mathbb{F}_q$, and $x \in \mathbb{F}_q$,

$$ {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \mid x \right) := \frac{q}{q-1} \sum_{\chi} \binom{A\chi}{\chi} \binom{B\chi}{C\chi} \chi(x). $$

Notice that the classical hypergeometric series also can be expressed in terms of binomial coefficients as

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| x \right) = C \sum_{k=0}^{\infty} \binom{a+k-1}{k} \binom{b+k-1}{c+k-1} x^k, $$

where $C = \left\{ \binom{b-1}{c-1} \right\}^{-1}$

Similarly, in general, observe that

$$ {}_{n+1}F_n\left(\begin{matrix} a_0, & a_2, & \cdots, & a_n \\ & b_1, & \cdots, & b_n \end{matrix}\middle| x\right) := C\sum_{k=0}^{\infty}\binom{a_0+k-1}{k}\binom{a_1+k-1}{b_1+k-1}\cdots\binom{a_n+k-1}{b_n+k-1}x^k, $$

where $C=\left\{\binom{a_1-1}{b_1-1}\cdots\binom{a_n-1}{b_n-1}\right\}^{-1}$. The observation above led directly to the following definition.

Definition

For characters $A_0, A_1, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ of $\mathbb{F}_q$ and $x \in \mathbb{F}_q$,

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \cdots, & A_n \\ & B_1, & \cdots, & B_n \end{matrix}\mid x\right) := \frac{q}{q-1}\sum_{\chi}\binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{B_n\chi}{B_n\chi}\chi(x). $$

Integral representation and its finite field analogue

Generalized hypergeometric series have the following inductive integral representation.

$$
\begin{aligned}
&{}_{n+1}F_n\left(\begin{matrix} a_0, & a_1, & \cdots, & a_n \\ & b_1, & \cdots, & b_n \end{matrix} \mid x\right) \\
&= \frac{\Gamma(b_n)}{\Gamma(a_n)\Gamma(b_n - a_n)} \int_0^1 {}_nF_{n-1}\left(\begin{matrix} a_0, & a_1, & \cdots, & a_{n-1} \\ & b_1, & \cdots, & b_{n-1} \end{matrix} \mid tx\right) \\
&\cdot t^{a_n}(1-t)^{b_n - a_n} \frac{dt}{t(1-t)}
\end{aligned}
$$

A finite field analogue for this result holds as follows:

Theorem

For characters $A_0, A_1, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ of $\mathbb{F}_q$ and $x \in \mathbb{F}_q$,

$$
\begin{aligned}
&{}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \cdots, & A_n \\ & B_1, & \cdots, & B_n \end{matrix} \mid x\right) \\
&= \frac{A_nB_n(-1)}{q} \sum_y {}_nF_{n-1}\left(\begin{matrix} A_0, & A_1, & \cdots, & A_{n-1} \\ & B_1, & \cdots, & B_{n-1} \end{matrix} \mid tx\right) \\
&\cdot A_n(y)\overline{A_n}B_n(1-y).
\end{aligned}
$$

From the definition of classical hypergeometric series, it is clear that if a numerator and a denominator parameter are equal, then the order of the series reduces by 1. Similar type of result is also expected in the finite field case. Also, if one of the numerator parameter is the trivial character ϵ, the series reduces to one lower order. We follow these notations.

Notations

For characters $A_1, A_2, \cdots, A_n, B_1, B_2, \cdots, B_n$, let

$$\binom{\overrightarrow{A}}{\overrightarrow{B}} := \prod_{k=1}^{n} \binom{A_k}{B_k},$$

and

$$F\left(\begin{matrix} C, & \overrightarrow{A} \\ & \overrightarrow{B} \end{matrix} \middle| x \right) := {}_{n+1}F_n\left(\begin{matrix} C, & A_1, & \cdots, & A_n \\ & B_1, & \cdots, & B_n \end{matrix} \middle| x \right)$$

Clearly, by the definition of classical hypergeometric series the order of the binomial coefficients is irrelevant, there are five cases to consider.

Case I

$$ {}_{n+2}F_{n+1}\left(\begin{matrix} \epsilon, & A, & \overrightarrow{B} \\ & C, & \overrightarrow{D} \end{matrix} \,\middle|\, x\right) = -\frac{1}{q}\overline{C}(x)\,{}_{n+1}F_{n}\left(\begin{matrix} A\overline{C}, & \overrightarrow{B}\,\overline{C} \\ & \overrightarrow{D}\,\overline{C} \end{matrix} \,\middle|\, x\right) + \epsilon(x)\binom{A}{C}\binom{\overrightarrow{B}}{\overrightarrow{D}}. $$

Case II

$$
\begin{aligned}
&{}_{n+3}F_{n+2}\left(\begin{matrix} A & \epsilon, & B, & \overrightarrow{C} \\ & D, & E, & \overrightarrow{F} \end{matrix} \middle| \, x\right) \\
&= A(-1)\overline{D}(x)\binom{D}{A}{}_{n+2}F_{n+1}\left(\begin{matrix} A\overline{D}, & B\overline{D}, & \overrightarrow{C}\overline{D} \\ & E\overline{D}, & \overrightarrow{D}\overline{C} \end{matrix} \middle| \, x\right) \\
&- \frac{1}{q}D(-1)\epsilon(x)\binom{B}{E}\binom{\overrightarrow{C}}{\overrightarrow{F}} \\
&+ \frac{q-1}{q^2}D(-1)\overline{E}(x){}_{n+1}F_n\left(\begin{matrix} B\overline{E}, & \overrightarrow{C}\overline{E} \\ & \overrightarrow{F}\overline{E} \end{matrix} \middle| \, x\right)\delta(A\overline{D}).
\end{aligned}
$$

Case III

$$
{}_{n+2}F_{n+1}\left(\begin{matrix} A, & B, & \overrightarrow{C} \\ & B, & \overrightarrow{D} \end{matrix}\;\middle|\; x\right) = -\frac{1}{q}\,{}_{n+1}F_{n}\left(\begin{matrix} A, & \overrightarrow{C} \\ & \overrightarrow{D} \end{matrix}\;\middle|\; x\right) + \overline{B}(x)\binom{A\overline{B}}{\overline{B}}\binom{\overrightarrow{C}\,\overline{B}}{\overrightarrow{D}\,\overline{B}}.
$$

Case IV

$$
{}_{n+3}F_{n+2}\left(\begin{array}{cccc} A & B, & C, & \overrightarrow{D} \\ & A, & E, & \overrightarrow{F} \end{array}\middle|\, x\right) = \binom{B}{A}{}_{n+2}F_{n+1}\left(\begin{array}{ccc} B, & C, & \overrightarrow{D} \\ & E, & \overrightarrow{F} \end{array}\middle|\, x\right)
$$

$$
-\frac{1}{q}\overline{A}(-x)\binom{C\overline{A}}{E\overline{A}}\binom{\overrightarrow{D}\overline{A}}{\overrightarrow{F}\overline{A}}
$$

$$
+\frac{q-1}{q^2}A(-1)\overline{E}(x){}_{n+1}F_n\left(\begin{array}{cc} C\overline{E}, & \overrightarrow{D}\overline{E} \\ & \overrightarrow{F}\overline{E} \end{array}\middle|\, x\right)\delta(B).
$$

Case V

$$
{}_{n+3}F_{n+2}\left(\begin{matrix} A & B, & C, & \overrightarrow{D} \\ & E, & B, & \overrightarrow{F} \end{matrix} \,\middle|\, x\right) = \binom{C\overline{E}}{B\overline{E}}\, {}_{n+2}F_{n+1}\left(\begin{matrix} A, & C, & \overrightarrow{D} \\ & E, & \overrightarrow{F} \end{matrix} \,\middle|\, x\right)
$$
$$
- \frac{1}{q} BE(-1)\overline{B}(x)\binom{A\overline{B}}{\overline{B}}\binom{\overrightarrow{D}\overline{B}}{\overrightarrow{F}\overline{B}}
$$
$$
+ \frac{q-1}{q^2} BE(-1)\, {}_{n+1}F_{n}\left(\begin{matrix} A, & \overrightarrow{D} \\ & \overrightarrow{F} \end{matrix} \,\middle|\, x\right)\delta(C\overline{E}).
$$

The classical hypergeometric series satisfies several transformation formulas. We shall discuss analogues over finite fields of certain transformation formulas due to Greene.

Finite field analogue of the Gauss Theorem

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{(Gauss Theorem)} $$

$$ {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \middle| 1\right) = A(-1)\binom{B}{\overline{A}C}. $$

Pfaff's transformation and the finite field analogue

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, x\right) = (1-x)^{-a}\,{}_2F_1\left(\begin{matrix} a, & c-b \\ & c \end{matrix} \,\middle|\, \frac{x}{x-1}\right) $$

(Pfaff's Transformation)

$$ {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \,\middle|\, x\right) = C(-1)\overline{A}(1-x)\,{}_2F_1\left(\begin{matrix} A, & C\overline{B} \\ & C \end{matrix} \,\middle|\, \frac{x}{x-1}\right) + A(-1)\binom{B}{\overline{A}C}\delta(1-x). $$

Euler's transformation and the finite field analogue

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, x\right) = (1-x)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a, & c-b \\ & c \end{matrix} \,\middle|\, x\right) $$

(Euler's Transformation)

$$ {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \,\middle|\, x\right) = C(-1)C\overline{AB}(1-x){}_2F_1\left(\begin{matrix} C\overline{A}, & C\overline{B} \\ & C \end{matrix} \,\middle|\, x\right) $$

$$ + A(-1)\binom{B}{\overline{A}C}\delta(1-x). $$

This are rather simpler transformations. Certain complicated transformation formulas also exists.

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & 1-a+b \end{matrix} \middle| x\right) = (1-x)_2^{-b}F_1\left(\begin{matrix} \frac{1}{2}b, & \frac{1}{2}+\frac{1}{2}b \\ & 1-a+b \end{matrix} \middle| \frac{4x}{(1+x)^2}\right) $$

Finite field analogue

$$ {}_2F_1\left(\begin{matrix} A, & B \\ & \overline{A}B \end{matrix} \middle| x\right) = \delta(1+x)\begin{cases} 0 & \text{if } B \text{ is not a square} \\ \binom{C}{A}\binom{\phi C}{A} & \text{otherwise} \end{cases} $$

$$ + \frac{A(-1)}{q}B\left(\frac{1+x}{x}\right) + \overline{B}(1+x)\frac{q}{q-1}\sum_{\chi}\binom{B\chi^2}{\chi}\binom{B\chi}{\overline{A}B\chi}\chi\left(\frac{x}{(1+x)^2}\right). $$

Clearly, the major term on the RHS does not represent a hypergeometric series. Greene solved this problem by imposing the condition that B is a square.

The analogy is therefore improved as

Finite field analogue

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} A, & B^2 \\ & \overline{A}B^2 \end{matrix}\,\middle|\, x\right) &= \delta(1+x)\left[\binom{B}{A} + \binom{\phi B}{A}\right] \\
&+ A(-1)\frac{q-1}{q}\epsilon(x)\epsilon(1+x)\delta(\phi B) \\
&+ \frac{q-1}{q^2}\binom{\phi}{\phi B}^{-1} A(-1)B^2\left(\frac{2}{1-x}\right)\epsilon(x)\epsilon(1+x)\delta(A\overline{B}) \\
&+ \binom{B}{A}\binom{\phi}{\phi B}^{-1} A(-1)B^2\left(\frac{2}{1-x}\right){}_2F_1\left(\begin{matrix} \phi B, & B \\ & \overline{A}B^2 \end{matrix}\,\middle|\, \frac{4x}{(1+x)^2}\right).
\end{aligned}
$$

Further Readings

Transformation for higher order hypergeometric series also exists. Interested readers can go through the paper by John Greene, *Hypergeometric functions over finite fields*, Trans. Amer. Math. Soc. **301** (1987), no. 1, 77-101.

The hypergeometric series over finite field(by J. Greene) is essentially defined in terms of binomial coefficients(which is related to the Jacobi sum, by definition). Dermot McCarthy [(2012)] gave another version of finite field analogue for the hypergeometric series purely in terms of Gauss sums.

McCarthy's version

For characters $A_0, A_1, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ of $\mathbb{F}_q$ and $x \in \mathbb{F}_q$,

$$
{}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \cdots, & A_n \\ & B_1, & \cdots, & B_n \end{matrix} \mid x\right) := \frac{1}{q-1}\sum_{\chi}\prod_{i=0}^{n}\frac{G(A_i\chi)}{G(A_i)}\prod_{j=1}^{n}\frac{G(\overline{B_j\chi})}{G(\overline{B_j})}G(\overline{\chi})\chi(-1)^{n+1}\chi(x).
$$

Interested persons can go through the paper by Dermot McCarthy, Transformations of well-poised hypergeometric functions over finite fields, Finite Fields Appl. **18**(6) (2012), 1133-1147.

- G. Andrews, R. Askey, and R. Roy, *Special functions*, vol. **71**, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge 1999.
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- D. McCarthy, *Transformations of well-poised hypergeometric functions over finite fields*, Finite Fields Appl. **18**(6) (2012), 1133-1147.

Thank You