n-rank of class group of parameterized families quadratic fields

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Contents

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Motivation

❀ $\mathbb{Z} \rightarrow$ has unique factorization of integers into primes. But, in $\mathcal{O}_K \rightarrow$ **not** necessarily has unique factorization of algebraic integers into primes (irreducibles).

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- $K = \mathbb{Q}(\sqrt{-5})$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$. In $\mathcal{O}_K$:

$$2 \times 3 = 6 = (1 + \sqrt{-5}) \times (1 - \sqrt{-5})$$

Now $2, 3, 1 \pm \sqrt{-5}$ are irreducibles in $\mathcal{O}_K \rightarrow$ 2 distinct factorizations of 6 $\Rightarrow \mathcal{O}_K$ is **not** UFD.

Class group

☆ Fractional ideal $\rightarrow$ a non-zero $\mathcal{O}_K$-module $\mathcal{I} \subset K$ such that $d\mathcal{I} \subset \mathcal{O}_K$ for some $d \in \mathcal{O}_K - \{0\}$.

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- Ideal class group $\rightarrow$

$$\mathfrak{C}_K := \frac{\text{The group of all fractional ideals of K}}{\text{The group of all principal fractional ideals of K}}$$

- Class number; h_K $\rightarrow$ order of $\mathfrak{C}_K$.

General problems

- $K \to$ number field
  $\mathfrak{C}_K \to$ Class group of K
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- Natural questions to ask:
  1. What is the size of $\mathfrak{C}_K$?
  2. What is the structure of $\mathfrak{C}_K$?
  3. Do these questions have a quantitative answer, depending, say, on the size of the discriminant of K?

Classical results on the size

- ☞ Assume that K runs through imaginary quadratic fields. In 1801, Gauss cojectured that:

$$\lim_{\mathrm{Disc(K)}\to -\infty} h_K = +\infty$$

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- LCNL: For given low class number (eg. 1, 2, and 3), Gauss gives lists of imaginary quadratic fields with the given class number and believed them to be complete. SOLVED COMPLETELY.
- Gauss also conjectured that there are ∞-ly many real quadratic fields with class number one. This problem is still open.

Gauss–Heilbronn

Carl Friedrich Gauss

Hans Arnold Heilbronn

A question about the structure

- ☆ If $n > 1$ is an integer and G is a finite abelian group, by n-rank of G, $rank_n G$, we mean the largest positive integer r such that G contains $(\mathbb{Z}/n\mathbb{Z})^r$ as a subgroup.

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Remark

For a given integer $n \geq 2$, $n \mid h_K \Leftrightarrow rank_n K \geq 1$.

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For a given integer $n \geq 2$, $n \mid h_K \Leftrightarrow rank_n K \geq 1$.

Conjecture (Folklore)

For an integer $n > 1$, the $rank_n \mathfrak{C}_K$ is unbounded when K runs through all quadratic fields.

Some notations

☆ x, y, n and $\mu \longrightarrow$ positive integers. We consider:

$$K_{x,y,n,\mu} = \mathbb{Q}(\sqrt{x^2 - \mu y^n})$$

with the conditions: $\gcd(x, y) = 1,\ y > 1$ and $x^2 \leq \mu y^n$.

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☆ $s \longrightarrow$ the $+$ve integer such that

$$x^2 - y^n = -s^2 D.$$

Results of Ankeny-Chowla and Soundararajan

☞ Ankeny and Chowla[1] proved that $h_{K_{x,3,n,1}}$ is divisible by n if

1. $x > 0$ is an even;
2. n is a sufficiently large and even;
3. $0 < x < \sqrt{(2.3^{n-1})}$ and
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[1] N. C. Ankeny & S. Chowla, *On the divisibility of the class number of quadratic fields*, Pacific J. Math., **5** (1955).

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- K. Soundararajan[2] studied the divisibility of $h_{K_{x,y,n,1}}$ by n under the condition $s < \sqrt{(y^n - x^2)/(y^{n/2} - 1)}$.

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Ankeny-Chowla

Nesmith Cornett Ankeny

Sarvadaman D. S. Chowla

Two results

Theorem (KC, AH, YK, PPP)

Let n an odd integer, p and q be distinct odd primes with $q^2 < p^n$. Let d be the square-free part of $q^2 - p^n$. Assume that $q \not\equiv \pm 1 \pmod{|d|}$. Then $n | h_{K_{p,q,n,1}}$ if

$$p^{n/3} \neq (2q+1)/3, (q^2+2)/3$$

whenever both $d \equiv 1 \pmod 4$ and $3 \mid n$.

K. Chakraborty, A. Hoque, Y. Kishi and P. P. Pandey, Divisibility of the class numbers of imaginary quadratic fields, J. Number Theory, **185** (2018) 339–348.

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whenever both $d \equiv 1 \pmod 4$ and $3 \mid n$.

Theorem (KC, AH, YK, PPP)

Let $n \geq 3$ be an odd integer not divisible by 3. For each odd prime q the class number of $K_{p,q,n,1}$ is divisible by n for all but finitely many p's.

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A question

For each odd prime $q \neq n$, let S_q denote the set odd primes $p \notin \{q, n\}$ such that $q^2 < p^n$ and the class number of $\mathbb{Q}(\sqrt{q^2 - p^n})$ is not divisible by n.

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Remark: Kohnen and Ono[3] gave an infinite family of imaginary quadratic fields with class number not divisible by a given prime number.

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Conjecture (Wada)

All the class groups of imaginary quadratic fields are either cyclic or of the type $C_{h_1} \times C_{h_2} \times C_{2^{r_1}} \times C_{2^{r_2}} \times \cdots \times C_{2^{r_k}}$.

H. Wada, *A table of ideal class groups of imaginary quadratic fields*, Proc. Japan Acad. **46** (1970), 401–403.

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- We[4] consider:

$$K_{x,y,n,2} = \mathbb{Q}(\sqrt{x^2 - 2y^n}).$$

- $\mathfrak{C}_{K_{5,11,27,2}} = C_{381006210618} \times C_6 \times C_6 \times C_2 \times C_2 \times C_2.$

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Conjecture (Chakraborty-Hoque)

Let p and q be two distinct odd primes. For each positive odd integer n and for each positive integer m such that m is not a n-root of any rational integer, there are infinitely many imaginary quadratic fields of the form $\mathbb{Q}(\sqrt{p^2 - mq^n})$ *whose class number is divisible by n.*

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$N(3, r, x) \longrightarrow$ The # of quadratic fields whose class group has 3-rank at least r and absolute discriminant $\leq x$.

Theorem (−)

$N(3, 3, x) \gg x^{\frac{1}{3}}$.

A. Hoque, *Parameterized families of quadratic fields with class groups of 3-rank at least* 3, Preprint

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③ $L_1 \neq L_2 \neq L_3$.

④ Counting such fields.

For suitable integers m, n, b and c, define $f(m, n, b, c) = 2nc(4m^3 - 216b^2nc)$. Assume that d is the square-free parts of $f(m, n, b, c)$. Then $rank_3\mathbb{Q}(\sqrt{d}) \geq 3$. $\Rightarrow$ ② and ③.

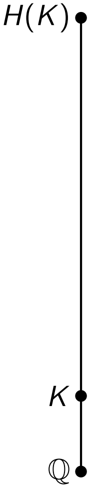
A. Hoque, *Parameterized families of quadratic fields with class groups of 3-rank at least* 3, Preprint

HCF

$K \rightarrow$ a number field

$$
\begin{array}{c}
K \\
| \\
\mathbb{Q}
\end{array}
$$

HCF



$K \rightarrow$ a number field

$H(K) \rightarrow$ Hilbert class field of K

HCF

$$\begin{array}{c} H(K) \\ | \\ K \\ | \\ \mathbb{Q} \end{array}$$

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$Gal(H(K)/K) \cong Cl(K)$

HCF

$$\begin{tikzcd}
H(K) \arrow[d, dash] \\
L \arrow[d, dash] \\
K \arrow[d, dash] \\
\mathbb{Q}
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$L \to$ unramified abelian extension

$|L : K| \mid \mathrm{Ord}(Gal(H(K)/K)) = h_K$

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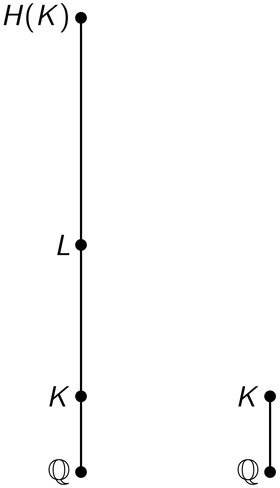

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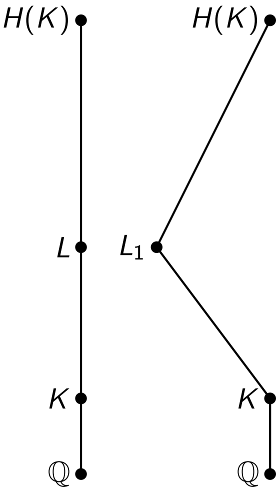

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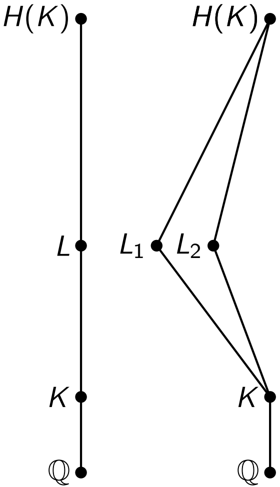

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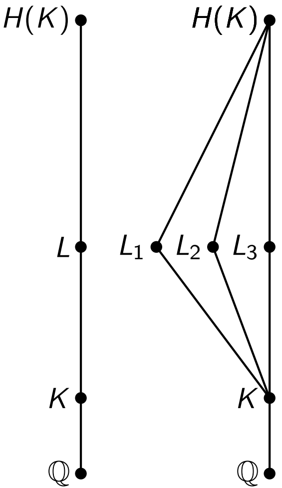

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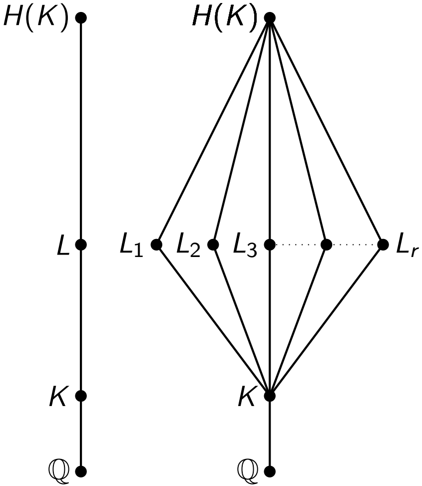

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Construction of unramified extensions

Theorem (Kishi-Miyanke)

For any integer u and v, let $g(Z) = Z^3 - uvZ - u^2$. If

(i) *$d = 4uv^3 - 27u^2$ is not a square in $\mathbb{Z}$,*

(ii) $\gcd(u, v) = 1$,

(iii) *$g(Z)$ is irreducible,*

(iv) *one of the following conditions holds:*

- (iv.a) $3 \nmid v$,
- (iv.b) $3 \mid v$, $\quad uv \not\equiv 3 \pmod 9$, $\quad u \equiv v \pm 1 \pmod 9$,
- (iv.c) $3 \mid v$, $\quad uv \equiv 3 \pmod 9$, $\quad u \equiv v \pm 1 \pmod{27}$,

then the normal closure of $\mathbb{Q}(\alpha)$, where α is a root of $g(Z)$, is a cyclic, cubic and unramified extension of $K = \mathbb{Q}(\sqrt{d})$; in particular, the class number of K is divisible by 3.

Y. Kishi and K. Miyake, *Parametrization of the quadratic fields whose class numbers are divisible by three*, J. NumberTheory **80** (2000), 209–217.

Splitting of primes

Lemma (–)

Let $g(Z) = Z^3 - uvZ - u^2 \in \mathbb{Z}[Z]$ *and* α *a root of* $g(Z)$.

(i) *If* $p \geq 3$ *is a prime such that* $p^{2r} \mid\mid u$ *and* $p \nmid v$, *then* p *splits in* $\mathbb{Q}(\alpha)$ *as* $p = \mathfrak{pqr}$ *or* $\mathfrak{pq}$ *according as* $\left(\frac{w}{p}\right) = 1$ *or* -1, *where* $w = uv/p^{2r}$.

(ii) *If* $p \geq 5$ *is prime and* $\Delta = 4uv^3 - 27u^2$ *such that* $p \nmid v\Delta$, *then* p *decomposes in* $\mathbb{Q}(\alpha)$ *as follows:*

$$(p) = \begin{cases} \mathfrak{pqr} & \textit{when } \left(\frac{\Delta}{p}\right) = 1 \textit{ and } f(x) \pmod{p} \textit{ has a root}, \\ \mathfrak{p} & \textit{when } \left(\frac{\Delta}{p}\right) = 1 \textit{ and } f(x) \pmod{p} \textit{ has no root}, \\ \mathfrak{pq} & \textit{when } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

(iii) 3 *is inert in* $\mathbb{Q}(\alpha)$ *if* $uv \equiv 1 \pmod{3}$.

Square-free sieve

Let $F(X,Y) = a_0X^r + a_1X^{r-1}Y + \cdots + a_{r-1}XY^{r-1}a_rY^r \in \mathbb{Z}[X,Y]$. Assume that M, N and T are integers with $T \geq 1$. For any positive real number x, let $R(x)$ denote the number of square-free integers d with $|d| \leq x$ for which there are integers m, n and z satisfying $m \equiv M \pmod{T}$, $n \equiv N \pmod{T}$ and $F(m,n) = dz^2$.

Lemma (Stewart–Top)

Let M, N and T be integers with $T \geq 1$. Let F be defined as above with non-vanishing discriminant and degree $r \geq 3$. Assume that the largest degree of an irreducible factor of F over $\mathbb{Q}$ is at most 6. Then for any large positive real number x, there exists a sufficiently large positive constant c, which depends on T and F, such that $R(x) > cx^{\frac{2}{r}}$.

C. L. Stewart and J. Top, *On ranks of twists of elliptic curves and power-free values of binary forms*, J. Amer. Math. Soc. **8** (1995), 943–973.

Completion of the proof

Recall that $f(m,n,b,c) = 2nc(4m^3 - 216b^2nc)$. Then $f(m,n,43,-553n) = 2n(m^2 - 9954mn + 33027372n^2)(4m^3 - 399384m^2n + 3975468336mn^2 - 13190603938848n^3)$.

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Finally, we choose $m \equiv 530881 \pmod{713370}$ and $n \equiv 120401 \pmod{713370}$ to complete the proof.

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- The Diophantine equations $cx^2 + d^m = \mu y^n, x, y \geq 1, m, n \geq 1$, $\mu \in \{1, 2, 4\}$, where c and d are fixed positive integers.


Thank
you !!!