Generalization of five q-series identities of Ramanujan and unexplored weighted partition identities

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(Joint work with Subhash Chand Bhoria and Pramod Eyyunni)

Outline

- Five q-series identities of Ramanujan

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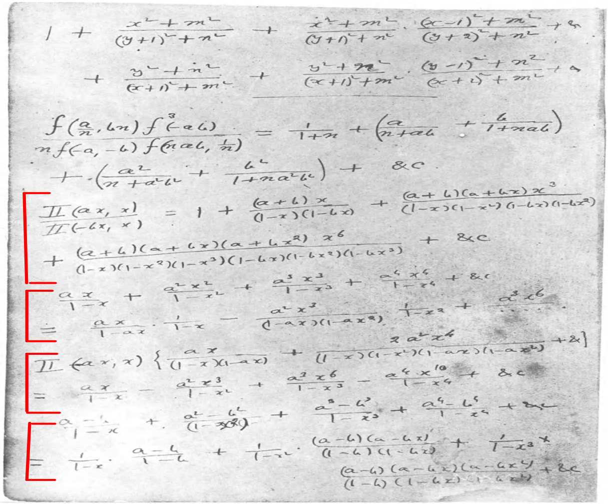
- Two-variable generalization of Bressoud and Subbarao's identity
- A new weighted partition identity connected to divisor function

Page 354 from the unorganized portion in Ramanujan's second notebook



$$1 + \frac{x^2+m^2}{(y+1)^2+n^2} + \frac{x^2+m^2}{(y+1)^2+n^2}\cdot\frac{(x-1)^2+m^2}{(y+2)^2+n^2} + \&c$$

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$$\frac{f(\frac{a}{n}, bn)\, f^3(-ab)}{n f(-a,-b)\, f(nab, \frac{1}{n})} = \frac{1}{1+n} + \left(\frac{a}{n+ab} + \frac{b}{1+nab}\right) + \left(\frac{a^2}{n+a^2b^2} + \frac{b^2}{1+na^2b^2}\right) + \&c$$

$$\frac{\Pi(ax,x)}{\Pi(-bx,x)} = 1 + \frac{(a+b)x}{(1-x)(1-bx)} + \frac{(a+b)(a+bx)x^3}{(1-x)(1-x^2)(1-bx)(1-bx^2)} + \frac{(a+b)(a+bx)(a+bx^2)x^6}{(1-x)(1-x^2)(1-x^3)(1-bx)(1-bx^2)(1-bx^3)} + \&c$$

$$\frac{ax}{1-x} + \frac{a^2x^2}{1-x^2} + \frac{a^3x^3}{1-x^3} + \frac{a^4x^4}{1-x^4} + \&c = \frac{ax}{1-ax}\cdot\frac{1}{1-x} - \frac{a^2x^3}{(1-ax)(1-ax^2)}\cdot\frac{1}{1-x^2} + \frac{a^3x^6}{[illegible]}$$

$$\Pi(-ax,x)\left\{\frac{ax}{(1-x)(1-ax)} + \frac{2a^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)} + \&c\right\} = \frac{ax}{1-x} - \frac{a^2x^3}{1-x^2} + \frac{a^3x^6}{1-x^3} - \frac{a^4x^{10}}{1-x^4} + \&c$$

$$\frac{a-b}{1-x} + \frac{a^2-b^2}{1-x^2} + \frac{a^3-b^3}{1-x^3} + \frac{a^4-b^4}{1-x^4} + \&c = \frac{1}{1-x}\cdot\frac{a-b}{1-b} + \frac{1}{1-x^2}\cdot\frac{(a-b)(a-bx)}{(1-b)(1-bx)} + \frac{1}{1-x^3}\cdot\frac{(a-b)(a-bx)(a-bx^2)}{(1-b)(1-bx)(1-bx^2)} + \&c$$

Page 355 from the unorganized portion in Ramanujan's second notebook

$$\frac{a}{1-x} + \frac{2a^2}{1-x^2} + \frac{3a^3}{1-x^3} + \frac{4a^4}{1-x^4} + \&c$$

$$= \frac{a}{1-x}\cdot\frac{1}{1-a} + \frac{a^2}{1-x^2}\cdot\frac{1-x}{(1-a)(1-ax)} + \frac{a^3}{1-x^3}\cdot\frac{(1-x)(1-x^2)}{(1-a)(1-ax)(1-ax^2)} + \&c$$

2. $$\frac{\psi(x)\psi(x^2)\psi(x^3)}{\psi(x^6)} - \phi^2(x^3) = 1 + 2\left(\frac{x}{1-x} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} - \frac{x^{11}}{1-x^{11}} + \&c\right)$$

$$\psi(x)\psi(x^3) + \psi(-x)\psi(-x^3) = 2\psi(x^4)\phi(x^6)$$

$$\phi(x)\phi(x^3) + \phi(-x)\phi(-x^3) = 2\left\{1 + 6\left(\frac{x^4}{1-x^4} - \frac{x^8}{1-x^8} + \frac{x^{16}}{1-x^{16}} - \&c\right)\right\}$$

$$1 + \left(\tfrac{1}{2}\right)^3 4x(1-x) + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^3 \{4x(1-x)\}^2 + \&c = z^2$$

$$1 + 4\cdot\left(\tfrac{1}{2}\right)^3 4x(1-x) + 7\cdot\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^3 \{4x(1-x)\}^2 + \&c = \frac{1}{1-2x}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\cdot\left(\tfrac{1}{2}\right)^3 + \frac{13}{4^2}\cdot\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{19}{4^3}\cdot\left(\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \&c$$

$$\frac{16}{\pi} = 5 + \frac{47}{64}\cdot\left(\tfrac{1}{2}\right)^3 + \frac{89}{64^2}\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{131}{64^3}\left(\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \&c$$

$$\frac{8(1+\sqrt{5})}{\pi} = (6+\sqrt{5}) + (66+19\sqrt{5})\cdot\left(\tfrac{1}{2}\right)^3\cdot\frac{(\sqrt{5}-1)^8}{64} + \&c$$

$$\frac{x}{1-x} + \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \frac{x^7}{1-x^7} + \cdots$$

$$= \frac{x}{1-x} + \frac{x^3}{1-x^2} + \frac{x^6}{1-x^3} + \frac{x^{10}}{1-x^4} + \frac{x^{15}}{1-x^5} + \cdots$$

Two of the five q-series identities

[1] B. C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer-Verlag, New York, 1991.

Notation:

$$
\begin{aligned}
(A)_0 &:= (A;q)_0 = 1, \\
(A)_n &:= (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \qquad n \geq 1, \\
(A)_\infty &:= (A;q)_\infty = \lim_{n\to\infty} (A;q)_n.
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Entry 1:

$$\frac{(-aq)_\infty}{(bq)_\infty} = \sum_{n=0}^{\infty} \frac{(-b/a)_n a^n q^{n(n+1)/2}}{(q)_n (bq)_n}. \qquad (I)$$

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Remaining three q-series identities

- **Entry 3:**

$$\sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{(1-q^n)(b)_n} = \sum_{n=1}^{\infty} \frac{a^n - b^n}{1-q^n}. \qquad (III)$$

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- All five identities proved by Bruce Berndt.

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- Now replace a by bq and then replace c by $-a/b$ to obtain **Entry 1**.

Proof of Entry 2

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Entry 2:

$$(aq)_\infty \sum_{n=1}^{\infty} \frac{na^n q^{n^2}}{(q)_n (aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a^n q^{n(n+1)/2}}{1-q^n}.$$

Proof: Use Entry 9 of Chapter 16 of Ramanujan's second notebook, that is,

$$(aq)_\infty \sum_{n=1}^{\infty} \frac{b^n q^{n^2}}{(q)_n (aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{b}{a}\right)_n a^n q^{n(n+1)/2}}{(q)_n}.$$

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- Differentiating with respect to b, we have

$$(aq)_\infty \sum_{n=1}^{\infty} \frac{n b^{n-1} q^{n^2}}{(q)_n (aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^n \frac{\mathrm{d}}{\mathrm{d}b} \left(\frac{b}{a}\right)_n a^n q^{n(n+1)/2}}{(q)_n}.$$

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- Now replace b by a and use the fact that

$$\frac{\mathrm{d}}{\mathrm{d}b} \left(\frac{b}{a}\right)_n \Big|_{b=a} = -\frac{1}{a}(q)_{n-1}$$

to complete the proof.

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Entry 3: For $a \neq 0, |a| < 1$, and $|b| < 1$, we have

$$F(a,b) := \sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{(1-q^n)(b)_n} = \sum_{n=1}^{\infty} \frac{a^n - b^n}{1-q^n}. \qquad (III)$$

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Proof: Note that

$$\begin{aligned} F(a,b) - F(aq,bq) &= \sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{(1-q^n)(b)_n} - \sum_{n=1}^{\infty} \frac{(b/a)_n (aq)^n}{(1-q^n)(bq)_n} \\ &= \sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{(1-q^n)(b)_{n+1}} \left[(1-bq^n) - q^n(1-b)\right] \\ &= \sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{(b)_{n+1}}. \end{aligned}$$

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- With the help of Entry 6 of Chapter 16, Bruce Berndt derived that

$$\sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{(b)_{n+1}} = \frac{a}{1-a} - \frac{b}{1-b}.$$

Prof of Entry 3: Continued...

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- Hence, for $n \geq 0$, we have

$$F(aq^n, bq^n) - F(aq^{n+1}, bq^{n+1}) = \frac{aq^n}{1-aq^n} - \frac{bq^n}{1-bq^n}. \qquad (1)$$

Prof of Entry 3: Continued...

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Now taking sum both sides on n, from 0 to infinity, and observe that $F(aq^n, bq^n)$ tends to 0 as $n \to \infty$.

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- Therefore, we have

$$F(a,b) = \sum_{n=0}^{\infty} \frac{aq^n}{1-aq^n} - \frac{bq^n}{1-bq^n} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} [(aq^n)^m - (bq^n)^m]$$

$$= \sum_{m=1}^{\infty} \left(\frac{a^m}{1-q^m} - \frac{b^m}{1-q^m} \right).$$

[2]K. Uchimura, *An identity for the divisor generating function arising from sorting theory*, J. Combin. Theory Ser. A **31** (1981) 131–135.

[3]J. C. Kluyver, Vraagstuk XXXVII (Solution by S.C. van Veen), Wiskundige Opgaven (1919), 92–93.

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1-q^n)(zq)_n} = \sum_{n=1}^{\infty} \frac{z^n q^n}{1-q^n}. \qquad (2)$$

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[2]K. Uchimura, *An identity for the divisor generating function arising from sorting theory*, J. Combin. Theory Ser. A **31** (1981) 131–135.

[3]J. C. Kluyver, Vraagstuk XXXVII (Solution by S.C. van Veen), Wiskundige Opgaven (1919), 92–93.

Beautiful partition-theoretic interpretation by Bressoud-Subbarao; and Fokkink-Fokkink-Wang (FFW)

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Weighted partition identity of Bressoud-Subbarao

• In the same paper, using purely combinatorial arguments, Bressoud and Subbarao[7] derived a more general identity involving the generalized divisor function,

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A generalization of **Entry 3**

Theorem (Ramanujan, Berndt)

For $|a| < 1$ *and* $|b| < 1$*, we have*

$$\sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{(1-q^n)(b)_n} = \sum_{n=1}^{\infty} \frac{a^n - b^n}{1-q^n}.$$

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Theorem (Dixit-M.)

Let a, b, c be three complex numbers such that $|a| < 1$ and $|cq| < 1$. Then

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Note that

$$\begin{aligned}
G(a,b;c) - G(aq,bq;c) &= \sum_{n=1}^{\infty} \frac{\left(\frac{b}{a}\right)_n a^n}{(1-cq^n)(b)_{n+1}} (1 - bq^n - q^n(1-b)) \\
&= \sum_{n=1}^{\infty} \frac{\left(\frac{b}{a}\right)_n a^n}{(1-cq^n)(b)_{n+1}} (1-q^n) \\
&= \sum_{n=1}^{\infty} \frac{\left(\frac{b}{a}\right)_n a^n}{(b)_{n+1}} \left(\frac{(1-cq^n)-(1-c)q^n}{1-cq^n} \right) \\
&= \sum_{n=1}^{\infty} \frac{\left(\frac{b}{a}\right)_n a^n}{(b)_{n+1}} - (1-c) \sum_{n=1}^{\infty} \frac{\left(\frac{b}{a}\right)_n (aq)^n}{(1-cq^n)(b)_{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{\left(\frac{b}{a}\right)_n a^n}{(b)_{n+1}} - \frac{(1-c)}{(1-b)} G(aq,bq;c).
\end{aligned}$$

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$$G(a, b; c) - \left(\frac{c-b}{1-b}\right) G(aq, bq; c) = \frac{a}{1-a} - \frac{b}{1-b}. \tag{7}$$

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- Multiply both sides by $(c-b)/(1-b)$ to get

$$\left(\frac{c-b}{1-b}\right) G(aq,bq;c) - \left(\frac{c-b}{1-b}\right)\left(\frac{c-bq}{1-bq}\right) G(aq^2,bq^2;c) \tag{8}$$

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Now letting $n \to \infty$, we have

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- However, one can observe that

$$ \frac{1}{(1-cq)} {}_3\phi_2\left(\begin{matrix} \frac{bq}{a}, q, cq \\ bq, cq^2 \end{matrix}; q, a\right) = \frac{(1-b)}{(a-b)} \sum_{n=1}^{\infty} \frac{\left(\frac{b}{a}\right)_n a^n}{(1-cq^n)(b)_n}, $$

$$ \frac{1}{(1-a)} {}_3\phi_2\left(\begin{matrix} \frac{b}{c}, q, a \\ bq, cq \end{matrix}; q, cq\right) = (1-b) \sum_{n=0}^{\infty} \frac{\left(\frac{b}{c}\right)_n (cq)^n}{(b)_n(1-bq^n)(1-aq^n)}. $$

- Using these two expressions and simplifying we can complete the proof.

A generalization of Ramanujan's fourth identity

[9] G. E. Andrews, F. G. Garvan, J. Liang, *Self-conjugate vector partitions and the parity of the* spt-*function*, Acta Arith. **158** (2013), 199–218.

Letting $a \to 0$ and replacing b by zq, we see that

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where $\quad \mathrm{FFW}(c,n) := \sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} \left(1 + c + \cdots + c^{s(\pi)-1}\right).$

[9] G. E. Andrews, F. G. Garvan, J. Liang, *Self-conjugate vector partitions and the parity of the* spt-*function*, Acta Arith. **158** (2013), 199–218.

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Implications of the main theorem

- Letting $a \to 0$ and $b = zq$, we get a two-variable generalization of the result of Andrews, Garvan and Liang,

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For $|cq| < 1$, we have

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- Now letting $d \to 0$ in (13), we arrive at a beautiful q-series identity of Andrews, namely,

Theorem (Andrews)

For $|cq| < 1$,

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- Multiplying by $\frac{(1-b/a)(1-c/d)ad}{(1-b)(1-cq)}$ on both sides and simplifying one can complete the proof.

A generalization of Bressoud and Subbarao's identity

Theorem (Bressoud-Subbarao)

For any integer $m \geq 0$ *and* $n \in \mathbb{N}$,

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j)^m = \sum_{d|n} d^m, \qquad (14)$$

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- But if we apply the same operator successively m-times on the left hand side, the final expression becomes complicated.

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Lemma

Let $k\in\mathbb{N}$, $r\in\mathbb{N}\cup\{0\}$ and let $F_0(a) := a^r(a+a^2+\cdots+a^k)$. Then, for each $m\in\mathbb{N}$,

$$\begin{aligned} D^m\left[F_0(a)\right] &= (r+1)^m a^{r+1} + (r+2)^m a^{r+2} + \cdots + (r+k)^m a^{r+k} \\ &= \sum_{j=1}^{k} (r+j)^m a^{r+j}. \qquad (17) \end{aligned}$$

- Now apply this lemma with $F_0(a) := a^r(a+a^2+\cdots+a^k)$, where $r = \ell(\pi) - s(\pi)$ and $k = s(\pi)$

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■ To prove this identity for negative integer m, we have to apply the integral operator $I[f(a)] := \int_0^a \frac{f(t)}{t} dt$ successively m-times on the both sides of the following lemma:

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Let $\mathcal{D}^*(n)$ be the collection of partitions of n, where only the largest part appears exactly twice and other parts are distinct and $\#(\pi) \geq 3$. Then,

$$d(n) = 1 + \left\lfloor \frac{n}{2} \right\rfloor - \sum_{\pi \in \mathcal{D}^*(n)} (-1)^{\#(\pi)-1} (s_2(\pi) - s(\pi)), \qquad (19)$$

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Partition $\pi \in \mathcal{D}^*(9)$	$\#(\pi)$	$s_2(\pi) - s(\pi)$	$(-1)^{\#(\pi)-1}(s_2(\pi) - s(\pi)$
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- Recently, Gupta and Kumar studied following q-series

$$\sum_{n=1}^{\infty} \frac{\left(\frac{q}{a}\right)_n a^n}{(1-q^n)^k (q)_n}, \quad k \in \mathbb{N}.$$

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Thank you for your attention!