Combinatorics of Stammering Tableaux

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Supervisors

Samuele Giraudo, Matthieu Josuat-Vergès

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For example the polynomials $x(x+1)\ldots(x+n-1)$ are used to count permutations in $\mathfrak{S}_n$ by number of cycles, i.e. the coefficient of x^i in this polynomial denotes the number of permutations with i cycles.

Another Combinatorial Family: Increasing Binary Trees

Definition

An increasing binary tree on n vertices is defined as an ordered rooted tree satisfying the following properties:

- Each vertex has atmost two children which are ordered, a left child and a right child i.e., each vertex has zero or one left children and zero or one right children.
- Each vertex is labelled differently using the labels $\{1, \ldots, n\}$ such that the label of a vertex is always strictly greater than any of its children. Thus, the root has label 1.

We shall refer to the vertices by their labels.

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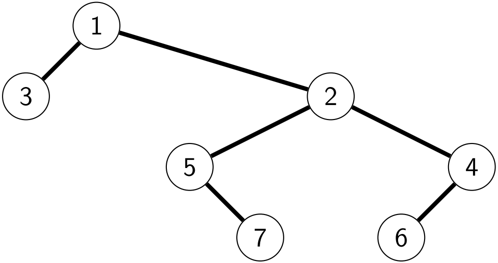

Figure: Increasing tree on 7 vertices.

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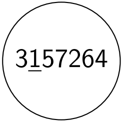
Example of Bijection: Increasing Binary Trees and Permutations



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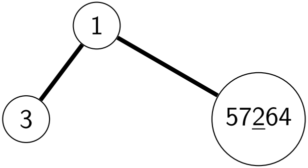

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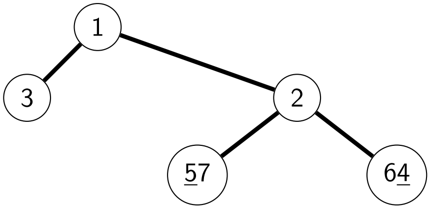

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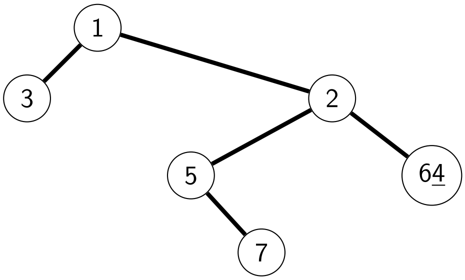

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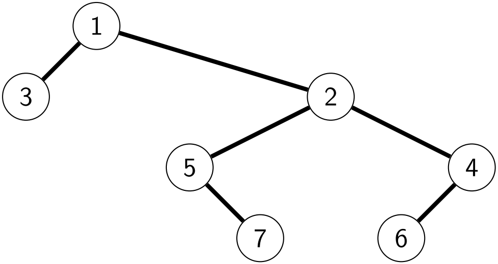

Figure: Increasing tree for permutation $\sigma = 3157264$.

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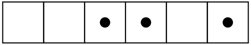

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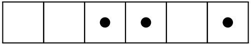

The possible transitions are:

- a particle moves to the right,
- a particle moves to the left,
- a new particle arrives from the left,
- a particle exits on the right.

These four transitions happen with different probabilities and that makes it a Markov chain. One is interested in computing stationary probabilities of this Markov chain (probability vector that is a eigenvector of the transition matrix of eigenvalue one.)

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By a partition λ we mean a non-increasing sequence of natural numbers $\lambda_1 \geq \lambda_2 \geq \dots$ such that there is some $k \in \mathbb{N}_{>0}$ such that $\lambda_n = 0$, $\forall n \geq k$.

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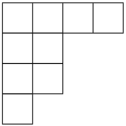
For example, $\lambda = (4, 2, 2, 1)$ is a partition of 9.

Some Definitions: Young Diagrams

We can denote a partition λ diagrammatically in the following representations:

- English: We put a row of λ_1 squared cells, just below it we put a row of λ_2 squared cells beginning just below the first cell of the previous row and so on. For example, for the partition $\lambda = (4, 2, 2, 1)$ the English representation is the following:



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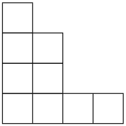

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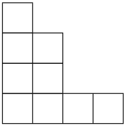

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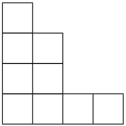

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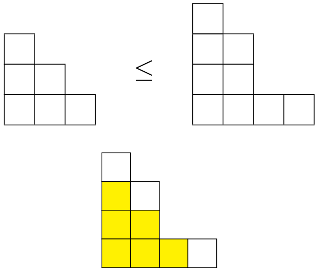
In terms of Young diagrams, $\lambda \leq \mu$ if the diagram of λ is contained in the diagram of μ.



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- if $\lambda \leq \mu$ and $\lambda \neq \mu$,
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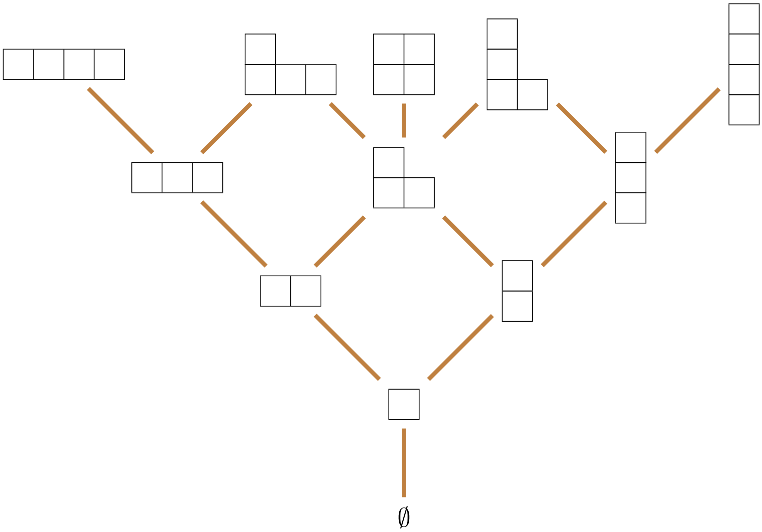
We can have a diagrammatic representation of the Young Lattice by drawing the Young diagrams from bottom to top where Young diagrams having the same number of cells are placed in the same row. We put edges in between two Young diagrams to denote cover relations. This is called the Haase diagram of this poset.

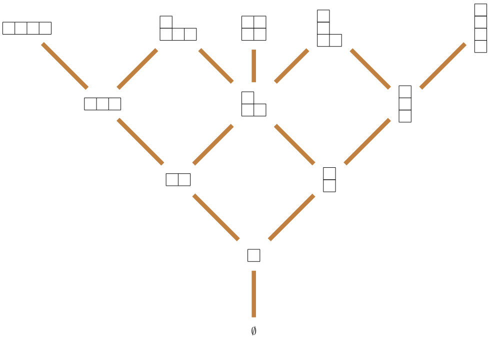
Young Lattice



Young Lattice upto 4^{th} gradation

Up and Down Operators

Let $K\langle\mathcal{Y}\rangle$ be the linear span of all partitions over a field K of characteristic zero, i.e a vector space over K where the basis elements are the partitions. Other than the identity operator I, we define two operators D and U on this vector space:



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$$D.\lambda = \sum_{\mu \lessdot \lambda} \mu$$

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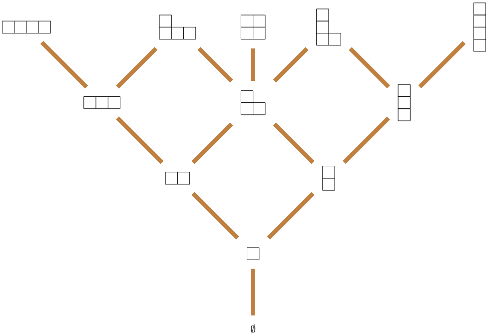
where $\lessdot$ is the covering relation in the poset $\mathcal{Y}$.



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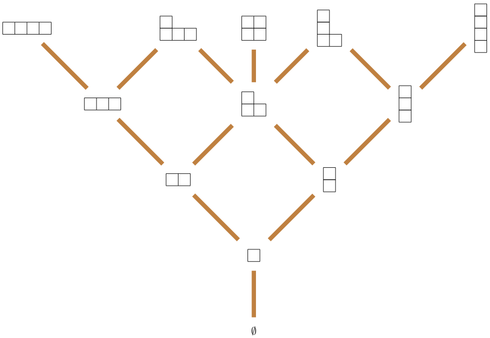

It turns out that

$$DU - UD = I$$

.

Standard Young Tableaux

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A standard Young tableau (SYT) is defined as a finite sequence of partitions $T = (\emptyset = \lambda^{(0)}, \ldots, \lambda^{(n)} = \lambda)$ where $\lambda^{(i)} \lessdot \lambda^{(i+1)}$ for all $i \in \{0, \ldots, n-1\}$.

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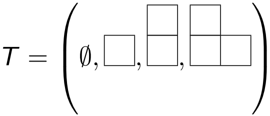

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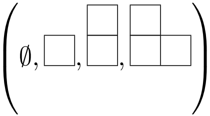
$T =$ 

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Stammering Tableaux are motivated from the commutation relation and from some kinds of tableaux such as oscillating tableaux, vacillating tableaux and hesitating tableaux. Here tableaux refers to some walks on the Young lattice.

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Definition (Josuat-Vergès, 2017)

A *stammering tableaux* of size n is a finite sequence of partitions, $(\lambda^{(0)}, \dots, \lambda^{(3n)})$ where $\lambda^{(0)} = \emptyset$ and $\lambda^{(3n)} = \emptyset$ such that

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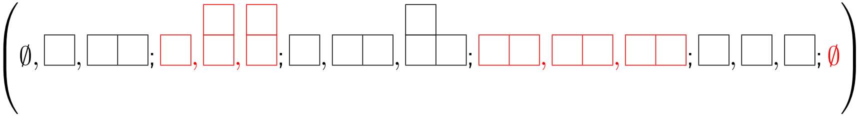

is a stammering tableaux of size 5.

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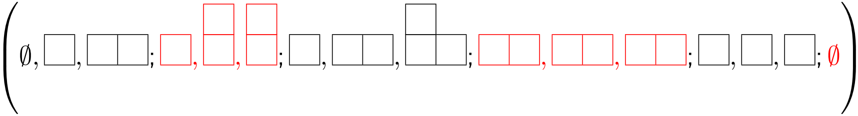
- if $i \equiv 0$ or $1 \pmod 3$ then either $\lambda^{(i)} \lessdot \lambda^{(i+1)}$ or $\lambda^{(i)} = \lambda^{(i+1)}$,
- if if $i \equiv 2 \pmod 3$ then $\lambda^{(i)} \gtrdot \lambda^{(i+1)}$.



is a stammering tableaux of size 5.
Let $\mathrm{Stam}^{(n)}_{\emptyset,\emptyset}$ denote the set of stammering tableaux of size n and shape λ. Let $T^{(n)}_{\emptyset,\emptyset}$ denote the cardinality of this set.

A Generalisation

We can generalise this definition by removing the conditions $\lambda^{(0)} = \emptyset$ and $\lambda^{(3n)} = \emptyset$ and having $\lambda^{(0)} = \mu$ and $\lambda^{(3n)} = \lambda$. We then say that the stammering tableaux has shape λ. The corresponding set and its cardinality are denoted by $\mathrm{Stam}_{\mu,\lambda}^{(n)}$ and $T_{\mu,\lambda}^{(n)}$ respectively.

Enumeration of Stammering Tableaux

Theorem (Josuat-Vergès, 2017)

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Corollary

Thus we have that when $\lambda = \emptyset$,

$$T^{(n)}_{\emptyset,\emptyset} = (n+1)!.$$

Rook Placements on Double Staircase

Definition (Josuat-Vergès, 2017)

Let $2\delta_n$ denote the partition $(2n, 2(n-1), \ldots, 2)$. We call this as the double staircase. A partial filling of the cells of the Young diagram of $2\delta_n$ with rooks is called a *rook placement in* $2\delta_n$ if there is exactly one rook in each row and atmost one rook in each column.

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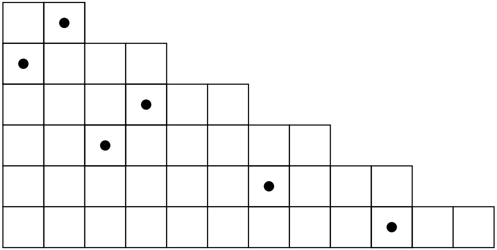

Figure: Rook placements on $2\delta_6$

Rook Placements on Double Staircase

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Rook Placements on Double Staircase

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There are three proofs provided in (Josuat-Vergès, 2017)

- A construction of Fomin, which generalises the Robinson-Schensted bijection, called growth diagrams.
- A variation of Schensted insertion.
- Viennot's shadow lines.

Projection from Stammering Tableaux to Oscillating Tableaux

Given a stammering tableaux T of size n, we can obtain an oscillating tableaux of length $2n$, $\Phi_n(T)$ from it by removing the consecutive repetitions of the same shape from T.

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For example for

$$T = \left(\emptyset, \ydiagram{1}, \ydiagram{2}; \ydiagram{1}, \ydiagram{1}, \ydiagram{2}; \ydiagram{1}, \ydiagram{2}, \ydiagram{2}; \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}; \ydiagram{1}, \ydiagram{1}, \ydiagram{1}; \emptyset\right)$$

we have

$$\Phi_5(T) = \left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1}, \emptyset\right)$$

Two Questions

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2. Given an oscillating tableaux S in the image of Φ_n, $\mathrm{Im}(\Phi_n)$, what is $\Phi_n^{-1}(S)$?

Equivalent Questions in Terms of Rook Placements

Via Fomin's local rules and growth diagrams, the map Φ_n is equivalent to taking a rook placement on the double staircase $2\delta_n$ with exactly one rook in each row and atmost one rook in each column, and removing the columns with no rooks and joining the remaining columns to make a Young diagram of n rows and n columns.

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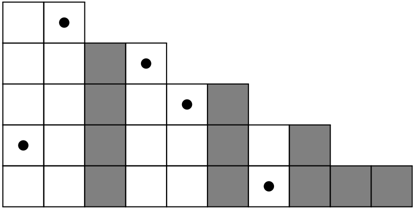
Thus instead of oscillating tableau we can talk about rook placements in double staircases and Young diagrams with n rows and n columns.

An Example of the Projection

For rook placement T in $2\delta_5$ in Figure below,



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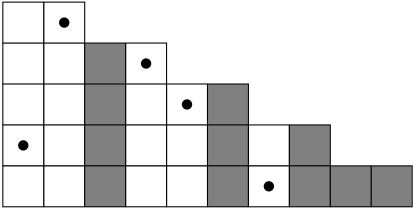
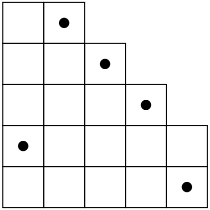


its projection $\Phi_5(T)$ is given by the rook placement in Figure below.



Definition (Block)

Given a Young diagram, a *block* is a maximal sequence of adjacent columns of the same height. The size of a block is the number of columns it has. The blocks of a rook placement are the blocks of its underlying Young diagram. For a rook placement T we denote the the number of blocks of size 1 of T as $\mathrm{bso}(T)$.

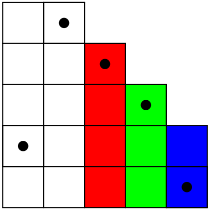


Figure: Rook placement on a Young diagram with 4 blocks, of which 1 block is of size 2 and the other blocks are of size 1.

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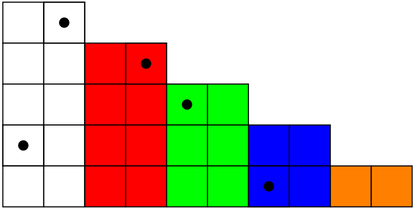
Lemma

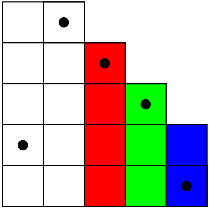
The size of a block of a rook placement $T \in \mathrm{Im}(\Phi_n)$ can be either 1 or 2.

Cardinality of the Preimage

Lemma

For $T \in \mathrm{Im}(\Phi_n)$, the cardinality of $\Phi_n^{-1}(T)$ is $2^{\mathrm{bso}(T)}$.





A New Sequence of Polynomials

Thus, we get the following identity:

$$(n+1)! = \sum_{T \in \mathrm{Im}(\Phi_n)} 2^{\mathrm{bso}(T)}.$$

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If we replace 2 with an indeterminate x we have the following definition of the sequences of polynomials:

$$p_n(x) = \sum_{T \in \mathrm{Im}(\Phi_n)} x^{\mathrm{bso}(T)}.$$

Thus, clearly $p_n(2) = (n+1)!$.

First few $p_n(x)$

n	$p_n(x)$
1	x
2	$x^2 + 2$
3	$x^3 + 8x$
4	$x^4 + 22x^2 + 16$
5	$x^5 + 52x^3 + 136x$
6	$x^6 + 114x^4 + 720x^2 + 272$
7	$x^7 + 240x^5 + 3072x^3 + 3968x$
8	$x^8 + 494x^6 + 11616x^4 + 34304x^2 + 7936$
9	$x^9 + 1004x^7 + 40776x^5 + 230144x^3 + 176896x$
10	$x^{10} + 2026x^8 + 136384x^6 + 1328336x^4 + 2265344x^2 + 353792$
11	$x^{11} + 4072x^9 + 441568x^7 + 6949952x^5 + 21953408x^3 + 11184128$

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3. We see that the non-zero coefficients of $p_n(x)$ matches with the triangle of numbers A101280 on OEIS.

Definition (Ascents and Descents)

The *descent* of a permutation of $\sigma \in \mathfrak{S}_n$ is a position $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$. An ascent is a position i such that $\sigma(i) < \sigma(i+1)$. Let $\operatorname{des}(\sigma)$ ($\operatorname{asc}(\sigma)$) denote the number of descents (ascents) of a permutation σ.

Some Statistics

Definition (Four statistics)

Let $\sigma \in \mathfrak{S}_n$. We will take the convention that $\sigma_0 = 0$ and $\sigma_{n+1} = 0$. The integer $\sigma_i \in \{1, \ldots, n\}$ is called:

- a *peak* if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$,
- a *valley* if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$,
- a *double ascent* if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$,
- a *double descent* if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

In the permutation $\sigma = 943127685$ valleys are 1, 6, the peaks are 7,8,9, the double descents are 3,4,5 and the double ascent is 2.

0	9	4	3	1	2	7	6	8	5	0
	p	dd	dd	v	da	p	v	p	dd	

Eulerian Polynomials

We can define the Eulerian polynomials as follows:

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Definition

The n^{th} Eulerian polynomial is defined as

$$S_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)}$$

It is a degree $n - 1$ polynomial.

Relation to Eulerian polynomials

Theorem (Foata and Strehl 1974)

The Eulerian polynomial S_{n+1} can be expressed as a linear sum of the polynomials $(x+1)^n, x(x+1)^{n-2}, x^2(x+1)^{n-4}, \dots$ as

$$S_{n+1}(x) = (x+1)^n + a_{n+1,1}x(x+1)^{n-2} + a_{n+1,2}x^2(x+1)^{n-4} + \dots$$

where

$$a_{n+1,i} = \{\sigma \in \mathfrak{S}_{n+1} : \sigma \text{ has no double ascents and there are } i \text{ ascents each of which correspond to a valley.}\}$$

The coefficients $a_{n+1,i}$ seem to be the same as the coefficients that we obtain in our polynomials $p_n(x)$.
The coefficients $a_{n+1,i}$ were observed to be non-negative in (Foata and Schützenberger, 1970).

Main Result

Theorem

We have the following identity:

$$x^{n/2} p_n \left(\frac{x+1}{\sqrt{x}} \right) = S_{n+1}(x).$$

Sketch of Proof

We are done if we can show that the coefficient of x^{n-2i} in $p_n(x)$ is $a_{n+1,i}$.

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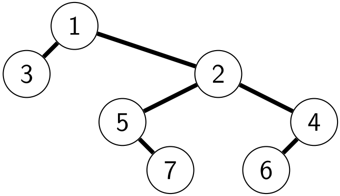
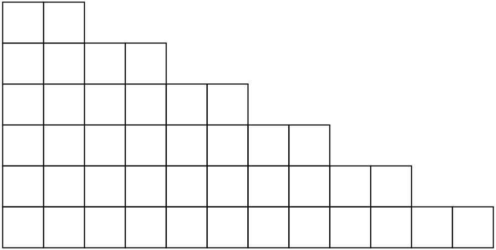
Note that the coefficient of x^{n-2i} in $p_n(x)$ counts the number of rook placements $T \in \mathrm{Im}(\phi_n)$ with $\mathrm{bso}(T) = n - 2i$. Thus, i is the number of blocks of size two.

Increasing Trees to Rook Placements





$\mathfrak{B}$: increasing binary trees of size $(n+1) \to$ rook placement on $2\delta_n$.
Rows of $2\delta_n$ enumerated from top to bottom, 1 through n and the blocks enumerated from left to right, 1 through n, in French notation.

1. If i is a left child of j put a rook in the left cell of block j and row $(i-1)$.
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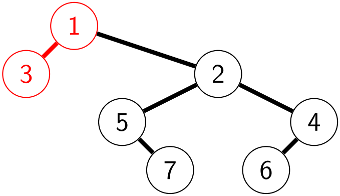

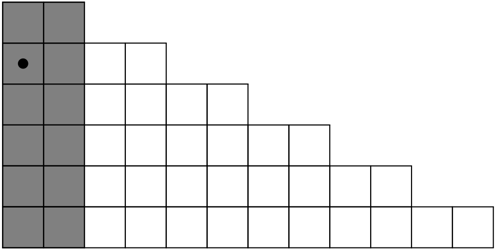

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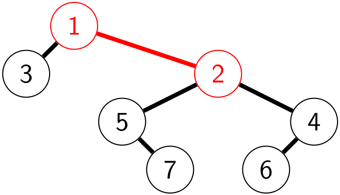

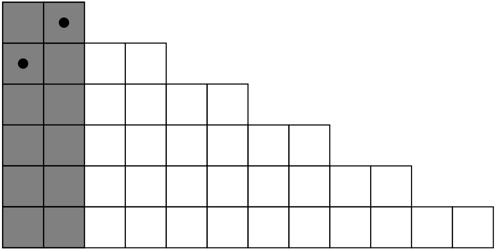

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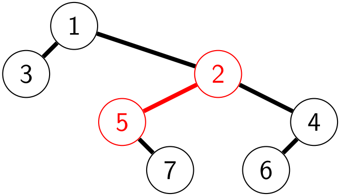

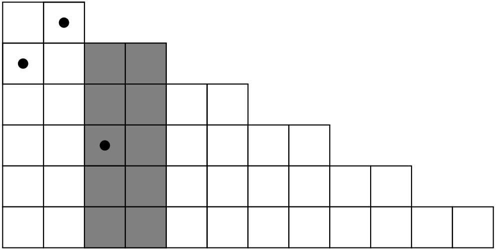

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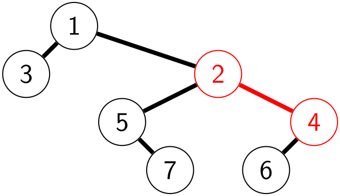

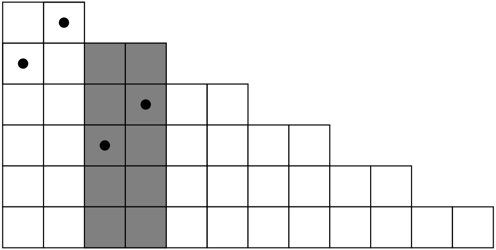

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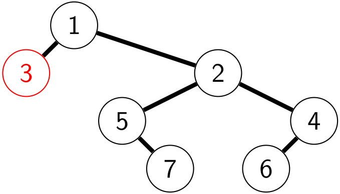

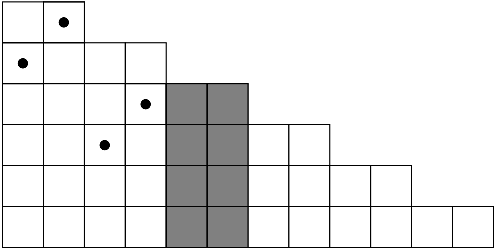

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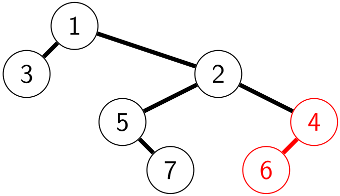

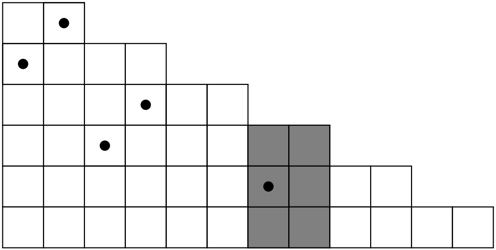

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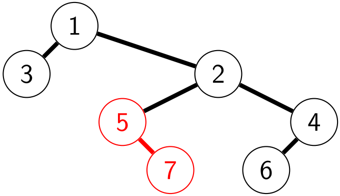

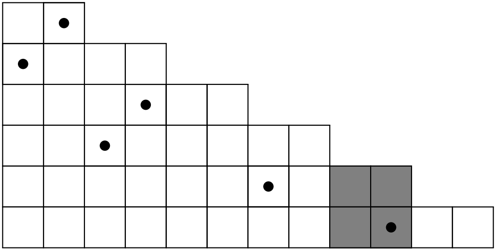

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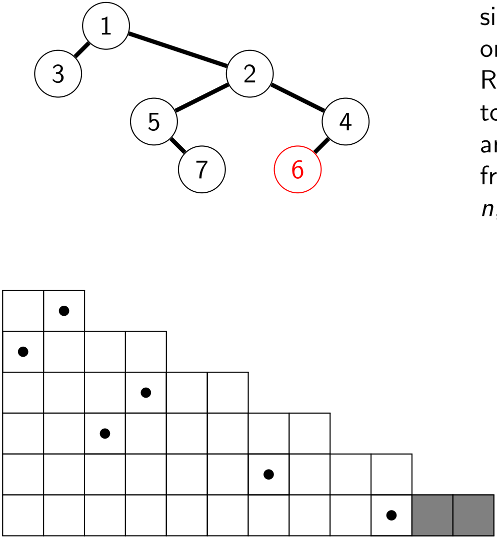

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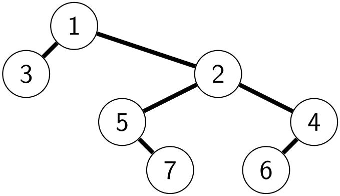

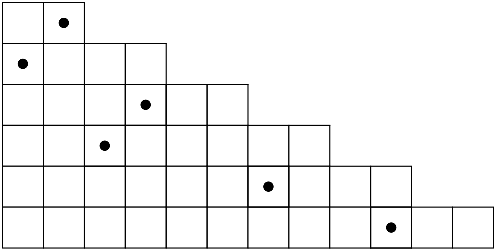

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$\mathfrak{B}$ is a bijection!

Increasing trees, Rook placements, Permutations

$$
\begin{array}{ccc}
\mathfrak{S}_{n+1} & & \\
\Big\Updownarrow \mathfrak{P} & & \\
\text{Increasing Trees of size } (n+1) & \overset{\mathfrak{B}}{\rightleftharpoons} & \text{Rook Placements on } 2\delta_n
\end{array}
$$

Back to Four Statistics and Comparison with Increasing Trees

In the permutation $\sigma = 943127685$ the valleys are 1, 6, the peaks are 7,8,9, the double descents are 3,4,5 and the double ascent is 2.

0	9	4	3	1	2	7	6	8	5	0
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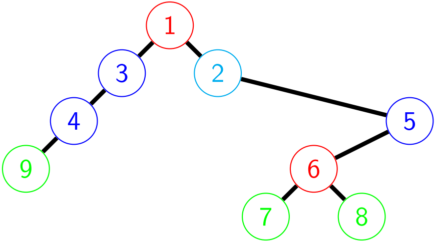

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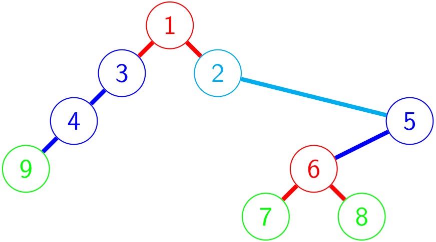
0	9	4	3	1	2	7	6	8	5	0
	p	dd	dd	v	da	p	v	p	dd	



Comparision of Statistics across three Families

Permutations on $n+1$ letters	Increasing Binary Trees on $n+1$ vertices	Rook placements on $2\delta_n$
Peaks	Leafs	Blocks with no rooks
Valleys	Vertices with two children	Blocks with two rooks
Double descents	Vertices with a left child and no right child	Blocks with rook in left column and no rook in right column
Double ascents	Vertices with right child and no left child	Blocks with a rook in right column and no rook in left column

Given $S \in \mathrm{Im}(\phi_n)$ there is a unique $T \in \phi_n^{-1}(T)$ in which each of the blocks is one of three forms:

- is empty,
- has exactly one rook in both of its columns,
- has exactly one rook in the left column and no rook in the right column.

Hence proved!

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Large Laguerre Profile of a Permutation

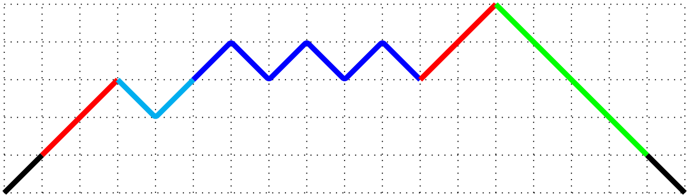
Definition ((Françon and Viennot, 1979))

For $\sigma \in \mathfrak{S}_n$ we take $\sigma_0 = \sigma_{n+1} = 0$. We define a function f from $[n-1]$ to $\{\searrow, \nearrow\}^2$. For $i \in [1, n]$, if

- if i is a valley then $f(i) = \nearrow\nearrow$
- if i is a peak then $f(i) = \searrow\searrow$
- if i is a double ascent then $f(i) = \searrow\nearrow$
- if i is a double descent then $f(i) = \nearrow\searrow$.

Now consider the path given by $\nearrow f(1)f(2)\dots f(n-1) \searrow$. This is a Dyck word. We call this Dyck word the *large profile* or the *large Laguerre profile* of the permutation σ and we shall denote it by $\nabla(\sigma)$.

The large profile of the permutation $\sigma = 943127685$ is drawn in Figure below. Here the valleys are 1, 6, the peaks are 7,8,9, the double descents are 3,4,5 and the double ascent is 2.



Dyck paths from Rook Placements

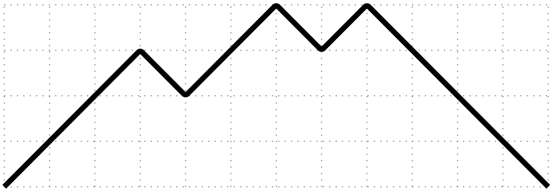
Given a rook placement R on $2\delta_n$ we define $d(R)$ of length $2n+2$ as

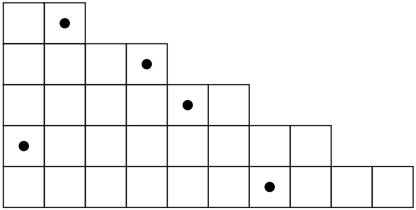
- the first step is $\nearrow$, the last step is $\searrow$,
- if $2 \leq i \leq 2n+1$, the i^{th} step is $\nearrow$ if the $(i-1)^{st}$ column of R contains a dot and $\searrow$ otherwise.

Lemma (Lemma 3.2 in (Josuat-Vergès, 2017))

$d(R)$ is a Dyck path of size $(n+1)$.

Example of Dyck path from Rook Placement





Sketch of a Bijection from Rook Placements on $2\delta_n$ to $\mathfrak{S}_{n+1}$

$$\mathfrak{S}_{n+1} \overset{\text{Chain of Large Profiles}}{\rightleftharpoons} (n+1)\text{-Chain of Dyck Shapes}$$

$$(n+1)\text{-Chain of Dyck Shapes} \overset{\zeta}{\rightleftharpoons} \text{Rook Placements on } 2\delta_n$$

This bijection between permutations and rook placements can also be used to prove the theorem.

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$$\begin{array}{ccc}
\mathfrak{S}_{n+1} & \xrightleftharpoons{\text{Chain of Large Profiles}} & (n+1)\text{-Chain of Dyck Shapes} \\
\Big\Updownarrow \psi & \circlearrowleft & \Big\Updownarrow \zeta \\
\text{Increasing Trees of size } (n+1) & \xrightleftharpoons{\mathfrak{B}} & \text{Rook Placements on } 2\delta_n
\end{array}$$

This bijection between permutations and rook placements can also be used to prove the theorem.
Also, the bijections commute!

Stirling Permutations

Definition (m-Stirling Permutations of size n)

An *m-Stirling permutation of size n* is a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_{mn}$ of the multiset $\{1^m, \ldots, n^m\}$ such that whenever for $i < j$, $\sigma_i = \sigma_j = l \in \{1, \ldots, n\}$, $\sigma_k \geq l$ for all $k \in \{i, \ldots, j\}$.

Let $\mathcal{Q}_n(m)$ denote the set of m-Stirling permutations of size n.
For example, 126624413355 is a 2-Stirling permutation.

Higher Arity Increasing Trees and Rook Placements

A m-Stirling Permutation can be represented with a $(m+1)$-ary increasing tree (first described in (Park, 1994) and explained in (Janson et al., 2011).).

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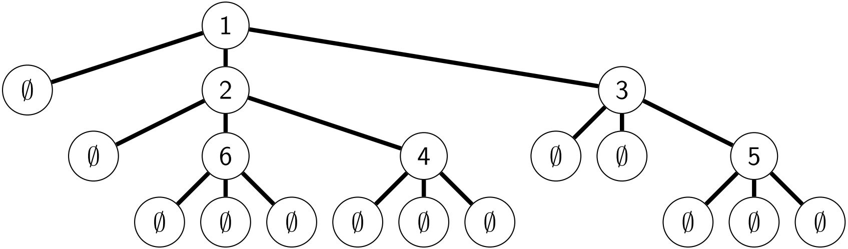


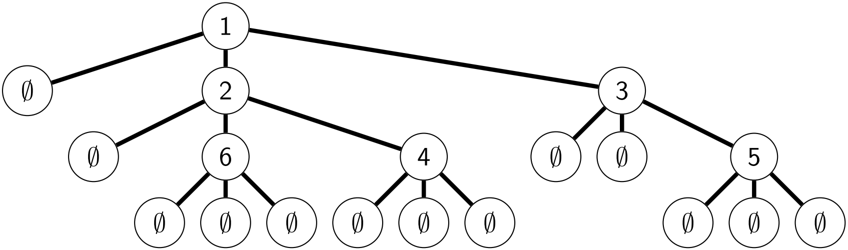
Ternary Increasing tree for Stirling permutation 126624413355

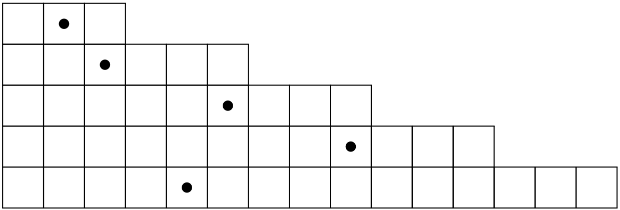
A Generalised Bijection

We easily can generalise the bijection $\mathfrak{B}$ to a bijection $\mathfrak{B}_m$ between increasing $(m+1)$-ary trees of size n and rook placements on $(m+1)\delta_{n-1}$ as follows:

1. If the vertex i has all empty $(\emptyset)$ children then keep the i^{th} block empty, *i.e.* without any rooks.
2. If the vertex i has a non-empty k^{th} child a then we put a rook in the k^{th} column of the i^{th} block in the $(a-1)^{\text{st}}$ row.

A Generalised Bijection





Some Related Work

A similar bijection between rook placements and increasing trees is mentioned in (Tewari, 2019). At this moment we do not know if the bijections are the same.

1. We have generalised the Laguerre profile of a permutation to Laguerre profile of a Stirling Permutations which uses rational Dyck paths.
2. We can enumerate the number of Stirling permutations which have the same Laguerre profile.
3. We can find the number of descents of an m-Stirling permutation by looking at the corresponding rook placement on a triple staircase.
4. We can prove a result of Bóna which says that that the plateau and descent statistics of 2-Stirling permutations have the same distribution. We do this by using rook placements on triple staircases.

Weight of a Stammering Tableau

Definition

For $T = (\lambda^{(0)}, \dots, \lambda^{(3n)}) \in \mathrm{Stam}^{(n)}_{\emptyset,\lambda}$, we define the weight of T to be

$$\mathrm{wt}(T) = \sum_{i=0}^{3n} |\lambda^{(i)}|.$$

For $j \in \{0, 1, 2\}$, we define the inner weight modulo j to be

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Thus, we have that

$$\mathrm{wt}(T) = \mathrm{inn}_0(T) + \mathrm{inn}_1(T) + \mathrm{inn}_2(T).$$

Polynomiality of Average Weights

Theorem

Let $|\lambda| = k$. *Then for all* $n \in \mathbb{N}$ *and* $n \geq k$ *we have that,*

$$\frac{1}{T^{(n)}_{\phi,\lambda}} \sum_{T \in \mathrm{Stam}^{(n)}_{\emptyset,\lambda}} \mathrm{wt}(T) = \frac{n^2 + 2(k+1)n + 3k}{2}$$

Sketch of Proof

We define some operators on the Young Lattice and inductively compute the weights. Follows after very long and ugly calculations!

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Aim: Find a bijective proof of this result.