A talk on

Congruences for ℓ-Regular Overpartition for $\ell \in \{5, 6, 8\}$

by

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Partition:

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. For example, $n = 4$ has five partitions, namely,

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

If $p(n)$ denote the number of partitions of n, then $p(4) = 5$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

- For any complex number a and q, we set

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \text{ for } n \geq 1,$$

$$\text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

- Ramanujan's general theta-function is given by:

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1.$$

Three special case of $f(a,b)$ are theta functions ϕ, ψ and f which are respectively defined by

$$\phi(q) := f(q;q) = \sum_{n=0}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}.$$

$$\psi(q) := f(q;q^3) = \sum_{k=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

and

$$f(-q) = f(-q;-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

Ramanujan established following beautiful congruences for $p(n)$:

$$p(5n+4) \equiv 0 \pmod{5},$$

$$p(7n+5) \equiv 0 \pmod{7},$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

• After the function $p(n)$, many other partition functions with certain restrictions are studied in recent times. Some partition functions and in the interest of this thesis are:

$\Rightarrow$ ℓ-regular partition.

$\Rightarrow$ ℓ-regular overpartition.

$\Rightarrow$ t-core partition.

$\Rightarrow$ t-core bipartition.

$\Rightarrow$ Restricted bipartition.

$\Rightarrow$ colour partition.

Partition k-tuples

• A partition k-tuple $(\lambda_1, \lambda_2, \cdots \lambda_k)$ of a positive integer n is a k-tuple of partitions $\lambda_1, \lambda_2, \cdots, \lambda_k$ such that the sum of all the parts equals n.

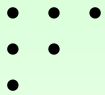
• If $k = 3$ then the 3-tuple $(\lambda_1, \lambda_2, \lambda_3)$ is called a partition triple of n.

• For example, $(\lambda_1 = \{3, 2\}, \lambda_2 = \{1, 1\}, \lambda_3 = \{1\})$ is a partition triple of 8 as $3 + 2 + 1 + 1 + 1 = 8$.

t-core bipartition

• A partition of n is called a t-core partition of n if none of its hook number is divisible by t.

• For example, the Ferrers-Young diagram of the partition 3+2+1 of 6 is



The nodes (1, 1), (1, 2), (1, 3), (2, 1), (2, 2) and (3, 1) have hook numbers 5, 3, 1, 3, 1 and 1, respectively. It is easily seen from above that the partition 3+2+1 of 6 is 4-core.

• A bipartition with t-core is a pair of partitions (λ, μ) such that λ and μ are both t - cores.

k-tuples t-core partitions

• A partition k-tuple $(\lambda_1, \cdots, \lambda_k)$ of a positive integer n with t-cores means that each λ_i is t-core.

r-Colour Partition

• A part in a partiton of n has r colours if there are r copies of n available and all of them are viewed as distinct objects.

• For any positive integer n and non-zero integer r, let $p_r(n)$ denote the number of partitions of n where each part has r distinct colours.

• For example, if each part in the partition of 3 has *TWO* colours, say red and green, then 2 colour partitions of 3 are given by

$$3_r, \quad 3_g, \quad 2_r+1_r, \quad 2_r+1_g, \quad 2_g+1_g, \quad 2_g+1_r,$$

$$1_r+1_r+1_r, \quad 1_g+1_g+1_g, \quad 1_r+1_g+1_g, \quad 1_r+1_r+1_g.$$

That is, $p_2(3)=10$.

Over partition and ℓ-regular partition

• An over partition of a positive integer n is a partition of n in which the first occurrence of each part can be over lined. For example, number of over partitions of $n = 3$ are

$$3, \quad \overline{3}, \quad 2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1}, \quad 1+1+1, \qquad \overline{1}+1+1.$$

So we have seen that number of over partition of 3 is 8.

• For any positive integer ℓ, ℓ-regular partition of a positive integer n is a partition of n such that none of its part is divisible by ℓ.

For example, the partition of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

So, it is not 3-regular or 2-regular.

ℓ-regular overpartition

- An ℓ-regular overpartition of a positive integer n is a partition of n in which the first occurrence of each part can be over lined with no part divisible by ℓ, where ℓ is a positive integer. For example, number of overpartitions for $n = 4$ is 14, namely,

$$4, \quad \overline{4}, \quad 3+1, \quad \overline{3}+1, \quad 3+\overline{1}, \quad \overline{3}+\overline{1}, \quad 2+2, \quad \overline{2}+2 \quad 2+1+1, \quad \overline{2}+1+1,$$

$$2+\overline{1}+1 \quad \overline{2}+\overline{1}+1, \quad 1+1+1+1, \qquad \overline{1}+1+1+1.$$

- Then the number of 3-regular over partition of 4 is 10, namely

$$4, \quad \overline{4}, \quad 2+2, \quad \overline{2}+2 \quad 2+1+1, \quad \overline{2}+1+1,$$

$$2+\overline{1}+1 \quad \overline{2}+\overline{1}+1, \quad 1+1+1+1, \qquad \overline{1}+1+1+1.$$

Generating function for ℓ-regular overpartition

- Shen (2016): If $\bar{A}_\ell(n)$ denote the number of as ℓ-regular overpartition of a positive integer n, then its generating function is given by

$$\sum_{n=0}^{\infty} \bar{A}_\ell(n) q^n = \frac{(q^\ell;q^\ell)_\infty^2 (q^2;q^2)_\infty}{(q;q)_\infty^2 (q^{2\ell};q^{2\ell})_\infty}.$$

- Shen (2016):

$$\bar{A}_3(4n+1) \equiv 0 \pmod 2$$
$$\bar{A}_3(4n+3) \equiv 0 \pmod 6$$
$$\bar{A}_3(9n+3) \equiv 0 \pmod 6$$
$$\bar{A}_3(9n+6) \equiv 0 \pmod{24}$$

• Chern (2016) proved some new infinite family of congruences modulo ℓ for $\bar{A}_\ell(n)$ where $\ell = 3, 5, 7$. Chern proved that, if $p \neq 5$ be a prime and k and n are non-negative integers, then

$$\bar{A}_5(p^{4k+3}(pn+i)) \equiv 0 \pmod{5}, i = 1, 2, 3, ...p-1$$

New Congruences for $\bar{A}_5(n)$

Theorem

Let $p \geq 5$ be a prime with $\left(\frac{-5}{p}\right) = -1$. Then for non-negative integers α and n and Legendre symbol $\left(\frac{\cdot}{\cdot}\right)$, we have

$$\sum_{n=0}^{\infty} \bar{A}_5\left(4p^{2\alpha}n + p^{2\alpha}\right) q^n \equiv 2(q;q)_\infty (q^5;q^5)_\infty \pmod{4}.$$

Corollary

Let $p \geq 5$ be an odd prime with $\left(\frac{-5}{p}\right) = -1$. Then for non-negative integers α and n, we have

$$\bar{A}_5\left(4p^{2\alpha+2}n + 4p^{2\alpha+1}j + p^{2\alpha+2}\right) \equiv 0 \pmod{4},$$

where $j = 1, 2, 3, \ldots, p-1$.

New Congruences for $\bar{A}_6(n)$

Theorem

Let p be an odd prime such that $\left(\frac{-2}{p}\right) = -1$. Then for non-negative integers α and n, we have

$$\sum_{n=0}^{\infty} \bar{A}_6\left(8p^{2\alpha}n + 3p^{2\alpha}\right) q^n \equiv 2\psi(q)\psi(q^2) \pmod{3}.$$

Corollary

Let p be an odd prime such that $\left(\frac{-2}{p}\right) = -1$. Then for non-negative integers α and n, we have

$$\bar{A}_6\left(8p^{2\alpha+2}n + 8p^{2\alpha+1}j + 3p^{2\alpha+2}\right) \equiv 0 \pmod{3},$$

where $j = 1, 2, 3, \ldots, p-1$.

New Congruences for $\bar{A}_6(n)$

Corollary

$$(i)\ \bar{A}_6(8n+5) \equiv 0 \pmod{3},$$
$$(ii)\ \bar{A}_6(8n+7) \equiv 0 \pmod{3}.$$

New Congruences for $\bar{A}_8(n)$

Theorem

$$(i)\ \bar{A}_8(12n+4i+1) \equiv 0 \pmod 3, \quad where \quad i = 1, 2.$$

$$(ii)\ \bar{A}_8(8n+k) \equiv 0 \pmod 4, \quad where \quad k = 3, 5.$$

$$(iii)\ \bar{A}_8(28n+4j) \equiv 0 \pmod 7, \quad where \quad j = 1, 2, 3, 4, 5, 6.$$

Theorem

Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for non-negative integers α and n

$$\sum_{n=0}^{\infty} \bar{A}_8\left(8p^{2\alpha}n + p^{2\alpha}\right) q^n \equiv 2(q;q)_\infty (q^2;q^2)_\infty \pmod 4.$$

Corollary

Let $p \geq 5$ *be a prime with* $\left(\frac{-2}{p}\right) = -1$*. Then for non-negative integers* α *and* n*, we have*

$$\bar{A}_8\left(8p^{2(\alpha+1)}n + 8p^{2\alpha+1}j + p^{2\alpha+2}\right) \equiv 0 \pmod{4},$$

where $j = 1, 2, 3, \ldots, p-1$.

Theorem

(*i*) $\bar{A}_8(4n+3) \equiv 0 \pmod{8}$.

(*ii*) $\bar{A}_8(8n+7) \equiv 0 \pmod{64}$.

(*iii*) $\bar{A}_8(24n+8i+7) \equiv 0 \pmod{3}$, *where* $i = 1, 2$.

(*iv*) $\bar{A}_8(56n+8j+7) \equiv 0 \pmod{7}$, *where* $j = 1, 2, 3, 4, 5, 6$.

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THANK YOU