Enumeration of Direct animals with Lattice paths

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Outline

1 history

2 Some properties of lattice path

3 Calculated $\mathcal{I}_m(n)$ for tables with few rows

4 Hankel matrix of $\mathcal{I}_n(n)$

5 Encoding perfect lattice paths with words

Overlays

A *lattice path* L in $\mathbb{Z}^d$ is any sequence $\nu_1, \nu_2, \ldots, \nu_k$ of points of $\mathbb{Z}^d$. The vectors $\nu_2 - \nu_1, \nu_3 - \nu_2, \ldots, \nu_k - \nu_{k-1}$ are called the *steps* of L. Lattice paths are studied by fixing a set of steps and an area $U \subseteq \mathbb{Z}^d$ where the paths live in. A typical problem to carry out is to count possible lattice paths in the given area U and a given length with steps in a given set $\mathbf{S} \subseteq \mathbb{Z}^d$.

Notations of this talk about of $S-$Lattice paths has been used by C. Krattenthaler and S. G. Mohanty:

C. Krattenthaler and S. G. Mohanty, Lattice path combinatorics - applications to probability and statistics. In norman L. Johanson, Campell B. Read, N. Balakrishnan, and Brani Vidakovic, Editors, Encyclopaedia of Statistical Sciences. Wiley, New York, Second Edition, 2003.

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Later Bousquet-Mélou and Conway and Corteel, Denise, give a bijective proofs to obtain algebraic equations satisfied by area generating function of directed lattice animals on infinite families of two-dimensional lattices.

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A *Dyck path* is a lattice path in $\mathbb{Z}^2$ starting from $(0, 0)$ and ending at a point $(2n, 0)$ $(n \geq 0)$ consisting of up-steps $(1, 1)$ and down-steps $(1, -1)$, which never passes below the x-axis.

You have seen that the number of Dyck paths is *Catalan numbers* $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$.

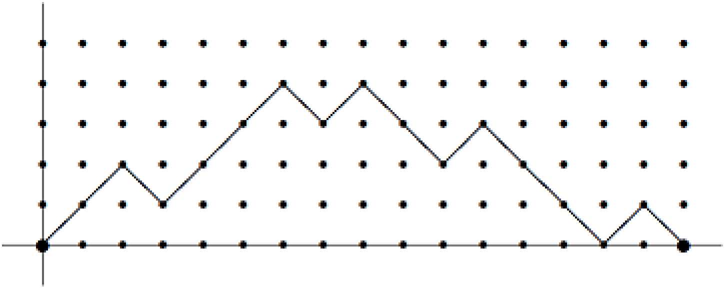

FIGURE: A lattice path from $(0; 0)$ back to the x-axis consisting of up-steps $(1; 1)$ and down-steps $(1; -1)$ never running below the x-axis.

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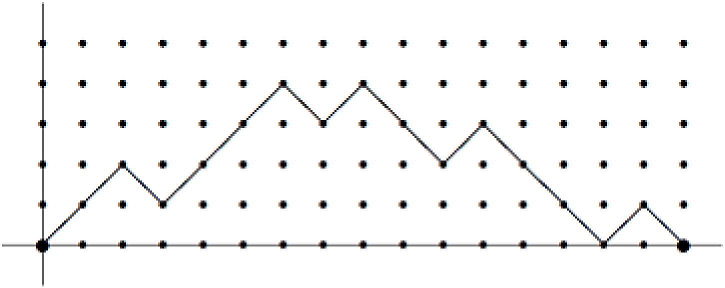

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Let me give another important example of lattice paths with three steps $(1,1),(1,-1)$ and $(1,0)$, which known Motzkin paths.
The Motzkin number M_n is defined as the number of Motzkin paths of length n.

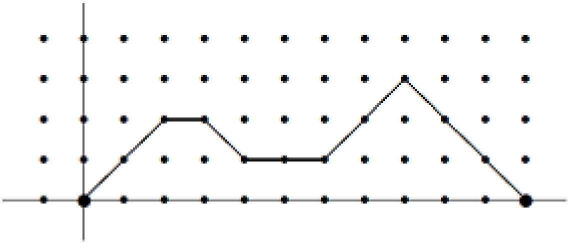

FIGURE: A Motzkin path is a lattice path from $(0;0)$ back to the x-axis consisting of up-steps $(1;1)$, level steps $(1;0)$, and down-steps $(1;-1)$, never running below the x-axis.

Overlays

Throughout this talk, $T_{m,n}$ stands for the $m \times n$ table in the first quadrant composed of mn unit squares, whose (x, y)-cell is located in the x^{th}-column from the left side and the y^{th}-row from the bottom side of $T_{m,n}$.

Also, for a set $\mathbf{S} \subseteq \mathbb{Z}^d$ of steps, $l((i,j) \to (s,t); \mathbf{S})$ denotes the number of all lattice paths in $T_{m,n}$ starting form the (i,j)-cell and ending at the (s,t)-cell with steps in $\mathbf{S}$, where $1 \leq i, s \leq n$ and $1 \leq j, t \leq m$.

Let us give an example:

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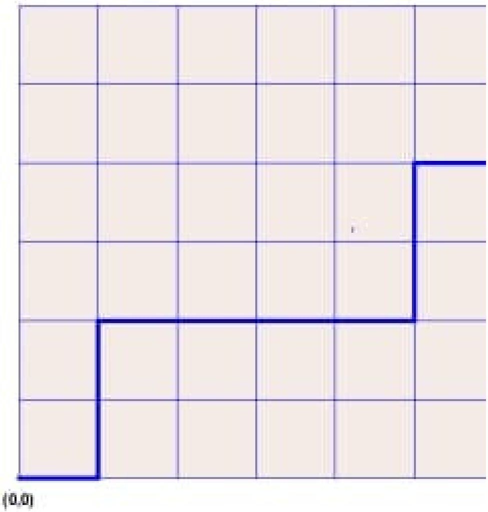

FIGURE: $l((0,0) \to (6,4); \mathbf{S})$, such that $\mathbf{S} = \{(1,0),(0,1)\}$.

Overlays

The paths we shall consider in this talk use the same set $\mathbf{S} = \{(1,1),(1,0),(1,-1)\}$ of steps as Motzkin paths but live in a bounded rectangular area, which we may assume to be $T_{m,n}$.

Notice that the number $l((1,1) \to (n,1); \mathbf{S})$ of all lattice paths in the table $T_{m,n}$ starting from the $(1,1)$-cell and ending at the $(n,1)$-cell using Motzkin steps $\mathbf{S} = \{(1,1),(1,0),(1,-1)\}$. The number of all such lattice paths is denoted by $\mathcal{I}_m(n)$ and called perfect lattice paths. Indeed,

$$\mathcal{I}_m(n) = \sum_{i,j=1}^{m} l((1,i) \to (n,j); \mathbf{S}).$$

Clearly, $l((1,i) \to (n,j)) = l((1,i') \to (n,j'))$ when $i + i' = m + 1$ and $j + j' = m + 1$.

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What does $\mathcal{I}_m(n)$ mean?

$\mathcal{I}_m(n)$ means the number of lattice paths starting from the first column and ending at the n column such that set $\mathbf{S} = \{(1,1),(1,0),(1,-1)\}$.

Do you have idea to calculating $\mathcal{I}_m(n)$?

Look at the following picture that is an example for $m = 3$ and $n = 2$:

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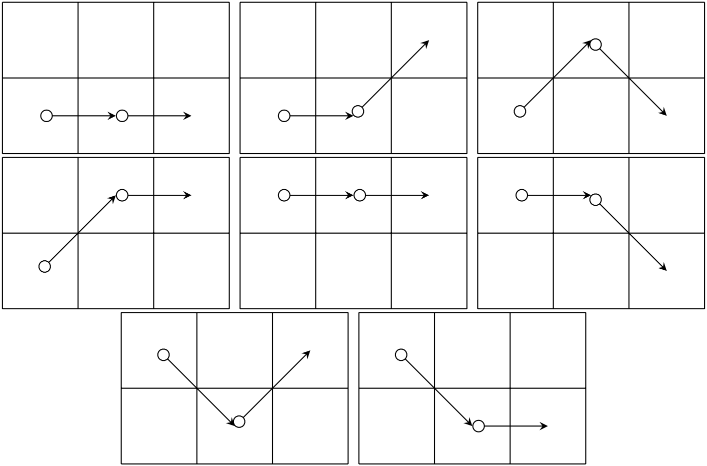

Figure: So, $\mathcal{I}_m(n) = 8$.

Overlays

we shall compute $\mathcal{I}_m(n)$ for $m = 1, 2, 3, 4$ and arbitrary positive integers n. Some values of the $\mathcal{I}_3(n)$ and $\mathcal{I}_4(n)$ are already given in A001333 and A055819, respectively.

Lemma

$\mathcal{I}_1(n) = 1$ and $\mathcal{I}_2(n) = 2^n$ for all $n \geq 1$.

Consider a table T with 3 rows and n columns. Let the number of all lattice paths from the first columns to the cell $(1, n-2)$ is x. Means

$n-2$	$n-1$	n
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x		
x		

$n-2$	$n-1$	n
x		
y		
x		

$n-2$	$n-1$	n
x	$x+y$	
y		
x	$x+y$	

$n-2$	$n-1$	n
x	$x+y$	
y	$x+x+y$	
x	$x+y$	

$n-2$	$n-1$	n
x	$x+y$	$3x+2y$
y	$x+x+y$	
x	$x+y$	$3x+2y$

$n-2$	$n-1$	n
x	$x+y$	$3x+2y$
y	$x+x+y$	$4x+3y$
x	$x+y$	$3x+2y$

Recall $\mathcal{I}_m(n)$ is the number of all lattice paths from the first column to the n columns which the steps come $\mathbf{S} = \{(1,1),(1,0),(1,-1)\}$.

What is the $\mathcal{I}_3(n-2) =?$ (means, the number of all lattice paths from the first columns to the $(n-2)$-th columns)

$n-2$	$n-1$	n
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$\mathcal{I}_3(n-2)=2x+y$	$\mathcal{I}_3(n-1)=4x+3y$	$\mathcal{I}_3(n)=10x+7y$

For integers a, b Let

$$\mathcal{I}_3(n) = a\mathcal{I}_3(n-1) + b\mathcal{I}_3(n-2)$$

Which lead

$$10x + 7y = a(4x+3y) + b(2x+y)$$

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To finding a and b, we have to solve to the following system equation

$$\begin{cases} 4a + 2b = 10, \\ 3a + 1b = 7. \end{cases} \tag{1}$$

It is easy to see $a = 2$ and $b = 1$. Thus the following linear recurrence exists for $\mathcal{I}_3$.

$$\mathcal{I}_3(n) = 2\mathcal{I}_3(n-1) + \mathcal{I}_3(n-2).$$

Since $\mathcal{I}_3(1) = \mathcal{Q}_2 = 3$ and $\mathcal{I}_3(2) = \mathcal{Q}_3 = 7$, it follows that $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$ for all $n \geq 1$, where $\mathcal{Q}_n$ is *Pell-Lucas* sequence.
Now, consider the coefficient matrix A

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix},$$

What is the determinant of the matrix A?

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It is easy to see $a = 2$ and $b = 1$. Thus the following linear recurrence exists for $\mathcal{I}_3$.

$$\mathcal{I}_3(n) = 2\mathcal{I}_3(n-1) + \mathcal{I}_3(n-2).$$

Since $\mathcal{I}_3(1) = \mathcal{Q}_2 = 3$ and $\mathcal{I}_3(2) = \mathcal{Q}_3 = 7$, it follows that $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$ for all $n \geq 1$, where $\mathcal{Q}_n$ is *Pell-Lucas* sequence.

Now, consider the coefficient matrix A

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix},$$

What is the determinant of the matrix A?

We have

$$|A| = -2 = -2^{\lfloor \frac{m}{2} \rfloor}.$$

Corollary

Let n be a positive integer. Then

$$\mathcal{I}_3(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \left[\binom{n-k+1}{k} + \binom{n-k}{k-1} \right] 2^{n-2k}.$$

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We repeat our way for $m = 4$, we have the following table

$n-2$	$n-1$	n
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y	$x+2y$	$3x+5y$
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x	$x+y$	$2x+3y$

Lemma

For all $n \geq 1$ we have $\mathcal{I}_4(n) = 2\mathcal{F}_{2n+1}$ such that $\mathcal{F}_n$ is $n-$th Fibonacci number.

Again, we have $\mathcal{I}_4(n-2) = 2x + 2y$, $\mathcal{I}_4(n-1) = 4x + 6y$, and $\mathcal{I}_4(n) = 10x + 16y$.

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Also,

$$A = \begin{bmatrix} 4 & 2 \\ 6 & 2 \end{bmatrix},$$

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Corollary

For all $n \geq 1$ we have

$$\mathcal{I}_4(n) = \sum_{k=0}^{n} (-1)^k \left[\frac{2n+1}{k} \binom{2n-k}{k-1} \right] 5^{n-k}. \quad (2)$$

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Conjecture

For a given $m \times n$ table A ($2n \geq m$), we have $|A| = -2^{\lfloor \frac{m}{2} \rfloor}$.

Definition

Let $T = T_{m,n}$ be the $m \times n$ table. For positive integers $1 \leq i, t \leq m$ and $1 \leq s \leq n$, the number of all perfect lattice paths from $(1, i)$ to (s, t) in T is denoted by $\mathcal{D}^i(s, t)$, that is, $\mathcal{D}^i(s, t) = l(1, i; s, t : \mathbf{S})$. Also, we put

$$\mathcal{D}_{m,n}(s, t) = \sum_{i=1}^{m} \mathcal{D}^i(s, t).$$

							1
						1	7
					1	6	27
				1	5	20	70
			1	4	14	44	133
		1	3	9	25	69	189
	1	2	5	12	30	76	196
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Table: Values of $\mathcal{D}^1(s, t)$. For example $\mathcal{D}^1(7, 4) = 44$ and $\mathcal{D}^1(4, 2) = 5$

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We may simple use $\mathcal{D}(s,t)$ for $\mathcal{D}_{m,n}(s,t)$. Also;

$$\mathcal{I}_m(n) = \mathcal{D}(n,1) + \mathcal{D}(n,2) + \cdots + \mathcal{D}(n,m).$$

For example:

$$\mathcal{I}_8(8) = 127 + 196 + 189 + 133 + 70 + 27 + 7 + 1 = 750.$$

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Consider $n = m$ (square table), we put $\mathcal{D}_n(s,t) := \mathcal{D}_{n,n}(s,t)$. So, it is easy to see that

$$\mathcal{D}_n(n,n) = \mathcal{D}_n(n-1,n) + \mathcal{D}_n(n-1,n-1).$$

where $\mathcal{D}_1(1,1) = 1, \mathcal{D}_2(2,2) = 2, \mathcal{D}_3(3,3) = 5, \mathcal{D}_4(4,4) = 13, \ldots$.

The values of $\mathcal{D}_n(n,n)$ is OEIS sequence A005773, where T is a square table. By the way, notice how the diagram for $\mathcal{D}_4(4,4) = 13$ is

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In linear algebra, a Hankel matrix (or catalecticant matrix), named after Hermann Hankel, is defined as following:

$$H_n^t = (a_{i+j+t})_{0\leq i,j\leq n-1} = \begin{bmatrix} x_t & x_{t+1} & x_{t+2} & \cdots & x_{t+n-1} \\ x_{t+1} & x_{t+2} & x_{t+3} & \cdots & x_{t+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{t+n-1} & x_{t+n} & x_{t+n+1} & \cdots & x_{t+2n-2} \end{bmatrix}$$

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We have interesting conjecture on Hankel determinant evaluation of $D(n,n)$:

Put $D_n(n,n) = d_n$. According to OEIS sequence A005773, some values of sequence d_n are
$1, 1, 2, 5, 13, 35, 96, 267, 750, 2123, 6046, 17303, 49721, 143365, \cdots$.
Then

$$H_n^1 = \det \begin{bmatrix} d_1 & d_2 & d_3 & \dots & d_n \\ d_2 & d_3 & d_4 & \dots & d_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n & d_{n+1} & d_{n+2} & \dots & d_{2n-1} \end{bmatrix} =?$$

Problem

How can we compute $\det(H_n^t)$*?*

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For example, H_6^1 is

$$\begin{pmatrix} 1 & 2 & 5 & 13 & 35 & 96 \\ 2 & 5 & 13 & 35 & 96 & 267 \\ 5 & 13 & 35 & 96 & 267 & 750 \\ 13 & 35 & 96 & 267 & 750 & 2123 \\ 35 & 96 & 267 & 750 & 2123 & 6046 \\ 96 & 267 & 750 & 2123 & 6046 & 17303 \end{pmatrix}$$

and $\det(H_6^1) = 1$. Actually, the work on the Hankel determinants began with computer experiments of the I.

Theorem (Krattenthaler and Yaqubi)

For all positive integers n

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Theorem (Krattenthaler and Yaqubi)

For all positive integers n and non-negative integers k, we have

$$\det H_n^1 = \begin{cases} (-1)^{n_1\binom{k+1}{2}}(xy)^{(k+1)^2\binom{n_1+1}{2}-n} & n = n_1(k+1), \\ (-1)^{n_1\binom{k+1}{2}}(xy)^{(k+1)^2\binom{n_1+1}{2}} & n = n_1(k+1)+1, \\ 0 & n \not\equiv 0 \pmod{k+1}. \end{cases} \tag{3}$$

For example, H_6^2 is

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Theorem (Krattenthaler and Yaqubi)

For all positive integers n and non-negative integers k, we have

$$\det H_n^2(\mathcal{D}) = \begin{cases} 2, & n \equiv 1 \pmod 6, \\ 1, & n \equiv 2, 0 \pmod 6, \\ -1, & n \equiv 3, 5 \pmod 6 \\ -2, & n \equiv 4 \pmod 6. \end{cases}$$

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Conjecture

For positive integers n, consider the Hankel matrix

$$H_n^0(\mathcal{D}) = \begin{bmatrix} 1 & 1 & 2 & 5 & \dots & d_n \\ 1 & 2 & 5 & 13 & \dots & d_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ d_n & d_{n+1} & d_{n+2} & d_{n+3} & \dots & d_{2n} \end{bmatrix}.$$

Then

$$\det H_n^0(\mathcal{D}) = \begin{cases} 0, & n \equiv 3 \pmod 6, \\ -1, & n \equiv 4, 5 \pmod 6, \\ 1, & n \equiv 2, 3 \pmod 6. \end{cases}$$

Overlays

Recently, I compute the Hankel determinant $H_n^1(d_n)$ by using of Riordan array group. An infinite triangular matrix $D = (d_{n,k})_{n,k\geq 0}$ is called a Riordan array if its columns k has generating function $g(t)f(t)^k$, where $g(t)$ and $f(t)$ are formal power series with $g_0 = 1,\ f_0 = 0$ and $f_1 \neq 0$. The Riordan array is denoted by $D = (g(t), f(t))$.

First, we need to find generating function of d_n. What is the generation function of the sequence d_n?

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Th generating function of $D(n,n)$ is given by

$$F(x) = \frac{1}{2}\sqrt{\frac{1+x}{1-3x}} - \frac{1}{2}.$$

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Theorem

For $n \geq 0$, Riordan array of the perfect lattice paths is

$$D = \left(\frac{1}{2}\sqrt{\frac{1+x}{1-3x}} - \frac{1}{2}, \frac{1-x-\sqrt{1-2x-3x^2}}{2x}\right)$$

$$= \begin{bmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 3 & 1 & & & \\ 13 & 9 & 4 & 1 & & \\ 35 & 26 & 14 & 5 & 1 & \\ & & \dots & & & \ddots \end{bmatrix}$$

where $\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ is the ordinary generating function of Motzkin number M_n.

I computed the Hankel determinant of $H_n^1(d_n)$ with another easy way according to Riordan array:

Consider square matrix

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It easy to see that

$$DD^T = \begin{bmatrix} 1 & 2 & 5 & 13 & \cdots \\ 2 & 5 & 13 & 35 & \cdots \\ 5 & 13 & 35 & 96 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} = 1 \tag{5}$$

Since

$$\det(DD^T) = \det(D)\det(D^T) = 1.$$

Conjecture

We say that a matrix D is totally positive if all its minors are non-negative. The Riordan array matrix of $D(n, n)$ is totally positive.

Michael Somos in OEIS sequence A005773 gives the following recurrence relation for $\mathcal{D}_n(n, n)$.

Theorem

Inside the square $n \times n$ table we have

$$n\mathcal{D}_n(n, n) = 2n\mathcal{D}_n(n-1, n-1) + 3(n-2)\mathcal{D}_n(n-2, n-2).$$

Problem

Find combinatorial bijection for Somos identity.

Theorem

For any positive integer n, we have

$$\mathcal{I}_n(n) = 3\mathcal{I}_{n-1}(n-1) + 3^{n-1} - 2\mathcal{D}_{n-1,n-1}(n-1, n-1).$$

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Utilizing two last theorems for $\mathcal{D}_n(n, n)$, we can prove a conjecture of Alexander R. Povolotsky posed in OEIS sequence A081113 as follows.
This identity has appeared first in E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani, Directed animals, forests and permutations, Discrete Math. 204 (1999), 41–71.

Conjecture

The following identity holds for the numbers $\mathcal{I}_n(n)$.

$$(n+3)\mathcal{I}_{n+4}(n+4) = 27n\mathcal{I}_n(n) + 27\mathcal{I}_{n+1}(n+1) - 9(2n+5)\mathcal{I}_{n+2}(n+2) + (8n+21)\mathcal{I}_{n+3}(n+3).$$

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We went a step further and we cam obtained generating function for $\mathcal{I}_m(n)$!

Lemma

Let $\{a_n\}$ be a sequence of numbers satisfying a linear recurrence relation

$$a_{n+k} = \alpha_1 a_n + \cdots + \alpha_k a_{n+k-1}$$

for all $n \geq 1$. Then the generating function of $\{a_n\}$ is given by

$$f(x) = \frac{\sum_{i=1}^{k-1} \alpha_{i+1} x^{k-i} f_i(x) - f_k(x)}{\sum_{i=1}^{k} \alpha_i x^{k-i+1} - 1},$$

where $f_i(x) = a_1 x + \cdots + a_i x^i$ for all $i \geq 1$.

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Theorem

The generating function of $\mathcal{I}_m(n)$ is given by

$$f_m(x) = \frac{\displaystyle\sum_{i=1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_{m,\lceil \frac{m}{2} \rceil}^{i+1} x^{\lceil \frac{m}{2} \rceil - i} f_m^i(x) - f_m^{\lceil \frac{m}{2} \rceil}(x)}{\sum_{i=1}^{k} \alpha_{m,\lceil \frac{m}{2} \rceil}^{i} x^{\lceil \frac{m}{2} \rceil - i + 1} - 1}$$

for any $m \geq 1$ and $1 \leq k \leq m$, respectively, where

$$f_m^i(x) = \mathcal{I}_m(1)x + \cdots + \mathcal{I}_m(i)x^i$$

Theorem

Also,

$$\alpha^i_{m,n} = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{i+j} \left(\binom{n-j}{j, i-2j} - \binom{n-j-2}{j, i-2j-2} \right)$$

if m is odd and

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Also, we have obtained several results.

Theorem

For any $m \leq 1$; the inner product of columns a and b of the $m \times \infty$ table equals $\mathcal{I}_m(a+b-1)$, that is

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Let T be table with n rows and m columns, where $n \leqslant 2m+3$. The number of all perfect lattice paths in the table T given by

$$\mathcal{I}_n(m) = \sum_{i=1}^{m} \mathcal{D}_{m+2,i} \times \mathcal{D}_{n-m+1,i} \ ;$$

Also, if n be odd number, then

$$\mathcal{I}_n(m) = \sum_{i=1}^{m} (\mathcal{D}_{\frac{n+1}{2},i})^2 \ .$$

Definition

The number of lattice paths from $(1,1)$-cell to (s,t)-cell $(1 \le s \le n$ and $1 \le t \le m)$, using just the two steps $(1,1)$ and $(1,-1)$, is denoted by $\mathcal{A}(s,t)$. In other words, $\mathcal{A}(s,t) = l((1,1) \longrightarrow (s,t) : \mathbf{S}')$, where $\mathbf{S}' = \{(1,1),(1,-1)\}$.

Mention, the number of lattice paths from the $(1,i)$-cell to the (s,t)-cell is denoted by $\mathcal{D}^i(s,t)$ where $\mathbf{S} = \{(1,0),(1,1),(1,-1)\}$. Indeed, $\mathcal{D}^i(s,t) = l((1,i) \rightarrow (s,t); \mathbf{S})$ and

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We label the steps of the set $\mathbf{S} = \{(1,1), (1,0), (1,-1)\}$ by letters $\mathtt{u} = (1,1)$, $\mathtt{r} = (1,0)$, and $\mathtt{d} = (1,-1)$; also if $\mathtt{h}$ is a letter of the word $\mathcal{W}$, order or size of $\mathtt{h}$ in $\mathcal{W}$ is the number of times the letter $\mathtt{h}$ appears in the word $\mathcal{W}$ and it is denoted by $|\mathtt{h}| = |\mathtt{h}|_{\mathcal{W}}$.

Clearly, $\mathcal{I}_m(n)$ is the number of words $a_1 a_2 \dots a_{n-1} a_n$ ($a_i \in \{1, \dots, m\}$) such that $|a_{i+1} - a_i| \leq 1$ for all $i = 1, \dots, n-1$. Following example; shows perfect lattice paths in $T_{2,3}$ and the corresponding words, where the i^{th} letter indicates the rows whose i^{th} point of the paths belongs to.

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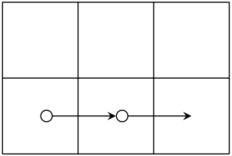

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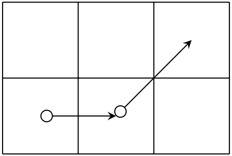

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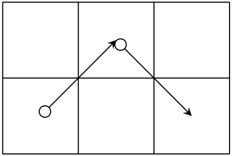

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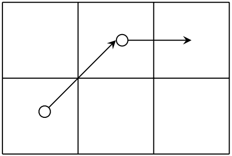

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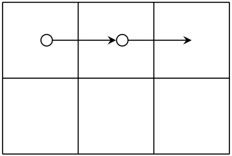

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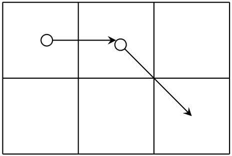

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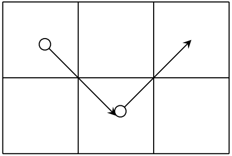

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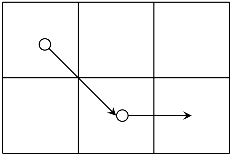

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The following tables Table illustrates the values of $\mathcal{A}(s,t)$ and $\mathcal{D}^1(s,t)$ for $1 \le s, t \le 8$.

							1
						1	7
					1	6	27
				1	5	20	70
			1	4	14	44	133
		1	3	9	25	69	189
	1	2	5	12	30	76	196
1	1	2	4	9	21	51	127

							1
						1	0
					1	0	6
				1	0	5	0
			1	0	4	0	14
		1	0	3	0	9	0
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TABLE: Values of $\mathcal{D}^1(s,t)$ (left), and values of $\mathcal{A}(s,t)$ (right)

For example, right table tells us $\mathcal{A}(6,1) = 5$ and the corresponding five words are

$$\mathrm{uuudd},\ \mathrm{uudud},\ \mathrm{ududu},\ \mathrm{uuddu},\ \mathrm{uduud}.$$

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$$\mathtt{uud},\ \mathtt{urr},\ \mathtt{rru},\ \mathtt{udu},\ \mathtt{rur}.$$

Theorem

For all $1 \le s, t \le m$, we have

$$\mathcal{D}^1(s,t) = \sum_{i=0}^{\lfloor \frac{s-t}{2} \rfloor} \binom{s-1}{s-t-2i} \mathcal{A}(t+2i, t).$$

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Example

From the left Table, we read $\mathcal{D}^1(8,4) = 133$. Using the last theorem , we can compute $\mathcal{D}^1(8,4)$ alternately as

$$\begin{aligned}
\mathcal{D}^1(8,4) &= \sum_{i=0}^{\lfloor \frac{8-4}{2} \rfloor} \binom{8-1}{8-4-2i} \mathcal{A}(2i+4,4) \\
&= \binom{7}{4}\mathcal{A}(4,4) + \binom{7}{2}\mathcal{A}(6,4) + \binom{7}{0}\mathcal{A}(8,4) \\
&= 35 \times 1 + 21 \times 4 + 1 \times 14 = 133.
\end{aligned}$$

Lemma

Inside the $n \times n$ table, we have

$$\mathcal{A}(s,t) = \frac{2t}{s+t}\binom{s-1}{\frac{s-t}{2}}.$$

for all $1 \leq s, t \leq n$.

So, the numbers $\mathcal{A}(s,t)$ are indeed computed as in the ballot problem were the paths can touch the $y = x$ line but never go above it. The number of such ballot paths from $(1,0)$ to (m,n) is $\frac{m-n+1}{m+1}\binom{m+n}{m}$.

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Lemma

Inside the square $n \times n$ table we have

$$\mathcal{D}_n = 3\mathcal{D}_{n-1} - \mathcal{M}_{n-2}.$$

where, $\mathcal{M}_n$ is n-th Motzkin number.

Utilizing the above recurrence relation, we prove the following theorem.

Theorem

Let T be a $n \times n$ table. Then

$$\mathcal{D}_{n-1} = \sum_{i=3}^{n+} \mathcal{M}_{i-3}\mathcal{D}_{n-i+1},$$

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LEMMA

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We call a sequence $\{p_n(x)\}_{n\geq 0}$ of polynomials over $\mathcal{D}$, where $p_n(x)$ is of degree n orthogonal if there exists a linear functional L on polynomials over $\mathcal{D}$ such that

$$L(p_n(x)p_m(x) := \begin{cases} 0 & \text{when } n \neq m \\ nonzero, & \text{when } n = m \end{cases}$$

Theorem

A sequence $p_n(x)$ of monic polynomials, $p_n(x)$ being of degree n, is orthogonal if and only if there exist sequence (b_n) and (λ_n), with $\lambda_n \neq 0$ for all $n \geq 1$, such that the three -term recurrence

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + \lambda_n p_{n-1}(x)$$

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We choose $b_i = 0$ and $\lambda_i = 1$ for all i. Then we have the three-term recurrence

$$xU_n(x) = U_{n+1}(x) + U_{n-1}(x)$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = x$. These polynomials are, up to reparametrization Chebyshev polynomials of the second kind. To see that, recall that the latter are defined by

$$U_n(cos\theta) = \frac{sin((n+1)\theta}{sin\theta}$$

Theorem

For positive integer n we have

$$F(r)=\sum_{n\geq} D^r(n,m)x^n := \begin{cases} \frac{U_r(\frac{1-x}{2x})U_{n-m}(\frac{1-x}{2x})}{xU_{n+1}(\frac{1-x}{2x})} & \text{when } r\leq m \\ \frac{U_m(\frac{1-x}{2x})U_{k-r}(\frac{1-x}{2x})}{xU_{n+1}(\frac{1-x}{2x})} & \text{when } r\geq m \end{cases}$$

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Also, we have another viewpoint to this question using matrix theory! Define the $n \times n$ tridiagonal matrix A as following

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}$$

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"Thanks for your attention"