Extremal inverse eigenvalue problems for matrices with a prescribed graph

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Inverse Eigenvalue Problem

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Remark

The same eigen data may give rise to a completely different IEP if the structure of the desired matrix is changed. In the same way, a slight change in the eigen data may give rise to a completely different IEP even though the structure of the required matrix is kept same.

Matrices of a graph and Graph of a matrix (Hogben [1])

Let G be an undirected simple graph with n vertices $v_1, v_2, \ldots, v_n$ and $A = (a_{ij})$ be an $n \times n$ symmetric matrix which is constructed such that for $i \neq j$, $a_{ij} \neq 0$ if v_iv_j is an edge and $a_{ij} = 0$ if v_iv_j is not an edge, then A is called *a matrix of the graph* G and G is called *the graph of the matrix* A.

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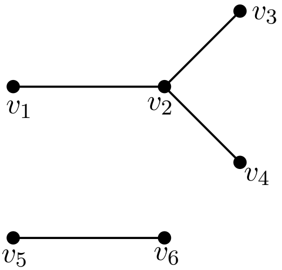
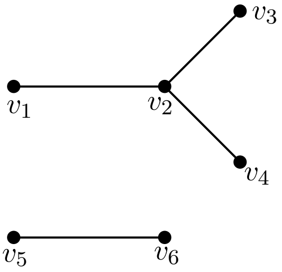
- There is no restriction on the diagonal elements.
- A given graph has infinite number of matrices associated with it but a given matrix has a unique graph.
- The set of all $n \times n$ symmetric matrices whose graph is G is denoted by $S(G)$.
- A matrix whose graph is a tree is called an *acyclic* matrix.

Illustration



Graph on 6 vertices

Illustration



Graph on 6 vertices

v_1

v_2

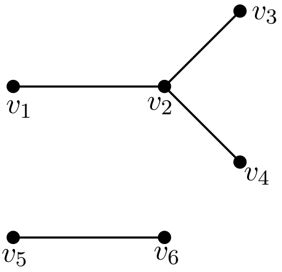
v_3

v_4

v_5

v_6

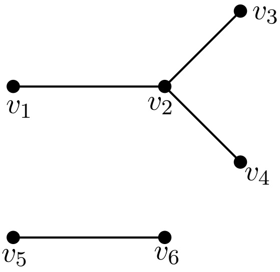
Illustration



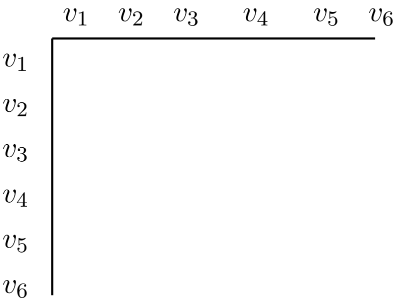
Graph on 6 vertices

	v_1	v_2	v_3	v_4	v_5	v_6
v_1						
v_2						
v_3						
v_4						
v_5						
v_6						

Illustration



Graph on 6 vertices



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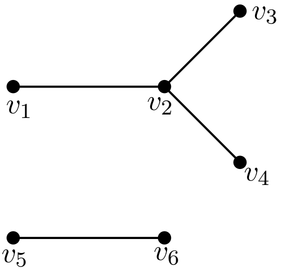
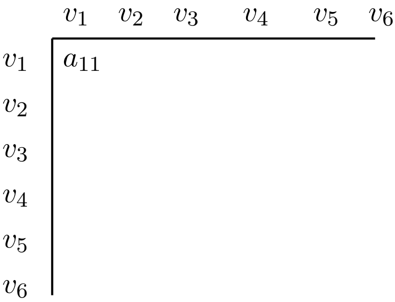
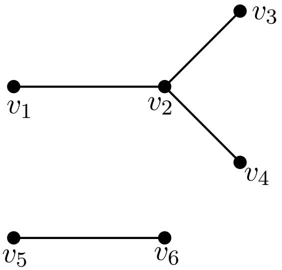


Graph on 6 vertices



A matrix of this graph

Illustration

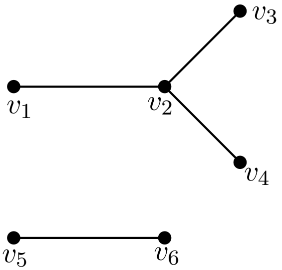


Graph on 6 vertices

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	a_{11}	a_{12}				
v_2						
v_3						
v_4						
v_5						
v_6						

A matrix of this graph

Illustration

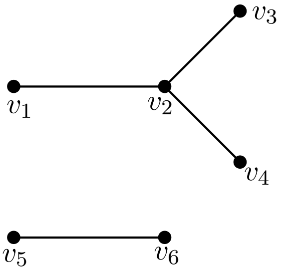


Graph on 6 vertices

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	a_{11}	a_{12}	0			
v_2						
v_3						
v_4						
v_5						
v_6						

A matrix of this graph

Illustration



Graph on 6 vertices

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	a_{11}	a_{12}	0	0	0	0
v_2						
v_3						
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A matrix of this graph

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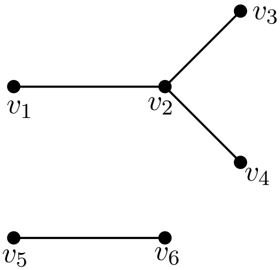
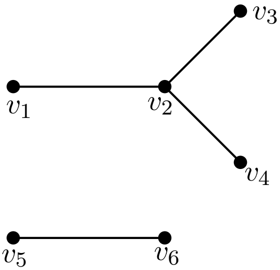


Graph on 6 vertices

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	a_{11}	a_{12}	0	0	0	0
v_2	a_{12}	a_{22}	a_{23}	a_{24}	0	0
v_3						
v_4						
v_5						
v_6						

A matrix of this graph

Illustration



Graph on 6 vertices

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	a_{11}	a_{12}	0	0	0	0
v_2	a_{12}	a_{22}	a_{23}	a_{24}	0	0
v_3	0	a_{23}	a_{33}	0	0	0
v_4	0	a_{24}	0	a_{44}	0	0
v_5	0	0	0	0	a_{55}	a_{56}
v_6	0	0	0	0	a_{56}	a_{66}

A matrix of this graph

Here $a_{12}, a_{23}, a_{24}, a_{56} \neq 0$.

Extremal IEP

Given a graph G on n vertices, $2n-1$ real numbers $\alpha_j, j = 1, 2, \ldots, n$ and $\beta_j, j = 1, 2, 3, \ldots, n$ with $\alpha_1 = \beta_1$, find a matrix $A \in S(G)$ such that for each $j = 1, 2, \ldots, n$, α_j and β_j are respectively the smallest and largest eigenvalues of A_j, the $j \times j$ leading principal submatrix of A.

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This problem was first studied by J. Peng et. al. [2] (2006) and then by H. Pickman et. al. [3, 4] (2007,2009) for the construction of arrow matrices and doubly arrow matrices.

Motivated by this, recently several authors studied the problem of constructing matrices whose graphs are certain types of trees, namely, brooms (D. Sharma and M. Sen [5]), dense centipedes (D. Sharma and M. Sen [6]), generalized stars (M. Heydari *et. al.* [7]), banana trees (M.B. Zarch *et. al.* [8]), double-starlike trees or double comets (M.B. Zarch and S.A.S. Fazeli [9]).

Special trees for which extremal IEP has been studied

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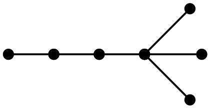
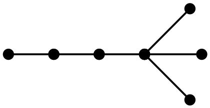


Broom

Special trees for which extremal IEP has been studied



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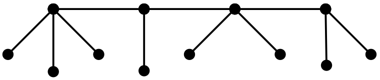

Dense Centipede

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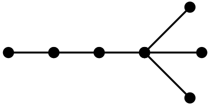
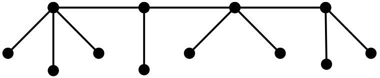
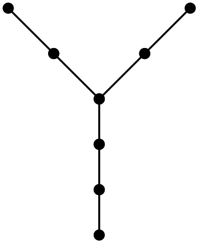


Broom



Dense Centipede



Generalized star

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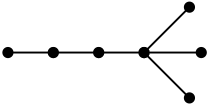
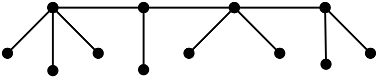


Broom



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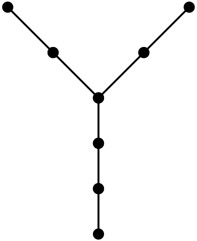
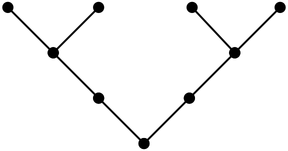
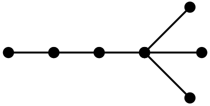


Generalized star

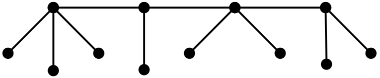


Banana Tree

Special trees for which extremal IEP has been studied



Broom



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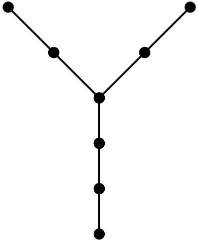
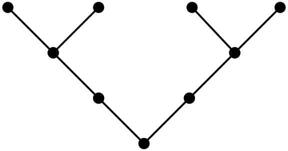
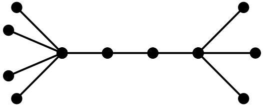


Generalized star



Banana Tree



Double comet

IEPT

Given a tree T on n vertices and $2n-1$ real numbers $\alpha_j, \beta_j,\ 1 \leq j \leq n$, with the convention $\alpha_1 = \beta_1$, find a matrix $A \in S(T)$ such that α_j and β_j are respectively the smallest and the largest eigenvalues of A_j.

Solving the extremal IEP for an arbitrary tree

The solutions obtained for special trees relied upon suitable ways of labelling the vertices of the trees so as to express the characteristic polynomials of the corresponding matrices and their leading principal submatrices in terms of simple recurrence relations.

Scheme of labelling

1. Start with any vertex of the unlabeled tree T_0 and label it as 1.
2. Select a vertex in T_0 adjacent to 1 and label it as 2.
3. In each subsequent step, label a vertex that is adjacent to one of the vertices that have already been labeled, maintaining the serial number of the new labels in the natural order.
4. Continue this process till all the vertices are labeled from 1 to n.

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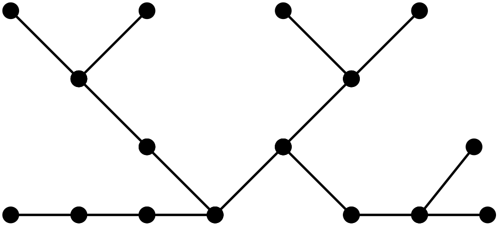
Remark

With this scheme of labelling, the labeled tree T thus obtained has the property that for each j, the subgraph induced by $\{1, 2, \ldots, j\}$ is connected i.e. also a tree.

Thus, for any matrix A in $S(T)$, graph of each $j \times j$ leading principal submatrix A_j of A is also a tree. We call such a matrix as highly acyclic matrix.

Illustration



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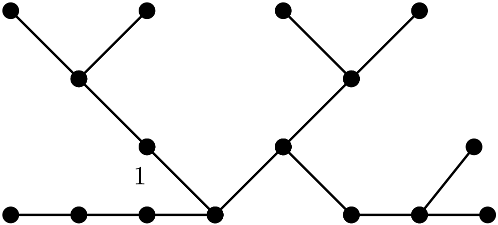

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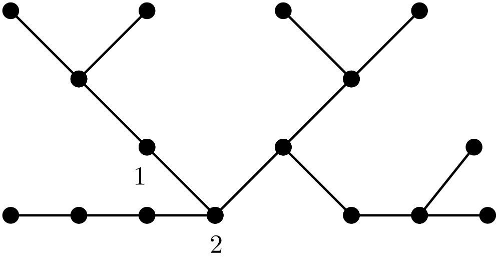

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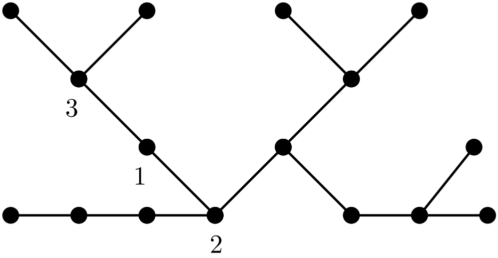

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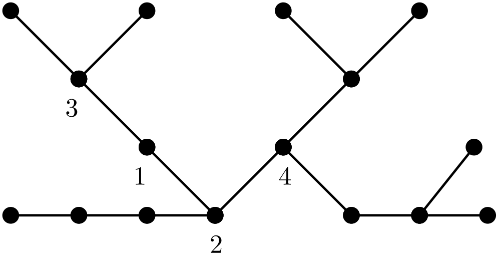

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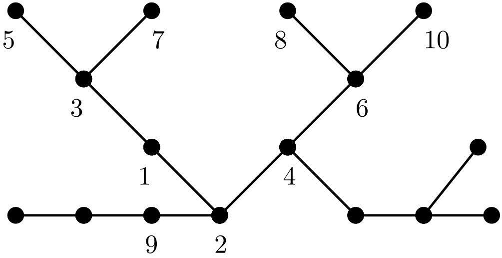

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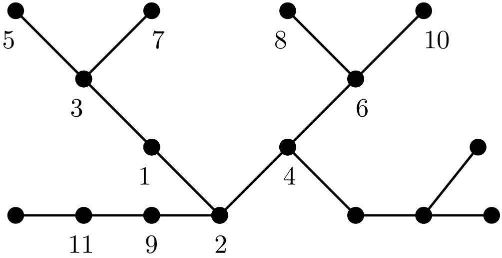

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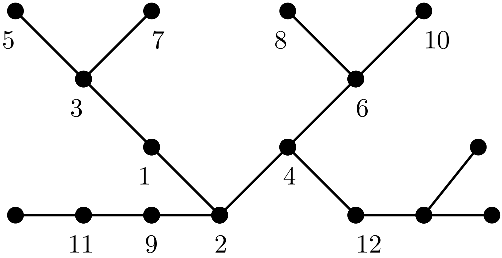

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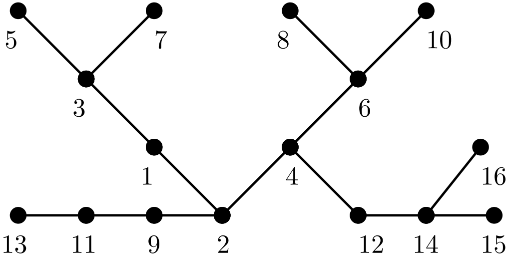

Figure: Scheme of labelling

Lemma 1

For $1 \leq j \leq n$, the vertex j is pendent in the subtree T_j of T induced by $\{1, 2, \ldots, j\}$.

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This allows us to define a function $v : \{2, 3, \ldots, n\} \rightarrow \{1, 2, \ldots, n-1\}$ given by $v(j) =$ the unique vertex among $1, 2, \ldots, j-1$ that is adjacent to j. Thus, for each $j = 1, 2, \ldots, n$, the only non-zero off-diagonal entry in the jth column of the $j \times j$ leading principal submatrix A_j of any matrix A in $S(T)$ is in the $v(j)$th row and the only non-zero off-diagonal entry in the jth row of A_j is in the $v(j)$th column.

Lemma 2

The characteristic polynomials $P_j(x)$ of A_j satisfy the following recurrence relation:

(i) $P_1(x) = x - a_1$;

(ii) $P_j(x) = (x - a_j)P_{j-1}(x) - b^2_{jv(j)}Q_j(x)$, $j = 2, 3, \ldots, n$.

where $Q_j(x)$ denote the characteristic polynomial of the principal submatrix of A_{j-1} obtained by deleting the row and the column indexed by $v(j)$. As a convention, $Q_2(x) = 1$.

Lemma 3

(Cauchy's interlacing theorem) Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of an $n \times n$ real symmetric matrix A, and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$ be the eigenvalues of an $(n-1) \times (n-1)$ principal submatrix B of A, then

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

Lemma 4

Let $P(x)$ be a monic polynomial of degree n with all real zeros and λ_{min} and λ_{max} be the smallest and largest zeros of P, respectively.

(i) If $\mu < \lambda_{min}$, then $(-1)^n P(\mu) > 0$.

(ii) If $\mu > \lambda_{max}$, then $P(\mu) > 0$.

(iii) If $P(\mu) < 0$, then $\mu < \lambda_{max}$.

Theorem 1

Let T be a tree labeled such that the adjacency matrix $A(T)$ is highly acyclic. Then, the IEPT has a solution if and only if

$$\alpha_n < \alpha_{n-1} < \cdots < \alpha_2 < \alpha_1 = \beta_1 < \beta_2 < \cdots < \beta_{n-1} < \beta_n.$$

Further, in that case, there is a unique solution $A \in S(T)$ with positive off-diagonal entries. Any other solution differs from A only by signs of some off-diagonal entries. The solution is given by

$$a_j = \frac{\alpha_j P_{j-1}(\alpha_j) Q_j(\beta_j) - \beta_j P_{j-1}(\beta_j) Q_j(\alpha_j)}{D_j}$$

$$b_{jv(j)}^2 = \frac{(\beta_j - \alpha_j) P_{j-1}(\alpha_j) P_{j-1}(\beta_j)}{D_j}.$$

where

$$D_j = P_{j-1}(\alpha_j) Q_j(\beta_j) - P_{j-1}(\beta_j) Q_j(\alpha_j).$$

The actual entries of the required matrix can be computed with SCILAB (or any other math software) by writing a computer program and feeding the eigen data and plugging in the adjacency matrix as inputs. The results and solutions which appeared in the papers [5–9] are just special cases of our result.

Numerical Example

Consider the generalized star T with 9 vertices shown in the figure below. This tree was considered in [7] with the eigen data $-60, -13, -8.83, -7.43, -2.7, 0.23, 2, 3.6, 4, 5.3, 10, 11.43, 12, 14.5, 15.64, 21, 45$. The authors had labeled it as

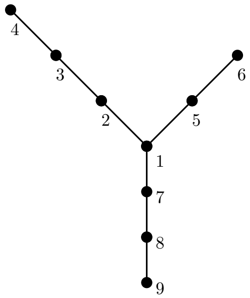

Figure: A labelling of GS(3,2,3)

Solution

The function v takes values $v(2) = v(5) = v(7) = 1, v(3) = 2, v(4) = 3, v(6) = 5, v(8) = 7, v(9) = 8$. By our program we obtain the solution as

$$A = \begin{pmatrix} 4.00000 & 0.72111 & 0 & 0 & 7.24597 & 0 & 7.24283 & 0 & 0 \\ 0.72111 & 4.90000 & 3.71950 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.71950 & 7.24042 & 3.88790 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.88790 & 4.04923 & 0 & 0 & 0 & 0 & 0 \\ 7.24597 & 0 & 0 & 0 & 5.24265 & 7.79528 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.79528 & 0.10873 & 0 & 0 & 0 \\ 7.24283 & 0 & 0 & 0 & 0 & 0 & 0.58485 & 14.33160 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 14.33160 & 8.43147 & 47.35109 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 47.35109 & -25.50266 \end{pmatrix}$$

which is the same as in [7].

A different labelling

We relabel the same tree as

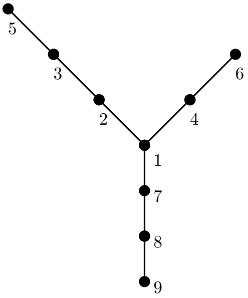

Figure: A labelling of GS(3,2,3)

New solution

The function v takes values $v(2) = v(4) = v(7) = 1, v(3) = 2, v(5) = 3, v(6) = 4, v(8) = 7, v(9) = 8$. By our program we obtain the solution as

$$A = \begin{pmatrix} 4.00000 & 0.72111 & 0 & 5.18175 & 0 & 0 & 8.70534 & 0 & 0 \\ 0.72111 & 4.90000 & 3.71950 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.71950 & 7.24042 & 0 & 5.47276 & 0 & 0 & 0 & 0 \\ 5.18175 & 0 & 0 & 7.73611 & 0 & 8.31937 & 0 & 0 & 0 \\ 0 & 0 & 5.47276 & 0 & 1.00253 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.31937 & 0 & -2.02590 & 0 & 0 & 0 \\ 8.70534 & 0 & 0 & 0 & 0 & 0 & -0.00739 & 13.41408 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 13.41408 & 9.55761 & 47.25667 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 47.25667 & -26.41688 \end{pmatrix}$$

Spectral constraints

The solution obtained satisfies the spectral constraints as seen below by computing the spectra of each leading principal submatrix.

$\sigma(A_9) = \{\mathbf{-\ 60}, -9.20564, -5.96072, -2.69053, 3.85537, 7.67418, 11.94503, 15.36881, \mathbf{45}\}$

$\sigma(A_8) = \{\mathbf{-\ 13}, -7.01439, -2.69380, 3.53984, 4.52978, 11.94470, 14.09725, \mathbf{21}\}$

$\sigma(A_7) = \{\mathbf{-\ 8.83}, -5.54438, -2.68935, 3.86974, 8.45463, 11.94512.\mathbf{15.64}\}$

$\sigma(A_6) = \{\mathbf{-\ 7.43}, -2.69630, 2.43860, 4.09610, 11.94476, \mathbf{14.5}\}$

$\sigma(A_5) = \{\mathbf{-\ 2.7}, 0.29007, 3.94664, 11.34234, \mathbf{12}\}$

$\sigma(A_4) = \{\mathbf{0.23}, 2.27603, 9.94049, \mathbf{11.43}\}$

$\sigma(A_3) = \{\mathbf{2}, 4.14042, \mathbf{10}\}$

$\sigma(A_2) = \{\mathbf{3.6}, \mathbf{5.3}\}$

$\sigma(A_1) = \{\mathbf{4}\}$

Dominant Solutions

We refer the solution A to the IEPT that has positive off-diagonal entries as the *dominant* solution. For each highly acyclic labelling of $\mathcal{T}$, we have a dominant solution of the corresponding IEPT. In this section, we discuss the number of dominant solutions that can be obtained from different highly acyclic labellings of $\mathcal{T}$.

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We refer the solution A to the IEPT that has positive off-diagonal entries as the *dominant* solution. For each highly acyclic labelling of $\mathcal{T}$, we have a dominant solution of the corresponding IEPT. In this section, we discuss the number of dominant solutions that can be obtained from different highly acyclic labellings of $\mathcal{T}$.

For $T \in \mathcal{T}$ let us denote by $\ell(T)$ the number of highly acyclic relabellings of T. The dominant solution A obtained for a highly acyclic relabelling T_0 of T is completely determined by the adjacency matrix of T_0. For any two highly acyclic relabellings of T, the dominant solutions coincide if and only if the corresponding adjacency matrices are identical. So, the number of highly acyclic relabellings of T giving rise to the same dominant solution A is $\left|Aut(T)\right|$, the number of automorphisms of T. Consequently, the number of dominant solutions obtained from different labellings of $\mathcal{T}$ is $\dfrac{\ell(T)}{\left|Aut(T)\right|}$.

We need to determine the number of ways of relabelling the vertices $\{1, \ldots, n\}$ of T such that for each $j = 1, 2, \ldots, n$, the subgraph induced by $\{1, 2, \ldots, j\}$ is a tree. Suppose T' is the resulting tree through such a relabelling. Note that n is pendent in the tree T', $n-1$ is pendent in the tree $T' - n$, and in general, j is pendent in the tree $T' - \{j+1, \ldots, n\}$. This observation facilitates a way for counting the number of highly acyclic relabellings by reducing the problem to counting the same for trees with lesser number of vertices.

Let $P^{(T)}$ be the set of pendent vertices of T. For $v \in P^{(T)}$ the number of highly acyclic relabellings of T with v as the nth vertex is $\ell(T - v)$. We obtain the following recurrence relation for $\ell(T)$.

$$\ell(T) = \sum_{v \in P^{(T)}} \ell(T - v).$$

Path P_n

Consider the path P_n on n vertices. Clearly, $\ell(P_1) = 1$. For $n \geq 2$, P_n has two pendent vertices, and deletion of each of them from P_n results P_{n-1}. Thus, by repeated application of the recurrence relation,

$$\ell(P_n) = 2\ell(P_{n-1}) = \cdots = 2^{n-1}\ell(P_1) = 2^{n-1}.$$

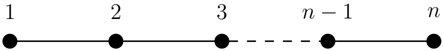

Figure: The path P_n

Since $\left|Aut(P_n)\right| = 2$, the number of dominant solutions obtained from the highly acyclic relabellings of P_n is 2^{n-2}.

Star S_n

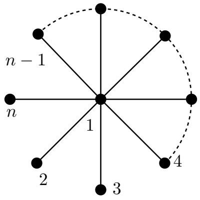

Figure: The star S_n

Let S_n be the star on n vertices, $n \geq 2$. $\ell(S_2) = \ell(P_2) = 2$. There are $n-1$ pendent vertices in S_n and deletion of each gives S_{n-1}. So,

$$\ell(S_n) = (n-1)\,\ell(S_{n-1}) = \cdots = (n-1)(n-2)\cdots 3 \cdot 2 \cdot \ell(S_2) = 2 \cdot (n-1)!.$$

Now, the permutations of the pendent vertices produce all automorphisms of S_n. So, $\left|Aut(S_n)\right| = (n-1)!$. So, the number of dominant solutions obtained from the highly acyclic relabellings of S_n is 2.

The broom (or comet) $B_{n,m}$

The broom $B_{n,m}$ has $n+m$ vertices of which $m+1$ vertices, namely, $1, n+1, n+2, \ldots, n+m$, are pendent vertices. Since $B_{n,1}$ is the path P_{n+1} and $B_{1,m}$ is the star S_{m+1}, we have $\ell(B_{n,1}) = 2^n$ and $\ell(B_{1,m}) = 2 \cdot m!$.

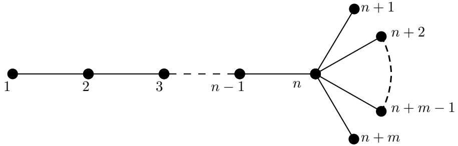

Figure: The broom $B_{n,m}$

For $n, m \geq 2$, the deletion of the pendent vertex 1 from $B_{n,m}$ produces $B_{n-1,m}$, and the deletion of each of the pendent vertices $n+1, \ldots, n+m$ produces $B_{n,m-1}$.We get the recurrence relation

$$\ell(B_{n,m}) = \ell(B_{n-1,m}) + m \cdot \ell(B_{n,m-1}).$$

By mathematical induction, we see that for $n, m \geq 2$ we have

$$\ell(B_{n,m}) = m! \sum_{i_{m-1}=1}^{n} \sum_{i_{m-2}=1}^{i_{m-1}} \cdots \sum_{i_1=1}^{i_2} 2^{i_1}.$$

Now, the permutations of the pendent vertices $n+1, \ldots, n+m$ produce all automorphisms of $B_{n,m}$. So, $\left|Aut(B_{n,m})\right| = m!$. Consequently, the number of dominant solutions obtained from the highly acyclic relabellings of $B_{n,m}$ is given by

$$\sum_{i_{m-1}=1}^{n} \sum_{i_{m-2}=1}^{i_{m-1}} \cdots \sum_{i_1=1}^{i_2} 2^{i_1}.$$

On going and future work

1. We have dealt with the case of extremal IEP for matrices whose graph is unicyclic.
2. The extremal IEP for an arbitrary connected graph is under study.
3. Finding the number of dominant solutions of IEPT for an arbitrary tree can be looked further into.

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Thank you