Combinatorial proofs of two Euler type Identities due to Andrews

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Abstract

Let $a(n)$ be the number of partitions of n such that the set of even parts has exactly one element, $b(n)$ be the difference between the number of parts in all odd partitions of n and the number of parts in all distinct partitions of n, and $c(n)$ be the number of partitions of n in which exactly one part is repeated. Beck conjectured that $a(n) = b(n)$ and Andrews, using generating functions, proved that $a(n) = b(n) = c(n)$.

What would we prove?

We give a combinatorial proof of Andrews' result. Our proof relies on bijections between a set and a multiset, where the partitions in the multiset are decorated with bit strings. We prove combinatorially a part of Beck's second identity,which was also proved by Andrews using generating functions.

So, let's start . . .

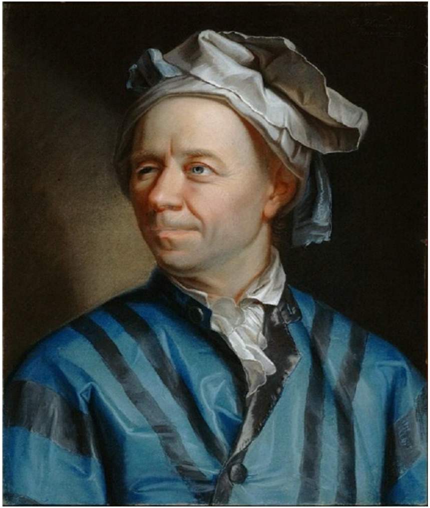
Introduction

1) Given a non negative integer n, a partition λ of n is a non increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \lambda_3 \dots \lambda_k)$ that add up to n. i.e.

$$\sum_{n=1}^{k} \lambda_n = n$$

The numbers λ_i are called the parts of λ and n is called the size of λ.
2) Let $O(n)$ be the set of partitions of n with all parts odd
3) Let $D(n)$ be the set of partitions of n with distinct parts
Then the Euler's Identity states that
$|O(n)| = |D(n)|$

Leonhard Euler



Some important definitions

1) Let $A(n)$ be the set of partitions of n such that the set of even parts has exactly one element.

2) Let $C(n)$ be the partitions of n in which exactly one part is repeated.

3) Let $a(n) = |A(n)|$ and $c(n) = |C(n)|$.

4) Let $b(n)$ is the difference between the number of parts in all partitions of $|O(n)|$ and the number of parts in all partitions in $D(n)$.

Example (On the definitions pertaining to the partitions)

The following are the partitions of 5:

5
4+1
3+2
3+1+1
2+2+1
2+1+1+1
1+1+1+1+1.

Now,

$A(5) = \{4+1, 3+2, 2+2+1, 2+1+1+1\};$
$|A(5)| = a(5) = 4$
$C(5) = \{3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1\};$
$|C(5)| = c(5) = 4$

$O(5) = \{5, 3+1+1, 1+1+1+1+1\}$
$D(5) = \{5, 4+1, 3+2, \}$

Clearly, $|O(5)| = |D(5)| = 3$ and hence one can verify the Euler's theorem for $n = 5$

Total no of parts in $O(5) = 9$ and
Total no of parts in $D(5) = 5$

Therefore, according to our definition, $b(5) = 9 - 5 = 4$

One can observe that, $a(5) = b(5) = c(5) = 4$

So that's what I am going to prove, this beautiful and fascinating result: $a(n) = b(n) = c(n)$ for $n \geq 1$.

Two Theorems

Let $n \geq 1$. We have,

1) $a(n) = b(n)$ (We will prove only this)

2) $c(n) = b(n)$

[The novelty of our approach would be the use of partitions decorated by bit strings.]

b(n) as the cardinality of a multiset of partitions

First we recall Glaisher's bijection. It is the map from the set of partitions with odd parts to the set of partitions with distinct parts which merges equal parts repeatedly. e.g.,for $n = 7$,

$$
\begin{aligned}
(7) &\xrightarrow{\varphi} (7) \\
(5, \underbrace{1, 1}) &\longrightarrow (5, 2) \\
(\underbrace{3, 3}, 1) &\longrightarrow (6, 1) \\
(3, \underbrace{\underbrace{1, 1}, \underbrace{1, 1}}) &\longrightarrow (4, 3) \\
(\underbrace{\underbrace{1, 1}, \underbrace{1, 1}}, \underbrace{1, 1}, 1) &\longrightarrow (4, 2, 1)
\end{aligned}
$$

Thus, each partition $\lambda \in O(n)$ has a part as its image $\varphi(\lambda) \in D(n)$.

When calculating $b(n)$, the difference between the number of parts in all odd partitions of n and the number of parts in all distinct partitions of n, we sum up the differences in the number of parts in each pair. $(\lambda, \varphi(\lambda))$.

Write each part μ_j of $\mu_j = \varphi(\lambda)$ as $\mu_j = 2^{k_j} \cdot m_j$ with m_j odd. Then, μ_j was obtained by merging 2^{k_j} parts in λ and thus contributes an excess of $2^{k_j} - 1$ parts to the difference. Therefore, the difference between the number of parts of λ and the number of parts of $\varphi(\lambda)$ is

$$\sum_{j=1}^{l(\varphi(\lambda))} (2^{k_j} - 1)$$

A definition

Definition: Given a partition μ with parts $\mu_j = 2^{k_j} \cdot m_j$ with m_j odd, the weight of the partition is defined as

$$wt(\mu) = \sum_{j=1}^{l(\mu)} (2^{k_j} - 1)$$

Thus, $wt(\mu) > 0$ if and only if μ contains at least one even part.

The multiset of partitions of a natural number, n

We denote by $MD(n)$ the multiset of partitions of n with distinct parts in which every partition $\mu \in D(n)$ appears exactly $wt(\mu)$ times. This discussion proves a proposition; an interpretation of b(n). Examples:

We find $MD(7)$. Now,

$$wt(4,2,1) = \sum_{j=1}^{l(4,2,1)} (2^{k_j} - 1) = \sum_{j=1}^{3} (2^{k_j} - 1) = 3 + 1 + 0 = 4$$

$$wt(4,3) = \sum_{j=1}^{l(4,3)} (2^{k_j} - 1) = \sum_{j=1}^{2} (2^{k_j} - 1) = 3 + 0 = 3$$

$$wt(6,1) = \sum_{j=1}^{l(6,1)} (2^{k_j} - 1) = \sum_{j=1}^{2} (2^{k_j} - 1) = 1 + 0 = 1$$

... continued

$$wt(5,2) = \sum_{j=1}^{l(5,2)} (2^{k_j} - 1) = \sum_{j=1}^{2} (2^{k_j} - 1) = 0 + 1 = 1$$

$$wt(7) = \sum_{j=1}^{l(7)} (2^{k_j} - 1) = \sum_{j=1}^{1} (2^{k_j} - 1) = 0$$

Therefore, $MD(7)=$

$\{(4,2,1),(4,2,1),(4,2,1),(4,2,1),(4,3),(4,3),(4,3),(6,1),(5,2)\}$

$|MD(7)| = 9$

Lets see whats b(7)?!

There are 15 partitions of 7:
7
6+1
5+2
5+1+1
4+3
4+2+1
4+1+1+1
3+3+1
3+2+2
3+2+1+1
3+1+1+1+1
2+2+2+1
2+2+1+1+1
2+1+1+1+1+1
1+1+1+1+1+1+1
From the above, you can find, that $b(7) = 9$

A Proposition

Let $n \geq 1$. Then, $b(n) = |MD(n)|$.

Bit strings

Bit String : A bit string ω is a sequence of letters from the set, $\{0, 1\}$. The length of a bit string ω, denoted l(ω), is the number of letters in ω. We refer to position i in ω as the i^{th} entry from the right, where the most right entry is counted as position 0.

For example the number 1407 can be partitioned as

$1407 = 768 + 384 + 105 + 96 + 25 + 12 + 9 + 6 + 2$. In some intermediate stage we would require to 'decorate' the particular part "384" such as:

$$\lambda = (768, 384_{0110}, 105, 96, 25, 12, 9, 6, 2).$$

Note that leading zeros are allowed and are recorded. Thus 010 and 10 are different bit strings even though they are the binary representation of the same number. We have $l(010) = 3$ and $l(10) = 2$. The empty bit string has length 0 and is denoted by ϕ.

Proof

Definition 1: A decorated partition is a partition μ with at least one even part in which one single even part, called the decorated part, has a bit string ω as an index. If the decorated part is $\mu_i = 2^k.m$, where $k \geq 1$ and m is odd, the index ω has length $0 \geq l(\omega) \geq k - 1$.

Definition 2: We denote by $DD(n)$ the set of decorated partitions of n with distinct parts.

Since a bit string contains units $\in \{0,1\}$, there are 2^t bit strings possible for a string of length t. Since the index ω has a length ranging from 0 to k-1, there are $\sum_{n=0}^{k-1} 2^n = (2^k - 1)$ distinct bit strings ω of length $0 \geq l(\omega) \geq k-1$. Thus, for each even part $\mu_i = 2^k.m$ of μ there are $2^k - 1$ possible indices and for each partition μ there are precisely wt(μ) possible decorated partitions with the same parts as μ. Then $|MD(n)| = |DD(n)|$ and therefore $b(n) = |DD(n)|$. Hence, $b(n) = |MD(n)|$; proved.

A combinatorial proof for $a(n) = b(n)$

We prove $a(n) = b(n)$ by establishing a one to one correspondence between $A(n)$ and $DD(n)$

$DD(n) \longrightarrow A(n)$

We describe a beautiful algorithm below:

• Let d_ω be the decimal representation of ω. Split part μ_i into $d_\omega + 1$ parts of size $2^{k-l(\omega)}m$ and parts of size m. Thus there will be $2^k - (d_w + 1)2^{k-l(\omega)}$ parts of size m. Since $d_\omega + 1 \leq 2^{l(\omega)}$, the resulting number of parts equal to m is not negative. Moreover, after the split there is at least one even part.

• Every part of size $2^t m$, with $t > k$ (if it exists), splits completely into parts of size $2^{k-l(\omega)}m$, i.e., into $2^{t-k+l(\omega)}$ parts of size $2^{k-l(\omega)}m$.

• Every other even part splits into odd parts of equal size. i.e., every part $2^u v$ with v odd, such that $2^u v \neq 2^s m$ for some $s \geq k$, splits into 2^u parts of size v.

The resulting partition, λ is in $A(n)$. Its set of even parts is $2^{k-l(\omega)}m$.

Example

Consider the decorated partition

$$\mu = \{96, 35, 34, 24_{01}, 6, 2\}$$
$$= \{2^5 \cdot 3, 35, 2 \cdot 17, (2^3 \cdot 3)_{01}, 2 \cdot 3, 2\} \in DD(197)$$

We have: $k = 3, m = 3, l(\omega) = 2, d_w = 1$.

• Part $24 = 2^3 \cdot 3$ splits into $d_\omega + 1 = 1 + 1 = 2$ parts of size $2^{k-l(\omega)} \cdot m = 2^{3-2} \cdot 3 = 2 \cdot 3 = 6$ and $2^k - ((d_w + 1)2^{k-l(\omega)})$ $= 2^3 - 2 \cdot 2 = 4$ parts of size 3.

• Part $96 = 2^5 \cdot 3$. So $t = 5 > k = 3$. Thus it splits into $2^{t-k+l(\omega)} = 2^{5-3+2} = 16$ parts of size $2^{k-l(\omega)} \cdot m = 2^{3-2} \cdot 3 = 6$

• All other even parts $(2^u v)$; $u < k$ split into odd parts,i.e. 2^u parts of size v.Thus 34 splits into 2 parts of size 17, part 6 splits into two parts of size 3, and part 2 splits into two parts of size 1.

The resulting partition is $\lambda = (35, 17^2, 6^{18}, 3^6, 1^2) \in A(197)$.

. . . continued

Hence the transformation maps the decorated partition

$\{96, 35, 34, 24_{01}, 6, 2\} \in DD(197)$

$\longrightarrow \{35, 17^2, 6^{18}, 3^6, 1^2\} \in A(197)$

$A(n) \longrightarrow DD(n)$

Start with partition $\lambda \in A(n)$. Then there is one and only one even number $2^k m, k \geq 1$,m odd, among the parts ofλ. Let f be the multiplicity of the even part in λ. As in Glaisher's bijection, we merge equal parts repeatedly until we obtain a partition μ with distinct parts. Since λ has an even part, μ will also have an even part.

Next, we determine the decoration of μ. Consider the parts μ_{j_i} of the form $2^{r_i} m$, with m odd and $r_i \geq k$. We have $j_1 < j_2 < \dots$. For notational convenience, set $\mu_{j_0} = 0$. Let h be the positive integer such that :

$$\sum_{i=0}^{h-1} \mu_{j_i} < f.2^k m < \sum_{i=0}^{h} \mu_{j_i} \qquad (1)$$

Then we will decorate the part $\mu_{j_h} = 2^{r_h} m$

To determine the decoration, let N_h be the number of parts $2^k m$ in λ that merged to form all parts of the form $2^r m > \mu_{j_h}$. Then

$$N_h = \frac{\sum_{i=0}^{h-1} \mu_{j_i}}{2^k m}$$

Then (1) becomes

$$2^k m N_h < f.2^k m \leq 2^k m N_h + 2^{r_h} m$$

$$\implies \mathrm{N}_h < f \leq N_h + 2^{r_h - k}$$

$$\implies 0 < \mathrm{f} - \mathrm{N}_h \leq 2^{r_h - k}$$

Let $d = f - N_h - 1$ & $l = r_h - k$. We have $0 < l \leq r_h - 1$. Consider the binary representation of d and insert leading 0's to form a bit string ω of length l. Decorate μ_{j_h} with ω.The resulting decorated partition is in DD(N).

Lets see an example

Consider the partition $\lambda = (35,\ 17^2, 6^{18}, 3^6, 1^2)$.
We have $k = 1, m = 3, f = 18$. Then we use the Glaishers bijection that produces the partition $\mu = (96,\ 35,\ 34,\ 24,\ 6,\ 2) \in MD(197)$.
The parts of the form $2^{r_i}.3$ with $r_i \geq 1$ are (96, 24, 6). Since $96 < 18 \times 6 \leq 96 + 24$ the decorated part will be $24 = 2^3 3$.
We have $N_h = 96/6 = 16$

To determine the decoration, let $d = 18 - 16 - 1 = 1$ & $l = 3 - 1 = 2$. The binary representation of d is 1 but to form a bit string of length 2, we introduce a leading 0. Thus the decoration is $\omega = 01$ and the resulting decorated partition is $(96,\ 35,\ 34,\ 24_{01}, 6, 2)$.

...continued

Similarly if we would have taken the partition $\lambda_1 = (35, 17^2, 6^{19}, 3^4, 1^2)$ then all the parameters would be the same as above with the exception that f=19 and so consequently, l=2 & d=2, which in binary representation is 10. The decorated part remains the same: 24_{10} and so the decorated partition in this case would be (96, 35, 34, $24_{10}, 6, 2$)

Summary

1) We aimed to prove that $a(n) = b(n)$

2) To prove that, We showed $b(n) = |MD(n)| = |DD(n)|$

3) Then we aimed to prove $|A(n)| = a(n) = |DD(n)|$

4) To prove 3), we tried to make a injective map that showed:

i) that for for any element in A(n) there is a image in DD(n) &

ii) For any element in DD(n) there is an injective preimage in A(n)

Thank You!