Translation surfaces with poles and meromorphic differentials

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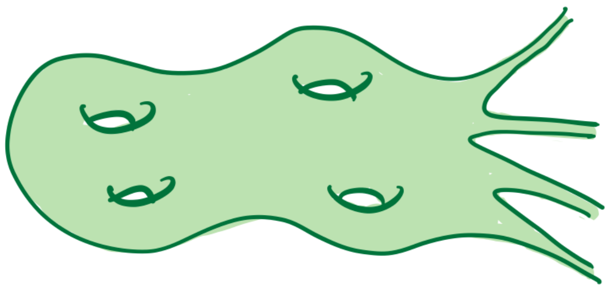
Overview

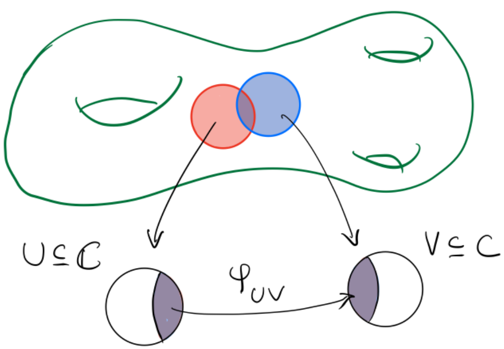
Definition

Let $S_{g,n}$ denotes a connected and oriented surface of genus $g \geq 0$ and $n \geq 0$ punctures.

Riemann surface

A Riemann surface X is the datum of a $1-$dimensional complex structure on a topological surface $S_{g,n}$, that is a maximal atlas $\mathcal{C}$ of charts taking values on $\mathbb{C}$ and the transition functions are biholomorphisms on their domains of definitions.





Two Riemann surfaces X and Y are equivalent if there exists a biholomorphism $f : X \to Y$.

Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of genus g and n punctures, *i.e.* the set of all biholomorphic equivalence classes $[X]$ of Riemann surfaces on $S_{g,n}$

Remark: From now on, with a little abuse of notation, X shall denotes an equivalent classes of Riemann surfaces.

Abelian differentials

Let X be a Riemann surface, that is a complex structure on $S_{g,n}$.

By abelian differential on a Riemann surface X we mean a *holomorphic* $1-$form ω with at most finite-order poles at the punctures, that we refer to as *meromorphic* differentials on $\overline{X}$.

$$\Omega(X) = \{\text{abelian differentials on } X\}$$

$$\Omega\mathcal{M}_{g,n} = \{(X,\omega) \mid X \in \mathcal{M}_{g,n} \text{ and } \omega \in \Omega(X)\}$$

Zeroes and poles of an abelian differential

An abelian differential ω in any local chart $z : U \to \mathbb{C}$ is of the form $\omega = f(z)dz$ where $f(z)$ is a meromorphic function.

Let z be a local parameter at P, $z(P) = 0$ and $\omega = \sum_{i=k_P}^{\infty} a_i z^i \, dz$.

P is *a zero* of ω if $k_P > 0$ and k_P is called the order of the zero.
P is *a pole* of ω if $k_P < 0$ and k_P is called the order of the pole.

Theorem

$$\sum_{P \text{ zeroes of } \omega} k_P \quad - \sum_{P \text{ poles of } \omega} k_P = 2g - 2.$$

Periods of abelian differentials

Let $(X, \omega) \in \Omega\mathcal{M}_{g,n}$. The period character of ω is the representation $\chi : H_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ defined as in

$$\gamma \longmapsto \int_\gamma \omega \in \mathbb{C}. \tag{1}$$

$$\textbf{Per} : \Omega\mathcal{M}_{g,n} \longrightarrow \operatorname{Hom}\Big(H_1\big(S_{g,n}, \mathbb{Z}\big),\ \mathbb{C}\Big)$$

maps (X, ω) to χ defined as (1).

Question: What about the image of the mapping **Per**? Stay tuned!

Translation surfaces: complex-analytic theory

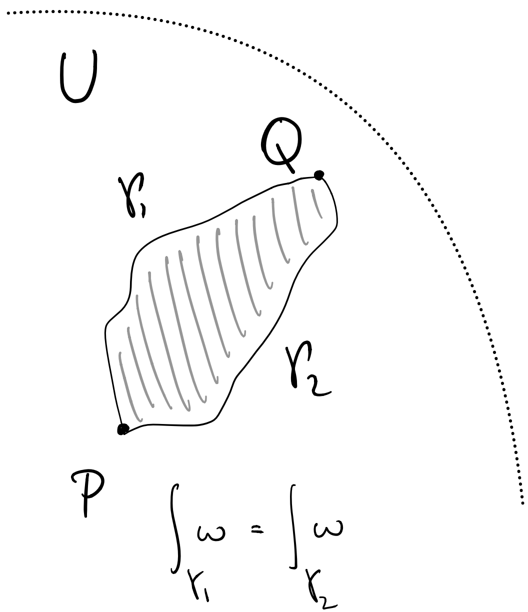
Complex-analytic definition

A translation surface with poles is the datum of a Riemann surface X and an abelian differential $\omega \in \Omega(X)$ with poles at the punctures of X.

Whenever $n = 0$, that means there are no punctures, we simply refer to a couple (X, ω) as Translation surface.

This is an abstract analytical definition and the name translation surface seems very well unmotivated. What do translations have to do with it?

Recall that a Riemann surface is the datum of a complex atlas $\mathcal{C}$.



Let $P \in (X, \omega)$ be any point and let U be a simply connected open neighborhood of P. Assume ω has not zero at P.

$$z(Q) = \int_P^Q \omega \quad \forall Q \in U.$$

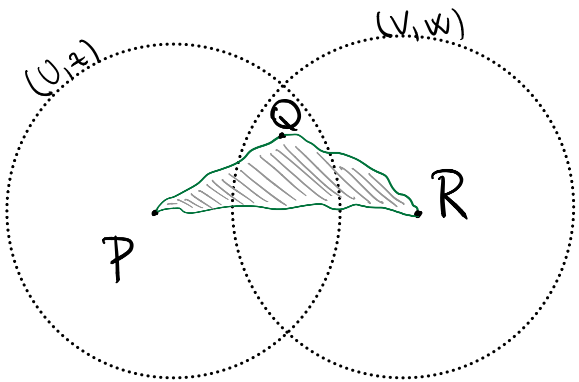
The couple (U, z) is a coordinate chart in $\mathcal{C}$.

Towards a gometric definition

Let Σ be zero locus of ω and define $\mathcal{T}' \subset \mathcal{C}$ to be the maximal collection of coordinate charts obtained by integrating ω on simply connected neighborhood of every point $P \in X \setminus \Sigma$.

Proposition

$X \setminus \Sigma$ admits a maximal atlas of charts to $\mathbb{C}$ whose transition maps are translations.



$\mathcal{T}' \subset \mathcal{C}$ is not maximal as complex atlas!!

$$w = \int_R^Q \omega = \int_P^Q \omega + \int_R^P \omega = z + c$$

Let P be a zero for ω and let U be an open simply connected neighborhood of P. Let k be the order of the zero. Then

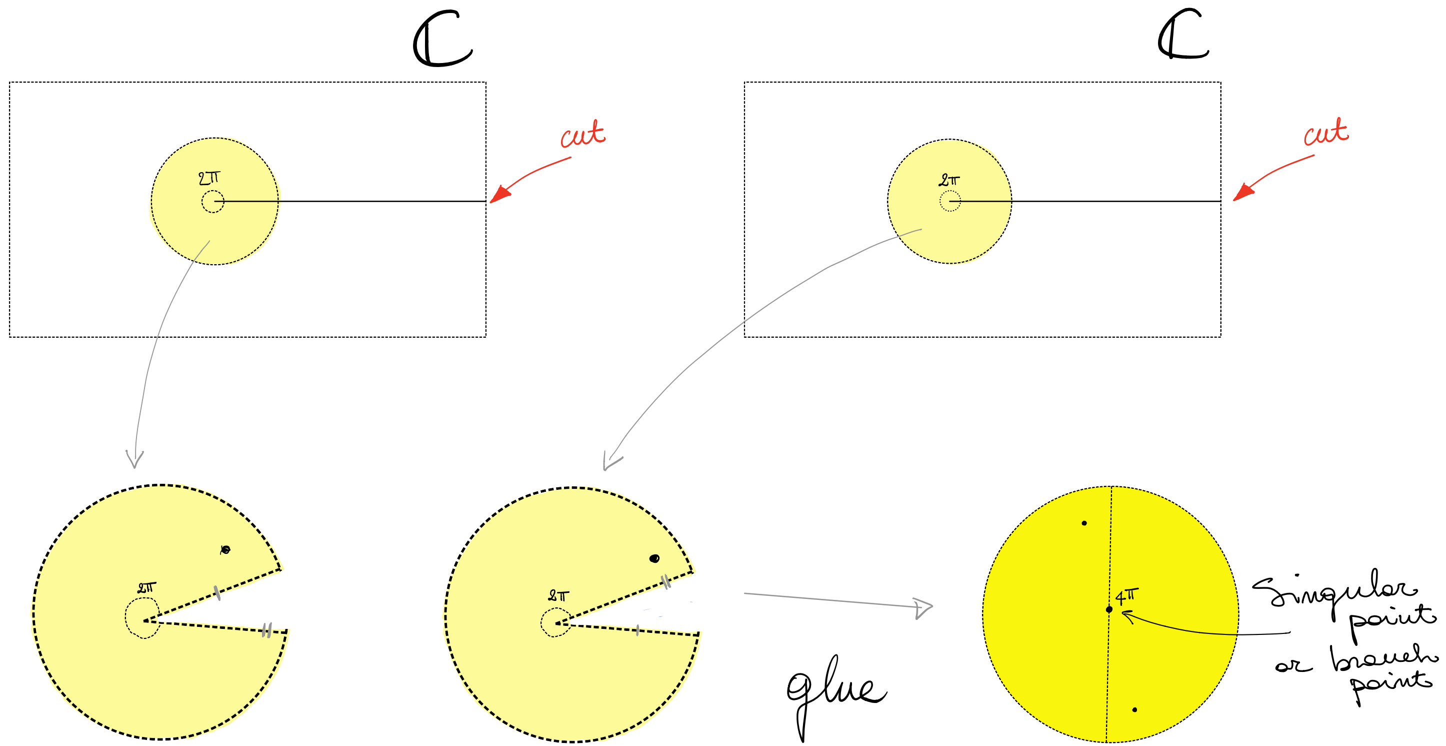
$$z(Q) = \int_P^Q \omega$$

yields a branched covering mapping $z \mapsto z^{k+1}$. The couple (U, z) is defined as *branched chart*. (A geometric interpretation in the next slide).

By adding all the branched charts, the atlas $\mathcal{T}'$ extends to a maximal atlas $\mathcal{T}$ of (possibly branched) charts on $\mathbb{C}$ such that transition functions are translations.

Geometric interpretation!

$k=1$



Two "PacMan"

The name translation surface is now motivated!!

Theorem

A translation surface with poles (X, ω) is equivalent to the datum of a maximal atlas $\mathcal{T}$ of (possibly branched) charts taking values on $\mathbb{C}$ and such that transition functions are translations.

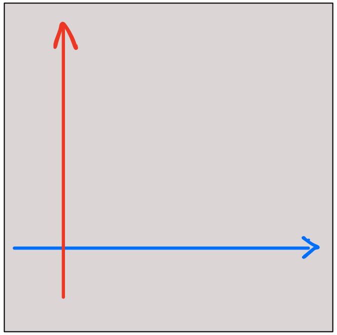
Idea of the proof. Let $(X, \omega) \in \Omega\mathcal{M}_{g,n}$ and let $\mathcal{C}$ be a maximal atlas for X. The abelian differential $\omega \in \Omega(X)$ defines $\mathcal{T}$ of branched charts. Vice versa, given $\mathcal{T}$, we extrapolate $\mathcal{T}'$ which extends to a complex atlas $\mathcal{C}$ on X. Then the local charts can be used to defined an abelian differential ω on X.

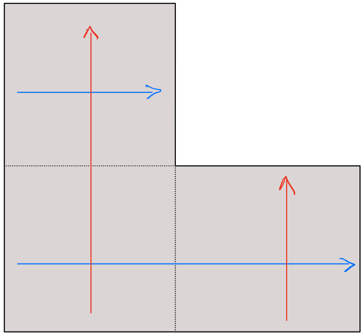
This interpretation is more geometric, but not fully satisfactory. We can do better!

Translation surfaces with poles

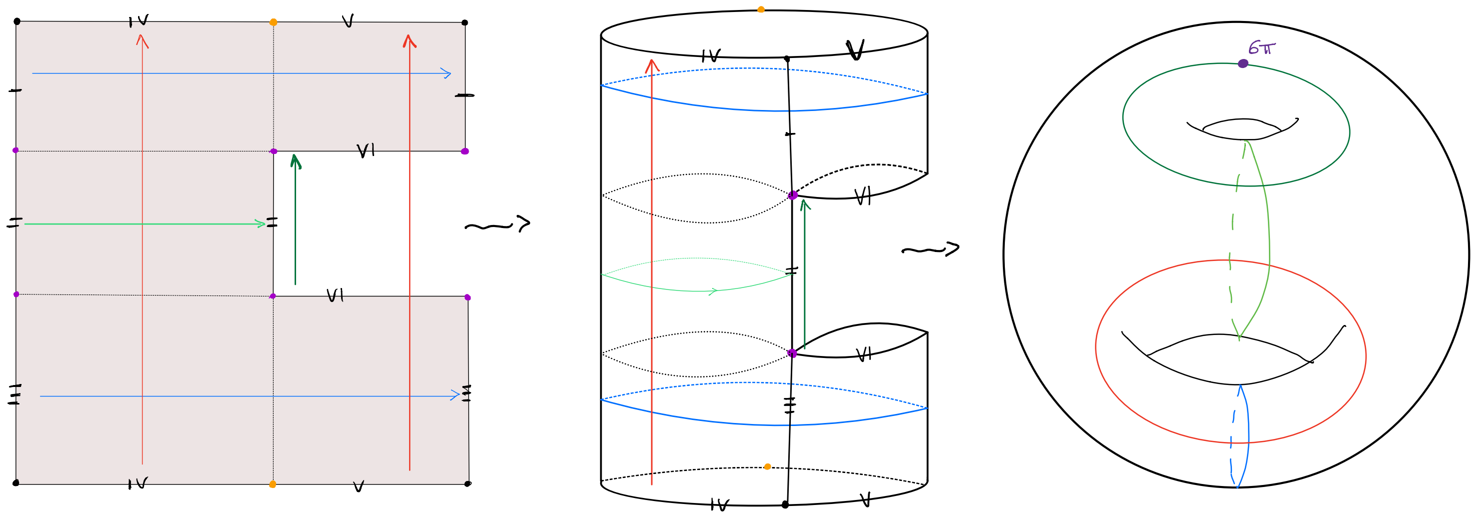
Geometric definition

A translation surface with poles is a surface obtained by gluing sides of (possibly disconnected and possibly non compact) polygon(s) on the complex plane by using translations of $\mathbb{C}$.

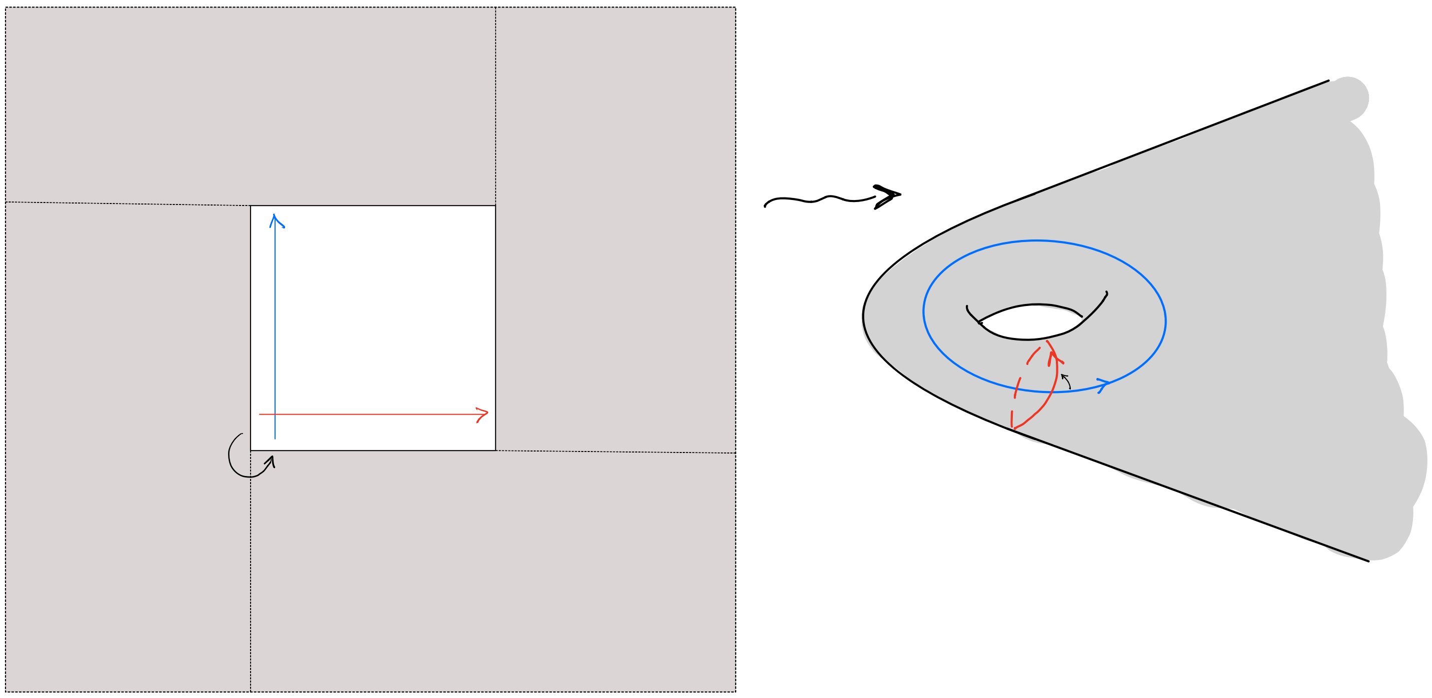




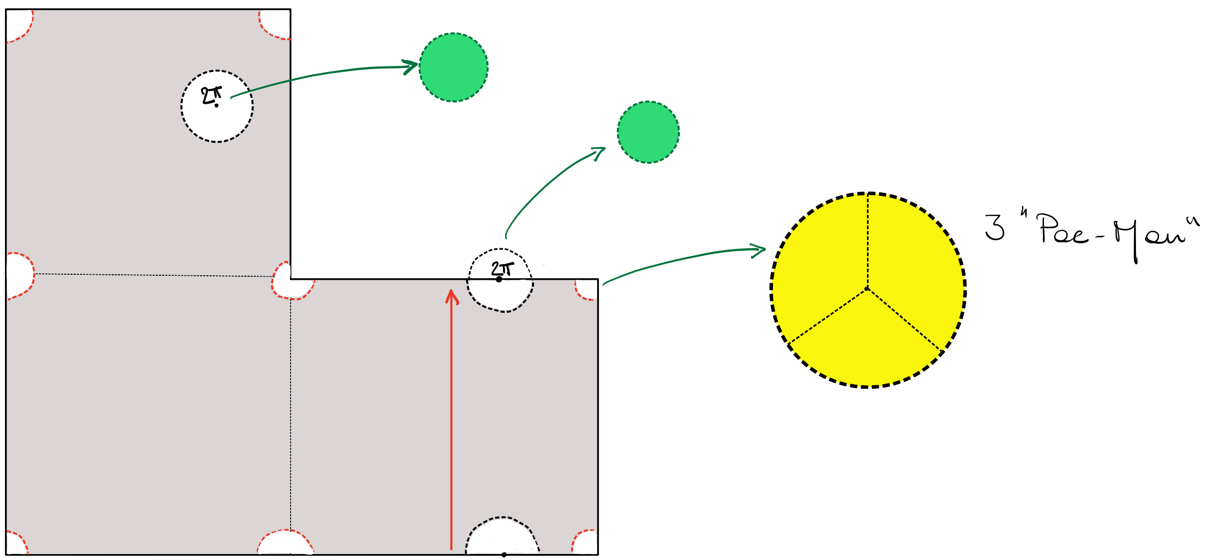
Translation surfaces by gluing polygons



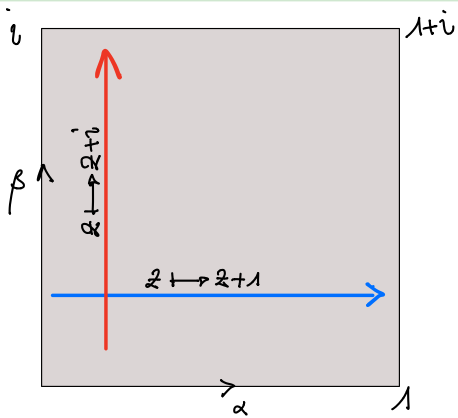
Translation surfaces by gluing polygons



Any translation surface (possibly with poles) yields a maximal atlas $\mathcal{T}$ of charts taking values on $\mathbb{C}$ and such that transition functions are translations. In turns, it yields a couple (X, ω).



The mapping associating to any curve the corresponding vector yields a representation $H_1(S_{g,n},\mathbb{Z}) \longrightarrow \mathbb{C}$ called holonomy.



$$\left.\begin{aligned} \chi(\alpha) &= 1 \\ \chi(\beta) &= i \end{aligned}\right\} \begin{array}{l} \text{define a representation} \\ \chi : H_1(S_1,\mathbb{Z}) \longrightarrow \mathbb{C} \end{array}$$

The holonomy representation defined above turns out the period character of the couple (X,ω) induced by the maximal atlas $\mathcal{T}$.

Recall that our goal is to determine the image of the mapping

$$\mathbf{Per} : \Omega\mathcal{M}_{g,n} \longrightarrow \mathrm{Hom}\Big(H_1(S_{g,n}, \mathbb{Z}),\, \mathbb{C}\Big)$$

Idea: Given a representation $\chi : H_1(S_{g,n}, \mathbb{Z}) \to \mathbb{C}$ we realize it as the holonomy of some translation surface (possibly with poles) obtained by gluing polygons. By construction, this appears as the period of some couple (X, ω).

In general, a representation $\chi : H_1(S_{g,n}, \mathbb{Z}) \to \mathbb{C}$ might be not realizable as the holonomy of some translation surface.

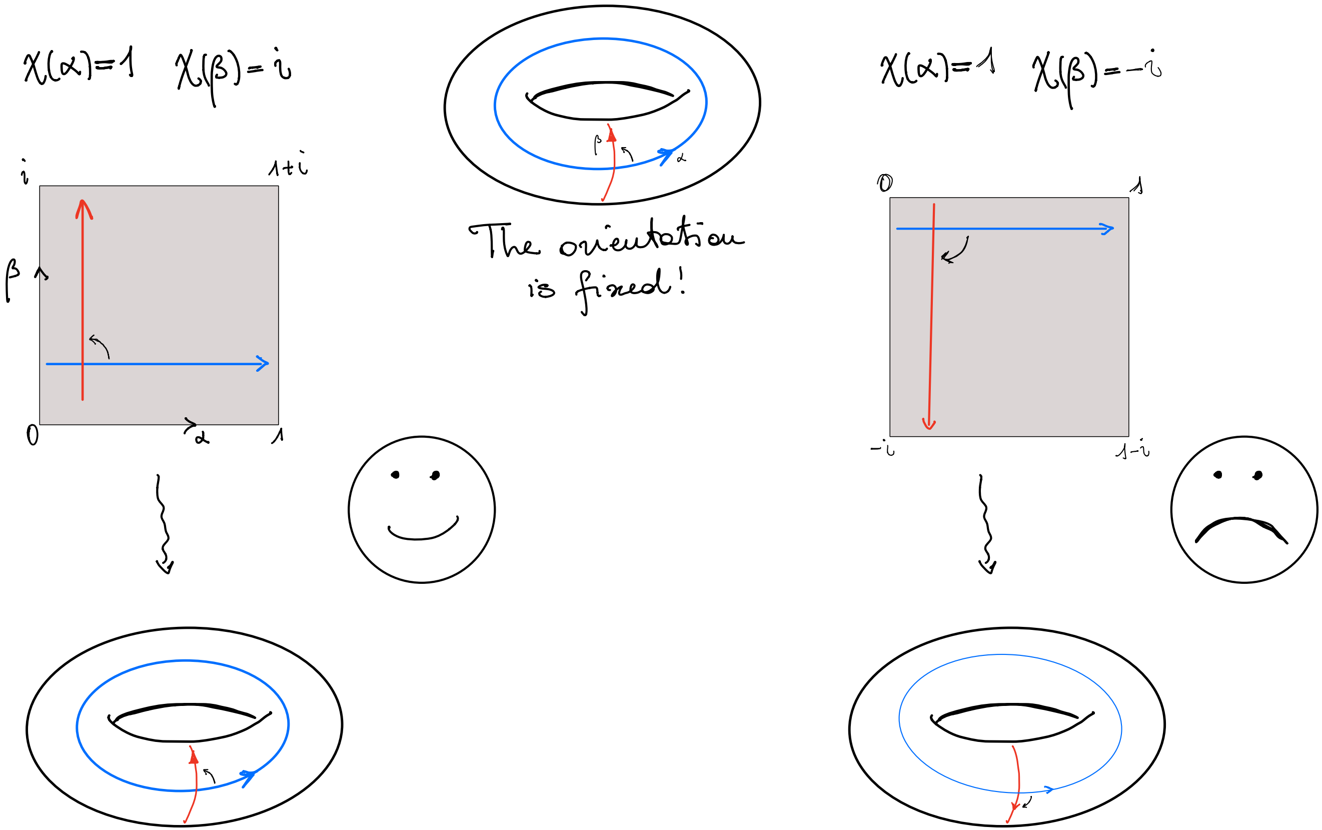
Haupt's Theorem

Given a representation $\chi : H_1(S_g, \mathbb{Z}) \longrightarrow \mathbb{C}$ there are two obstructions for χ to appear as the period of some abelian differential $\omega \in \Omega(X)$ for some complex structure X on S_g.

1. $\mathrm{vol}(\chi) = \sum_{i=1}^{g} \Im\left(\overline{\chi(\alpha_i)}\chi(\beta_i)\right) > 0$, with respect to some standard symplectic basis $\{\alpha_i, \beta_i\}_{1 \le i \le g}$,
2. If $\chi(\Gamma) = \Lambda$ is a lattice in $\mathbb{C}$, then $\mathrm{vol}(\chi) \geq 2\mathrm{Area}(\mathbb{C}/\Lambda)$

These conditions are necessary and sufficient!

A non-realizable representation



Surfaces with punctures

Theorem (Chenakkod-F.-Gupta, 2020)

Let $n \geq 1$. Every representation $\chi : H_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ is the period of some couple (X, ω). Equivalently, the period mapping

$$\mathbf{Per} : \Omega\mathcal{M}_{g,n} \longrightarrow \mathrm{Hom}\Big(H_1\big(S_{g,n}, \mathbb{Z}\big), \mathbb{C}\Big)$$

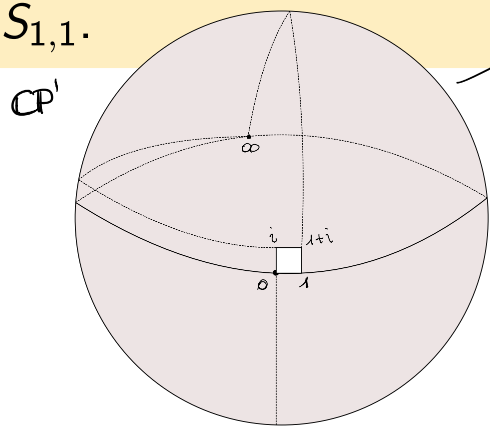
is surjective.

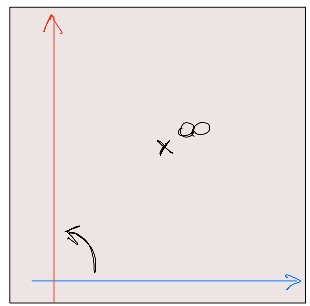
A special case

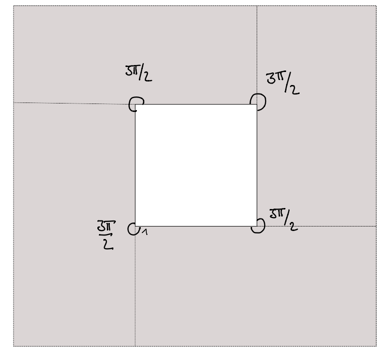
Any representation

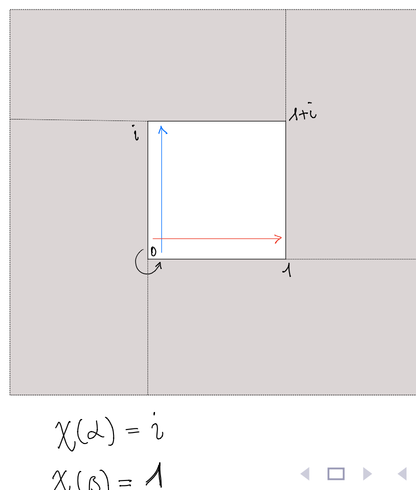
$$\chi : \mathbb{Z}^2 \cong H_1(S_{1,1}, \mathbb{Z}) \to \mathbb{C}$$

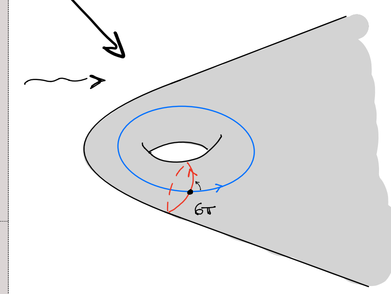
appears as the period character of an abelian differential of a couple (X, ω) on $S_{1,1}$.









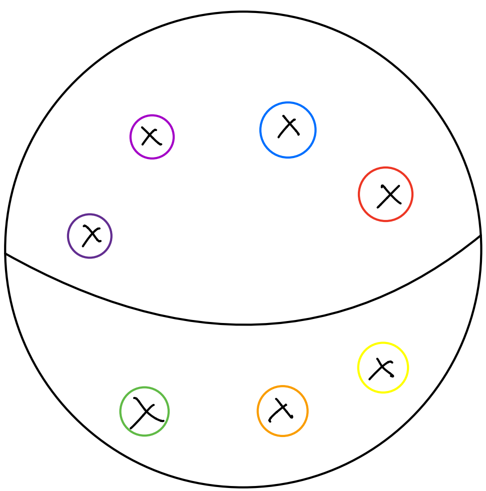


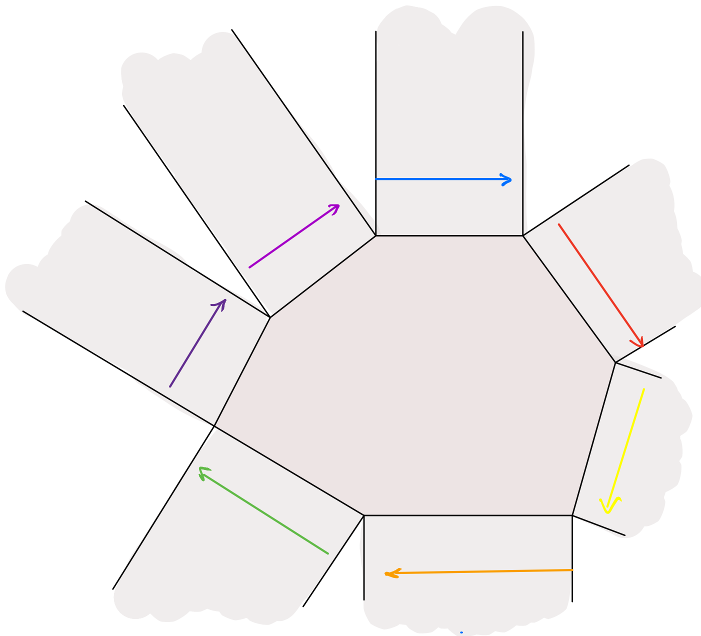
Another special case

Any character

$$\chi : H_1(S_{o,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$$

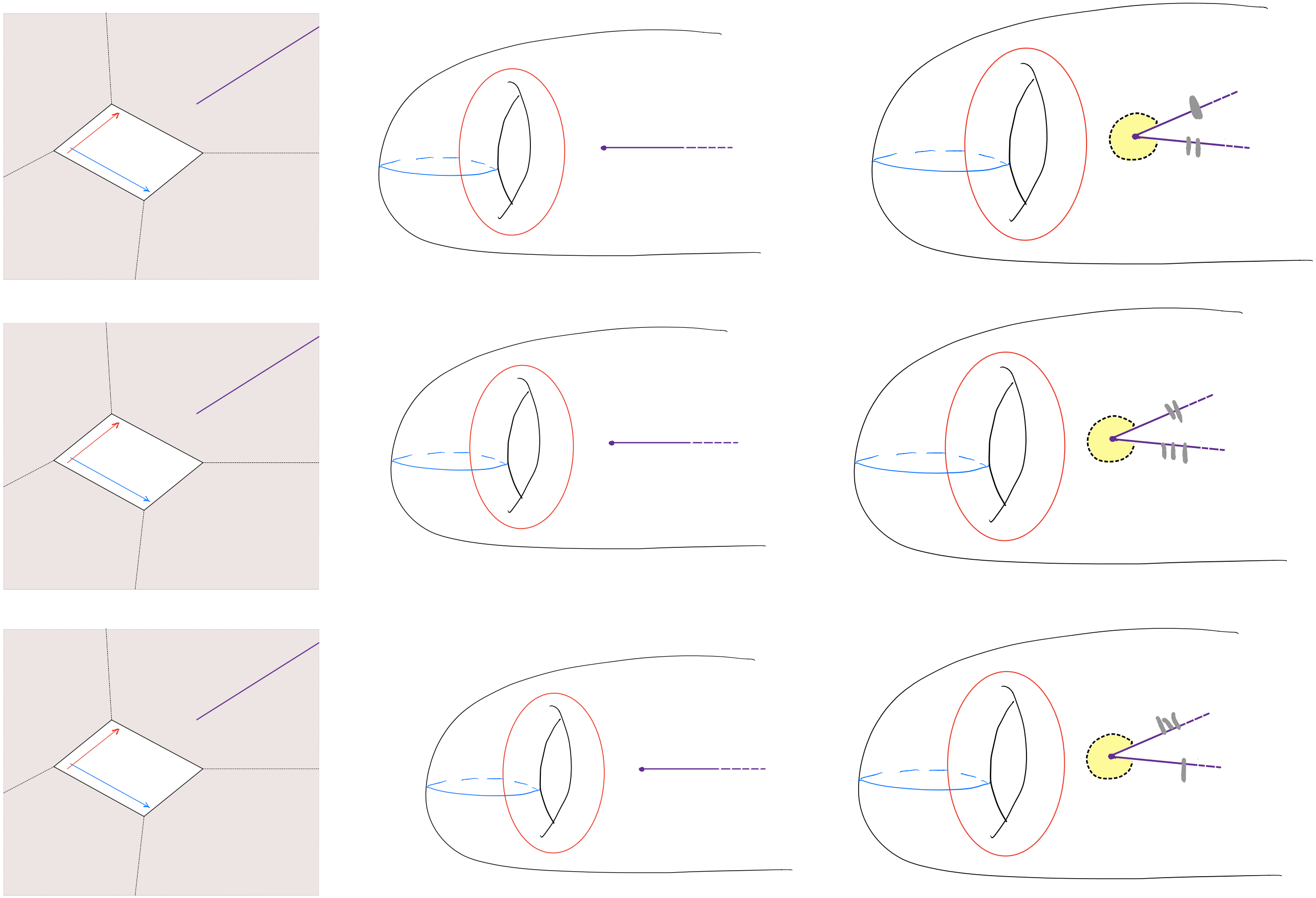
arises as the period of a couple (X, ω) on $S_{0,n}$.

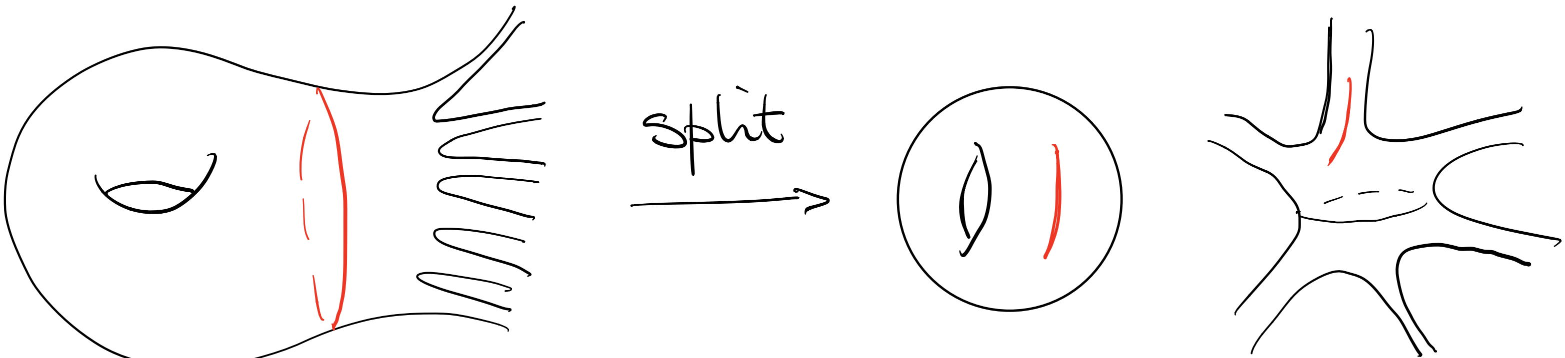




Any surface $S_{g,n}$ splits as the direct sum of lower-complexity pieces. In particular, it splits as the direct sum of $S_{g,0}$ and $S_{0,n}$. In turns $S_{g,0}$ splits as the direct sum of g-torus.

Let Σ be a such a piece. $H_1(\Sigma, \mathbb{Z}) \to H_1(S_{g,n}, \mathbb{Z})$ is an injection and the post-composition with χ yields a representation $\rho_\Sigma : H_1(\Sigma, \mathbb{Z}) \to \mathbb{C}$. We realized ρ_Σ for every piece Σ of the decomposition. These pieces, once glued together properly, yield a translation surface with poles (X, ω) on $S_{g,n}$ and ω will have period χ by construction.

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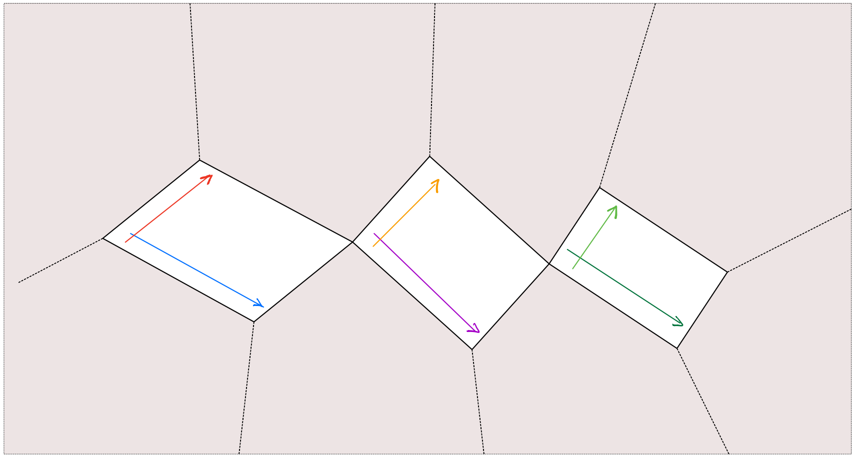
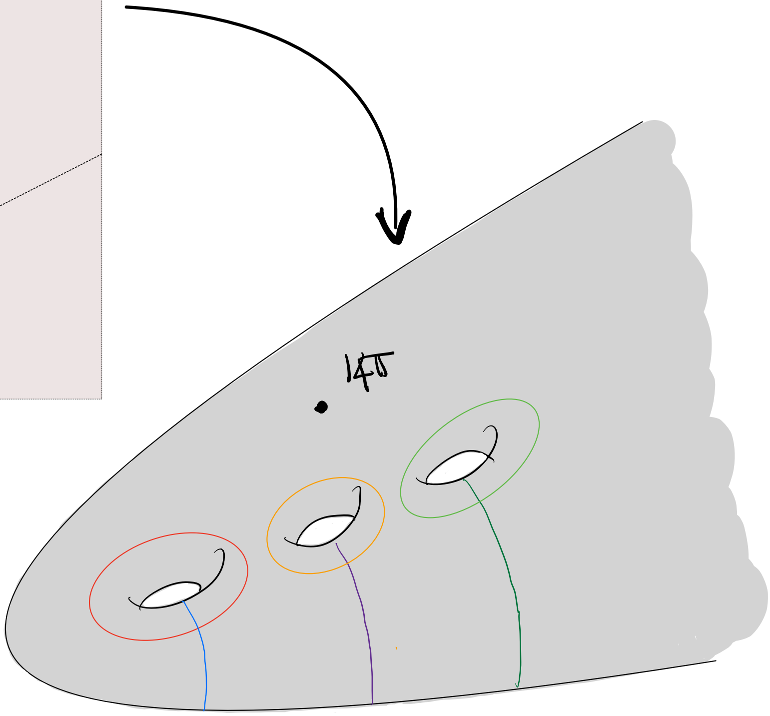




Sum of exterior
angle is 14π

Related results

Let $\mathcal{H}(d_1, \ldots, d_k; p_1, \ldots, p_n) \subset \Omega\mathcal{M}_{g,n}$ be the stratum of couples (X, ω) such that ω has exactly k zeros of orders $d_1, \ldots, d_k$ and n poles of orders $p_1, \ldots, p_n$. Of course,

$$\sum d_i - \sum p_j = 2g - 2$$

Consider the period mapping restricted to such a stratum:

$$\mathbf{Per} : \mathcal{H}(d_1, \ldots, d_k; p_1, \ldots, p_n) \longrightarrow \mathrm{Hom}\Big(H_1\big(S_{g,n}, \mathbb{Z}\big), \mathbb{C}\Big)$$

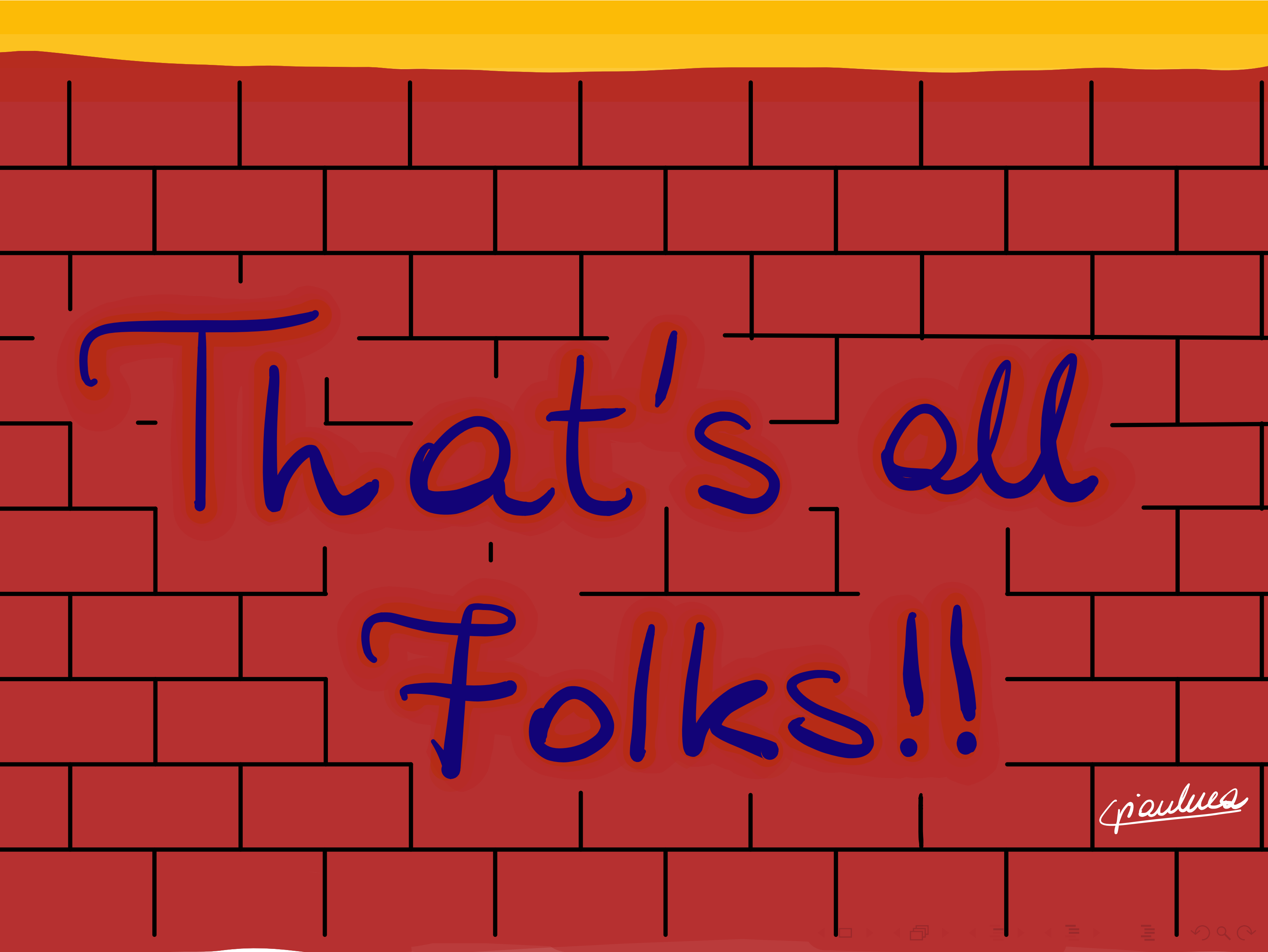
Theorem (Chenakkod-F.-Gupta, 2021)

Assume $n \geq 2$. If all the $p_i = 2$, then it is surjective. In the case $n = 1$, it is surjective if and only if $p \geq 3$.

In all other cases there are representation which are not realizable in that prescribed stratum. For instance, if a pole has order 1, the period is nothing but the residue of the pole which cannot be trivial! Hence, if at least one poles is prescribed of order one, then the trivial representation cannot be realized.

As an artefact of our definition, every puncture of $S_{g,n}$ is a pole for an abelian differential ω.

We may allow punctures to appear as removable singularities for an abelian differential. In this case, we may wonder when a representation $\chi : H_1(S_{g,n}, \mathbb{Z}) \to \mathbb{C}$ appears as the holonomy of some couple (X, ω) where all zero and poles are taken at the puncture. In this case, we can notice that, according to our definitions, the atlases $\mathcal{T}' = \mathcal{T}$. This case has been handled by F.-Gupta in a companion work.


That's all
Folks!!