Families of Congruences of Fractional Partition Functions Modulo Powers of Primes

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Partitions

$$4 = 4, \quad 4 = 3 + 1, \quad 4 = 2 + 2, \quad 4 = 2 + 1 + 1, \quad 4 = 1 + 1 + 1 + 1.$$

Partitions

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For each of the sums,

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1).$$

These above sequences are called the *partitions* of 4 and the summands/terms are called the *parts* of the partitions of 4. In general,

A partition $\lambda := (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k)$ of a positive integer n, is a finite non-increasing sequence of positive integers (the $\lambda_i s$) such that $n = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_k$.

The partition function and the generating function

$p(n)$:= Number of partitions of a positive integer n.

The generating function for the number of partitions of n, $p(n)$ was given by L. Euler (1707–1783).

Look at the following binomial expansions,

$$\frac{1}{1-q} = 1 + q^1 + q^2 + q^3 + \cdots = 1 + q^1 + q^{1+1} + q^{1+1+1} + \cdots$$
$$\frac{1}{1-q^2} = 1 + q^2 + q^4 + q^6 + \cdots = 1 + q^2 + q^{2+2} + q^{2+2+2} + \cdots$$

So that

$$\prod_{j=1}^{\infty} \frac{1}{1-q^j} = (1 + q^1 + q^{1+1} + \cdots) \cdot (1 + q^2 + q^{2+2} + \cdots) \cdot (1 + q^3 + q^{3+3} + \cdots) \cdots$$

Now, if we want get the coefficient of q^3 from the series expansion of

$$\prod_{j=1}^{\infty} \frac{1}{1-q^j} = (1 + q^1 + q^{1+1} + \cdots) \cdot (1 + q^2 + q^{2+2} + \cdots) \cdot (1 + q^3 + q^{3+3} + \cdots) \cdots,$$

then the contributors are

$$q^3,\ q^{2+1},\ \text{and}\ q^{1+1+1}.$$

Therefore,

$$\text{Coefficient}(q^3) = p(3).$$

In general,

$$\text{Coefficient}(q^n) = p(n).$$

So,

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}; \qquad p(0) = 1.$$

S. Ramanujan (1887–1920) first found

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

N.B. Integer power of the generating function of $p(n)$, viz. $\left(\prod \frac{1}{1-q^j}\right)^n$ generates the n-colored partitions.

Thought: What if we raise $\prod \frac{1}{1-q^j}$ to a rational number t!

Question: Can we interpret the coefficients in the series expansion of $\left(\prod \frac{1}{1-q^j}\right)^t$ combinatorially? Is it worth studying the coefficients?

For complex numbers a, and q such that $|q| < 1$,

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n := \lim_{n\to\infty}\prod_{j=0}^{n}(1-aq^j) = \prod_{j=0}^{\infty}(1-aq^j),$$

and

$$E_n := (q^n;q^n)_\infty = \prod_{j=1}^{\infty}(1-q^{nj}).$$

For example,

$$E_1 := (1-q)\cdot(1-q^2)\cdot(1-q^3)\cdots .$$

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$$\sum_{n=0}^{\infty} p_t(n)q^n = E_1^t.$$

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$$E_1^{-1/6} = 1 + \frac{1}{2 \cdot 3} q + \frac{19}{2^3 \cdot 3^2} q^2 + \frac{343}{2^4 \cdot 3^4} q^3 + \frac{11305}{2^7 \cdot 3^5} q^4 + \cdots$$

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S. T. Ng [*Undergraduate Thesis*, Singapore, 2003] proved, for all $n \geq 0$,

$$p_{-2/3}(19n + 9) \equiv 0 \pmod{19}.$$

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Theorem (Chan and Wang [*Acta Arith.*, Vol. 187(1), 2019])

For any integer n and prime ℓ, let $\mathrm{ord}_\ell(n)$ denote the integer k such that $\ell^k \mid n$ and $\ell^{k+1} \nmid n$. Let $t = a/b$, where $a, b \in \mathbb{Z}$, $b \geq 1$ and $\gcd(a, b) = 1$. Then

$$\mathrm{denom}(p_t(n)) = b^n \prod_{\ell \mid b} \ell^{\mathrm{ord}_\ell(n!)}.$$

N.B. The denominators of both $p_t(n)$ and t have the same prime divisors.

Theorem (Chan and Wang [*Acta Arith.*, Vol. 187(1), 2019])

Suppose $a, b, d \in \mathbb{Z}$, $b \geq 1$ and $\gcd(a, b) = 1$. Let ℓ be a prime divisor of $a + db$ and $0 \leq r < \ell$. Suppose d, ℓ and r satisfy any of the following conditions:

1. $d = 1$ and $24r + 1$ is a quadratic non-residue modulo ℓ;
2. $d = 3$ and $8r + 1$ is a quadratic non-residue modulo ℓ or $8r + 1 \equiv 0 \pmod{\ell}$;
3. $d \in \{4, 8, 14\}$, $\ell \equiv 5 \pmod{6}$ and $24r + d \equiv 0 \pmod{\ell}$;
4. $d \in \{6, 10\}$, $\ell \geq 5$ and $\ell \equiv 3 \pmod{4}$ and $24r + d \equiv 0 \pmod{\ell}$;
5. $d = 26$, $\ell \equiv 11 \pmod{12}$ and $24r + d \equiv 0 \pmod{\ell}$.

Then, for $n \geq 0$,

$$p_{-a/b}(\ell n + r) \equiv 0 \pmod{\ell}.$$

For example,

$$\begin{aligned} p_{-1/3}(5n + r) &\equiv 0 \pmod{5}, \quad r \in \{2, 3, 4\}, \\ p_{-2/3}(5n + r) &\equiv 0 \pmod{5}, \quad r \in \{3, 4\}, \\ p_{-1/2}(7n + r) &\equiv 0 \pmod{7}, \quad r \in \{2, 4, 5, 6\}. \end{aligned}$$

Chan and Wang conjectured 17 congruences for elementary proofs. Some of them are in the following theorem.

Theorem (Chan and Wang [*Acta Arith.*, Vol. 187(1), 2019])

For $n \geq 0$, we have

$$\begin{aligned} p_{1/2}(125n+r) &\equiv 0 \pmod{25}, && r \in \{38, 63, 88, 113\},\\ p_{2/3}(25n+r) &\equiv 0 \pmod{25}, && r \in \{19, 24\},\\ p_{1/4}(25n+r) &\equiv 0 \pmod{25}, && r \in \{14, 24\},\\ p_{1/4}(25n+19) &\equiv 0 \pmod{125}, &&\\ p_{-1/3}(25n+r) &\equiv 0 \pmod{125}, && r \in \{18, 23\},\\ p_{-3/4}(25n+r) &\equiv 0 \pmod{25}, && r \in \{13, 23\},\\ p_{-3/4}(25n+18) &\equiv 0 \pmod{125}, && \end{aligned}$$

and

$$p_{-3/4}(125n+r) \equiv 0 \pmod{3125}, \quad r \in \{93, 118\}.$$

The Rogers-Ramanujan continued fraction is defined as

$$\mathcal{R}(q) := \frac{q^{1/5}}{1} \, {}_{+} \, \frac{q}{1} \, {}_{+} \, \frac{q^2}{1} \, {}_{+} \, \frac{q^3}{1} \, {}_{+} \cdots, \quad |q| < 1,$$

which has the following well-known q-product representation

$$\mathcal{R}(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

In some of the next slides

$$R(q) := \frac{q^{1/5}}{\mathcal{R}(q)}.$$

A very important dissection

n-dissection of E_1 (Berndt, [*Ramanujan's notebook part III*, Springer, 1991])

For integer $n \geq 1$ with $n \equiv \pm 1 \pmod 6$, if $n = 6g + 1$, where $g \geq 1$, then

$$E_1 = E_{n^2}\left((-1)^g q^{(n^2-1)/24} + \sum_{j=1}^{(n-1)/2} (-1)^{j+g} q^{(j-g)(3j-3g-1)/2} \frac{(q^{2jn};q^{n^2})_\infty (q^{n^2-2jn};q^{n^2})_\infty}{(q^{jn};q^{n^2})_\infty (q^{n^2-jn};q^{n^2})_\infty}\right),$$

while if $n = 6g - 1$, where $g \geq 1$, then

$$E_1 = E_{n^2}\left((-1)^g q^{(n^2-1)/24} + \sum_{j=1}^{(n-1)/2} (-1)^{j+g} q^{(j-g)(3j-3g+1)/2} \frac{(q^{2jn};q^{n^2})_\infty (q^{n^2-2jn};q^{n^2})_\infty}{(q^{jn};q^{n^2})_\infty (q^{n^2-jn};q^{n^2})_\infty}\right).$$

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while if $n = 6g - 1$, where $g \geq 1$, then

$$E_1 = E_{n^2}\left((-1)^g q^{(n^2-1)/24} + \sum_{j=1}^{(n-1)/2} (-1)^{j+g} q^{(j-g)(3j-3g+1)/2} \frac{(q^{2jn};q^{n^2})_\infty (q^{n^2-2jn};q^{n^2})_\infty}{(q^{jn};q^{n^2})_\infty (q^{n^2-jn};q^{n^2})_\infty}\right).$$

For example, when $n = 5$

$$E_1 = E_{25}\left(R(q^5) - q - \frac{q^2}{R(q^5)}\right).$$

Theorem (Baruah and Das)

For all $n \geq 0$, we have

$$\begin{aligned} p_{-1/6}(25n+r) &\equiv 0 \pmod{25}, && r \in \{9, 14, 19, 24\}, \\ p_{1/6}(125n+r) &\equiv 0 \pmod{25}, && r \in \{96, 121\}, \\ p_{-5/6}(125n+r) &\equiv 0 \pmod{25}, && r \in \{95, 120\}, \\ p_{5/6}(25n+r) &\equiv 0 \pmod{125}, && r \in \{15, 20\}, \end{aligned}$$

and

$$p_{5/6}(125n+r) \equiv 0 \pmod{625}, \qquad r \in \{65, 70\}.$$

N.B. The method used to find the above theorem lets us prove all the conjectural congruences modulo powers of 5 by Chan and Wang.

Fractional partition functions: A general family of congruences

Theorem (Baruah and Das)

Let $\ell \geq 5$ be a prime and $k > 1$ and s be positive integers such that $s \leq \lfloor k/2 \rfloor$. Then, for all $n \geq 0$, we have

$$p_{-(\ell^k - b)/b}\left(\ell^{2s} \cdot n + \ell^{2s-1} \cdot r + \frac{(\ell - 24\lfloor \ell/24 \rfloor)\ell^{2s-1} - 1}{24}\right) \equiv 0 \pmod{\ell^{k-2s+1}},$$

where $0 \leq r < \ell,\ r \neq \lfloor \ell/24 \rfloor$, and $(\ell, b) = 1$.

Theorem (Baruah and Das)

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where $0 \leq r < \ell$, $r \neq \lfloor \ell/24 \rfloor$, and $(\ell, b) = 1$.

Some cases, when $\ell = 5$, $b = 1567$, and $k = 5$

$$p_{-1558/1567}(5^2 n + 5r + 1) \equiv 0 \pmod{5^4}, \quad r \in \{1, 2, 3, 4\},$$
$$p_{-1558/1567}(5^4 n + 5^3 r + 26) \equiv 0 \pmod{5^2}, \quad r \in \{1, 2, 3, 4\}.$$

N.B. The sequences $(5^2 n + 5r + 1)$ and $(5^4 n + 5^3 r + 26)$ do not have common terms.

Fractional partition functions: Congruences modulo powers of 5, 7, and 11

Using dissections of E_1^2, we find

Theorem (Baruah and Das)

Let $k > 1$ and s be positive integers such that $s \leq \lfloor k/2 \rfloor$. Then, for all $n \geq 0$, we have

$$p_{-(5^k-2b)/b}\left(5^{2s} \cdot n + 5^{2s-1} \cdot r + \frac{5^{2s}-1}{12}\right) \equiv 0 \pmod{5^{k-2s+1}}, \quad r \in \{1, 2, 3, 4\},$$

$$p_{-(7^k-2b)/b}\left(7^{2s} \cdot n + 7^{2s-1} \cdot r + \frac{7^{2s}-1}{12}\right) \equiv 0 \pmod{7^{k-2s+1}}, \quad r \in \{1, 2, \ldots, 6\},$$

and

$$p_{-(11^k-2b)/b}\left(11^{2s} \cdot n + 11^{2s-1} \cdot r + \frac{11^{2s}-1}{12}\right) \equiv 0 \pmod{11^{k-2s+1}}, \quad r \in \{1, 2, \ldots, 11\},$$

where b's in the above congruences are co-prime to the moduli.

N.B. Similar congruences do not hold true for prime 13.

Using dissections of E_1^3 and E_1^4, we find

Theorem (Baruah and Das)

Let k, m and s be positive integers such that $s \leq m+1$. Then, for all $n \geq 0$, we have

$$p_{-(3^{k+m}-3b)/b}\left(3^{2s}\cdot n + 3^{2s-1}\cdot r + \frac{3^{2s}-1}{8}\right) \equiv 0 \pmod{3^{k+m-s+1}}, \quad r \in \{1,2\},$$

$$p_{-(5^{k+m}-3b)/b}\left(5^{2s}\cdot n + 5^{2s-1}\cdot r + \frac{5^{2s}-1}{8}\right) \equiv 0 \pmod{5^{k+m-s+1}}, \quad r \in \{1,2,3,4\},$$

$$p_{-(5^{k+m}-4b)/b}\left(5^{2s}\cdot n + 5^{2s-1}\cdot r + \frac{5^{2s}-1}{6}\right) \equiv 0 \pmod{5^{k+m-s+1}}, \quad r \in \{1,2,3,4\},$$

and

$$p_{-(7^{k+m}-3b)/b}\left(7^{2s}\cdot n + 7^{2s-1}\cdot r + \frac{7^{2s}-1}{8}\right) \equiv 0 \pmod{7^{k+m-s+1}}, \quad r \in \{1,2,\ldots,6\},$$

where b's in the above congruences are co-prime to the moduli.

Using dissections of E_1^6, E_1^8, and E_1^{14}, we find

Theorem (Baruah and Das)

Let $k > 1$ and s be positive integers such that $s \leq \lfloor k/2 \rfloor$. Then, for all $n \geq 0$, we have

$$p_{-(3^k-6b)/b}\left(3^{2s}\cdot n + 3^{2s-1}\cdot r + \frac{3^{2s}-1}{4}\right) \equiv 0 \pmod{3^k}, \quad r \in \{1,2\},$$

$$p_{-(5^k-8b)/b}\left(5^{2s}\cdot n + 5^{2s-1}\cdot r + \frac{2\cdot 5^{2s-1}-1}{3}\right) \equiv 0 \pmod{5^k}, \quad r \in \{0,2,3,4\},$$

$$p_{-(5^k-14b)/b}\left(5^{2s}\cdot n + 5^{2s-1}\cdot r + \frac{11\cdot 5^{2s-1}-7}{12}\right) \equiv 0 \pmod{5^k}, \quad r \in \{0,1,3,4\},$$

and

$$p_{-(7^k-6b)/b}\left(7^{2s}\cdot n + 7^{2s-1}\cdot r + \frac{3\cdot 7^{2s-1}-1}{4}\right) \equiv 0 \pmod{7^k}, \quad r \in \{0,2,3,4,5,6\},$$

where b's in the above congruences are co-prime to the moduli.

Theorem (Baruah and Das)

Let $k > 1$ be an odd integer. Then, for all $n \geq 0$, we have

$$p_{-(3^k-6b)/b}\left(3^k \cdot n + \frac{3^{k+1}-1}{4}\right) \equiv 0 \pmod{3^k},$$
$$p_{-(5^k-8b)/b}\left(5^k \cdot n + \frac{2 \cdot 5^k-1}{3}\right) \equiv 0 \pmod{5^k},$$
$$p_{-(5^k-14b)/b}\left(5^k \cdot n + \frac{11 \cdot 5^k-7}{12}\right) \equiv 0 \pmod{5^k},$$

and

$$p_{-(7^k-6b)/b}\left(7^k \cdot n + \frac{3 \cdot 7^k-1}{4}\right) \equiv 0 \pmod{7^k},$$

where b's in the above congruences are co-prime to the moduli.

For any non-zero rational number t and integer $r > 1$, we define

$$\sum_{n=0}^{\infty} p_{[1,r;t]}(n) q^n = (E_1 E_r)^t.$$

For instance,

$$(E_1E_3)^{1/6} = 1 - \frac{1}{2\cdot 3}q - \frac{17}{2^3\cdot 3^2}q^2 - \frac{451}{2^4\cdot 3^4}q^3 - \frac{6191}{2^7\cdot 3^5}q^4 - \frac{12053}{2^8\cdot 3^6}q^5 - \frac{2845933}{2^{10}\cdot 3^8}q^6 + O\left(q^7\right).$$

Therefore, it is also meaningful to explore congruences for $p_{[1,r;t]}(n)$ modulo powers of prime ℓ such that $\ell \nmid$ denominator of t.

Theorem (Baruah and Das)

Suppose $a, b, d \in \mathbb{Z}$, $b \geq 1$ and $(a, b) = 1$. Let ℓ be an odd prime divisor of $a + db$ and $0 \leq r < \ell$. Suppose d, ℓ and r satisfy any of the following two conditions:

1. $d = 2$, $\ell \equiv 3 \pmod 4$, and $4r + 1 \equiv 0 \pmod \ell$,
2. $d = 3$, $\ell \equiv 5$ *or* $7 \pmod 8$, and $8r + 3 \equiv 0 \pmod \ell$.

Then, for all $n \geq 0$,

$$p_{[1,2;-a/b]}(\ell n + r) \equiv 0 \pmod \ell.$$

Theorem (Baruah and Das)

Suppose $a, b, d \in \mathbb{Z}, b \geq 1$ and $(a, b) = 1$. Let ℓ be an odd prime divisor of $a + db$ and $0 \leq r < \ell$. Suppose d, ℓ and r satisfy any of the following two conditions:

1. $d = 2$, $\ell \equiv 3 \pmod 4$, and $4r + 1 \equiv 0 \pmod \ell$,
2. $d = 3$, $\ell \equiv 5$ *or* $7 \pmod 8$, and $8r + 3 \equiv 0 \pmod \ell$.

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$$d = 3, \ell \equiv 5 \textit{ or } 11 \pmod{12} \textit{ and } 2r + 1 \equiv 0 \pmod \ell.$$

Then, for all $n \geq 0$,

$$p_{[1,3;-a/b]}(\ell n + r) \equiv 0 \pmod \ell.$$

Theorem (Baruah and Das)

Suppose $a, b, d \in \mathbb{Z}, b \geq 1$, and $(a, b) = 1$. Let ℓ be an odd prime divisor of $a + db$ and $0 \leq r < \ell$. Suppose d, ℓ, and r satisfy any of the following two conditions:

1. $d = 2$, $\ell \equiv 3 \pmod 4$, and $12r + 5 \equiv 0 \pmod \ell$,
2. $d = 3$, $\ell \equiv 3 \pmod 4$, and $8r + 5 \equiv 0 \pmod \ell$.

Then, for all $n \geq 0$,

$$p_{[1,4;-a/b]}(\ell n + r) \equiv 0 \pmod \ell.$$

Theorem (Baruah and Das)

Suppose $a, b, d \in \mathbb{Z}$, $b \geq 1$, and $(a, b) = 1$. Let ℓ be an odd prime divisor of $a + db$ and $0 \leq r < \ell$. Suppose d, ℓ, and r satisfy any of the following two conditions:

1. $d = 2$, $\ell \equiv 3 \pmod 4$, and $12r + 5 \equiv 0 \pmod \ell$,
2. $d = 3$, $\ell \equiv 3 \pmod 4$, and $8r + 5 \equiv 0 \pmod \ell$.

Then, for all $n \geq 0$,

$$p_{[1,4;-a/b]}(\ell n + r) \equiv 0 \pmod \ell.$$

Theorem (Baruah and Das)

For integer $k \geq 1$ and all $n \geq 0$, we have

$$p_{[1,4;-(5^k-3b)/b]}(5n + r) \equiv 0 \pmod 5, \quad \textit{where } (5, b) = 1 \textit{ and } r \in \{2, 3\}.$$

Theorem (Baruah and Das)

Let $k > 1$ and s be positive integers such that $s \leq \lfloor k/2 \rfloor$. Then, for all $n \geq 0$, we have

$$p_{[1,2;-(3^k-b)/b]}\left(3^{2s}\cdot n + 3^{2s-1}\cdot r + \frac{3^{2s}-1}{8}\right) \equiv 0 \pmod{3^{k-2s+1}}, \quad r \in \{1,2\},$$

$$p_{[1,2;-(5^k-b)/b]}\left(5^{2s}\cdot n + 5^{2s-1}\cdot r + \frac{5^{2s}-1}{8}\right) \equiv 0 \pmod{5^{k-2s+1}}, \quad r \in \{1,2,3,4\},$$

$$p_{[1,2;-(7^k-b)/b]}\left(7^{2s}\cdot n + 7^{2s-1}\cdot r + \frac{7^{2s}-1}{8}\right) \equiv 0 \pmod{7^{k-2s+1}}, \quad r \in \{1,2,\ldots,6\},$$

$$p_{[1,3;-(5^k-b)/b]}\left(5^{2s}\cdot n + 5^{2s-1}\cdot r + \frac{5^{2s}-1}{6}\right) \equiv 0 \pmod{5^{k-2s+1}}, \quad r \in \{1,2,3,4\},$$

$$p_{[1,4;-(7^k-b)/b]}\left(7^{2s}\cdot n + 7^{2s-1}\cdot r + \frac{11\cdot 7^{2s-1}-5}{24}\right) \equiv 0 \pmod{7^{k-2s+1}}, \quad r \in \{0,2,3,\ldots,6\},$$

$$p_{[1,3;-(11^k-b)/b]}\left(11^{2s}\cdot n + 11^{2s-1}\cdot r + \frac{5\cdot 11^{2s-1}-1}{6}\right) \equiv 0 \pmod{11^{k-2s+1}}, \quad r \in \{0,2,3,\ldots,10\},$$

$$p_{[1,4;-(11^k-b)/b]}\left(11^{2s}\cdot n + 11^{2s-1}\cdot r + \frac{7\cdot 11^{2s-1}-5}{24}\right) \equiv 0 \pmod{11^{k-2s+1}}, \quad r \in \{0,1,3,4,\ldots,10\},$$

where b's in the above congruences are co-prime to the moduli.

Using dissections of $(E_1E_2)^2$ and $(E_1E_3)^2$, we find

Theorem (Baruah and Das)

Let k, m, and s be positive integers such that $s \leq m+1$. Then, for all $n \geq 0$, we have

$$p_{[1,2;-(3^{k+m}-2b)/b]}\left(3^{2s}\cdot n + 3^{2s-1}\cdot r + \frac{3^{2s}-1}{4}\right) \equiv 0 \pmod{3^{k+m-s+1}}, \quad r \in \{1,2\}$$

and

$$p_{[1,3;-(5^{k+m}-2b)/b]}\left(5^{2s}\cdot n + 5^{2s-1}\cdot r + \frac{2\cdot 5^{2s-1}-1}{3}\right) \equiv 0 \pmod{5^{k+m-s+1}}, \quad r \in \{0,2,3,4\},$$

where b's in the above congruences are co-prime to the moduli.

Using dissections of $(E_1E_2)^2$, $(E_1E_2)^5$, and $(E_1E_3)^3$, we find

Theorem (Baruah and Das)

Let $k>1$ and s be positive integers such that $s\leq\lfloor k/2\rfloor$. Then, for all $n\geq 0$, we have

$$p_{[1,2;-(3^k-5b)/b]}\left(3^{2s}\cdot n+3^{2s-1}\cdot r+\frac{7\cdot 3^{2s-1}-5}{8}\right)\equiv 0 \pmod{3^k},\quad r\in\{0,2\},$$
$$p_{[1,2;-(5^k-3b)/b]}\left(5^{2s}\cdot n+5^{2s-1}\cdot r+\frac{7\cdot 5^{2s-1}-3}{8}\right)\equiv 0 \pmod{5^k},\quad r\in\{0,2,3,4\},$$

and

$$p_{[1,3;-(5^k-3b)/b]}\left(5^{2s}\cdot n+5^{2s-1}\cdot r+\frac{5^{2s-1}-1}{2}\right)\equiv 0 \pmod{5^k},\quad r\in\{0,1,3,4\},$$

where b's in the above congruences are co-prime to the moduli.

Theorem (Baruah and Das)

Let $k > 1$ be an odd integer. Then, for all $n \geq 0$, we have

$$p_{[1,2;-(3^k-5b)/b]}\left(3^k \cdot n + \frac{7 \cdot 3^k - 5}{8}\right) \equiv 0 \pmod{3^k},$$
$$p_{[1,2;-(5^k-3b)/b]}\left(5^k \cdot n + \frac{7 \cdot 5^k - 3}{8}\right) \equiv 0 \pmod{5^k},$$

and

$$p_{[1,3;-(5^k-3b)/b]}\left(5^k \cdot n + \frac{5^k - 1}{2}\right) \equiv 0 \pmod{5^k},$$

where b's in the above congruences are co-prime to the moduli.

References

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Thanks