Hard and Easy Instances of L-Tromino Tilings [1]

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Online Seminar

[1]Lecture Notes in Comp. Sc. (LNCS), Vol. 11355 (Springer) (2019).
Theoretical Computer Science (Elsevier), Vol. 815 (2020).

Outline

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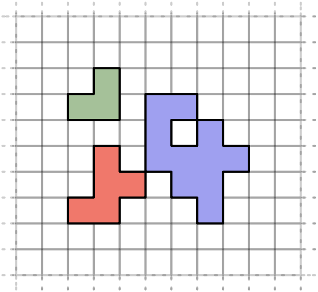

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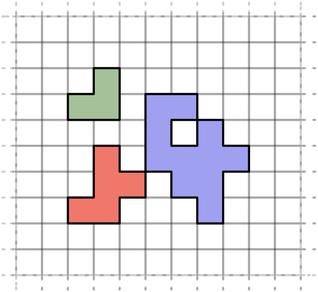

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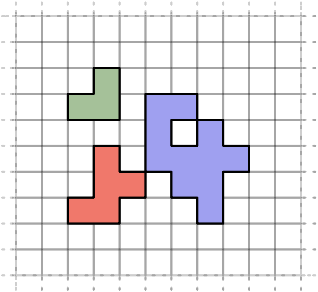

- Every cell (square) is fixed in a square lattice.
- Two cell are adjacent if the Manhattan distance is 1.

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1. Introduction
   - Polyominoes
   - L-Tromino Tiling Problem
   - Computational Complexity
2. Tiling of the Aztec Rectangles
   - Aztec Rectangle
   - Aztec Rectangle with a single defect
   - Tiling Aztec Rectangle with unbounded number of defects
3. 180-Tromino Tiling
   - A rotation constraint
   - Forbidden Polyominoes
4. Open Problems

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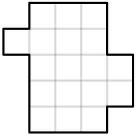
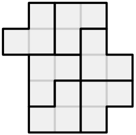


(a) A region R



(b) A tiling of region R

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- If the time complexity is polynomial in the input parameters, then we say that a problem can be solved in Polynomial time.

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- Example: Subgraph isomorphism problem.

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- T. Horiyama, T. Ito, K. Nakatsuka, A. Suzuki and R. Uehara (2012) constructed a one-one reduction from 1-in-3 SAT.

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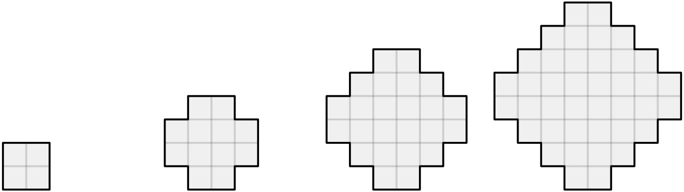

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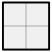
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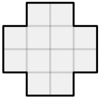
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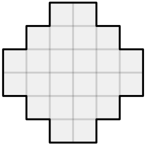
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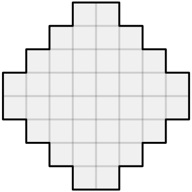
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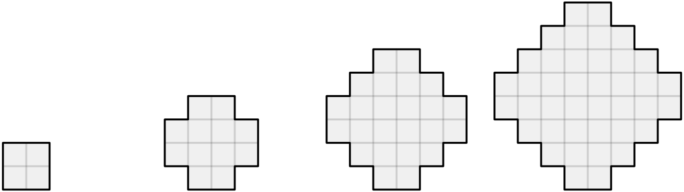

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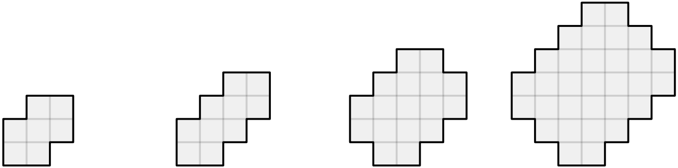

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Tiling Aztec Rectangle (cont'd)

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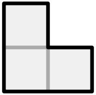
Each piece of L-tromino **covers 3 cells**.



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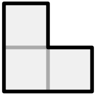

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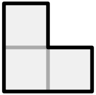

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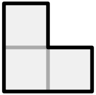

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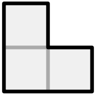

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$$|\mathcal{AR}_{a,b}| = a(b+1) + b(a+1).$$

Theorem

An Aztec rectangle $\mathcal{AR}_{a,b}$ has a tiling with L-trominoes
$\iff |\mathcal{AR}_{a,b}| \equiv 0 \pmod 3$
$\iff (a, b)$ *is equal to* $(3k, 3k')$ *or* $(3k-1, 3k'-1)$ *for some* $k, k' \in \mathbb{N}$.

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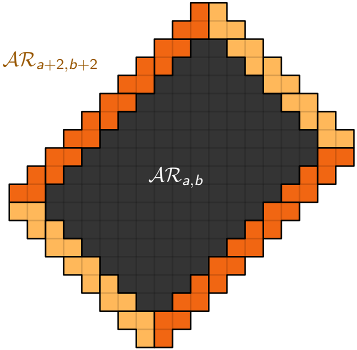

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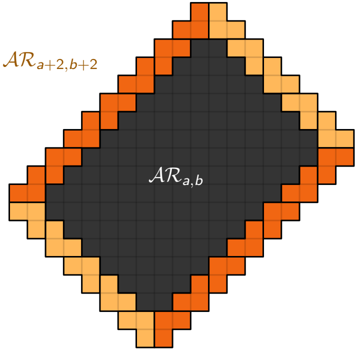

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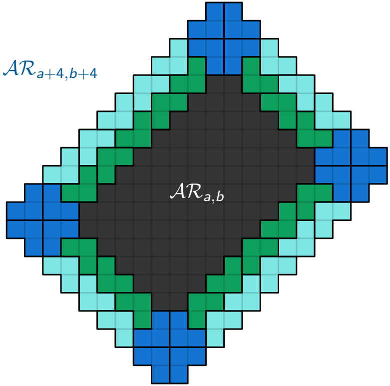

(b) Pattern 2

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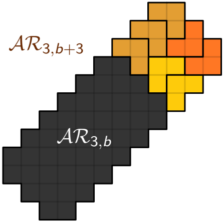

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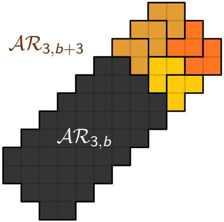

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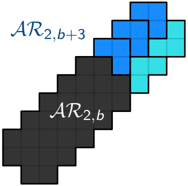

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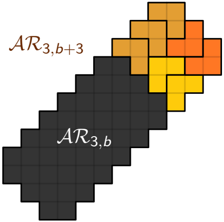

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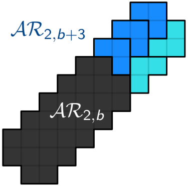

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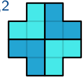
Base case: $\mathcal{AR}_{2,2}$ and $\mathcal{AR}_{3,3}$.



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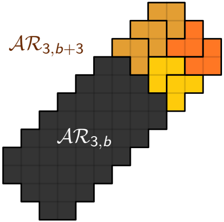

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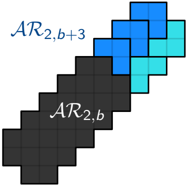

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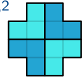

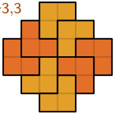

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5. Return R.

□

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- Steps 3.2 and 4.2 can be done in time $O(b)$.
- Giving a total time complexity of $O(b^2)$.

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   - A rotation constraint
   - Forbidden Polyominoes
4. Open Problems

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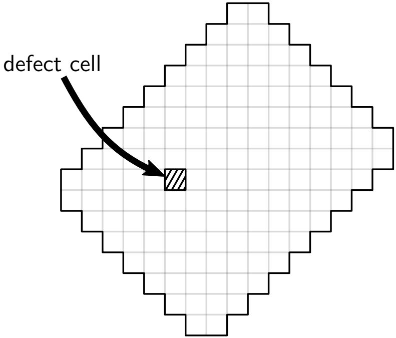

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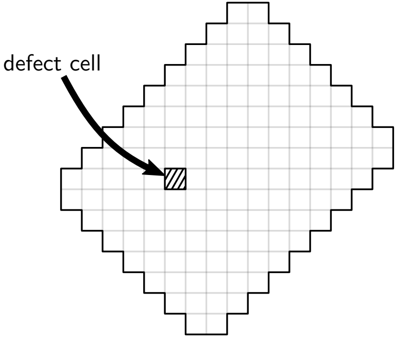


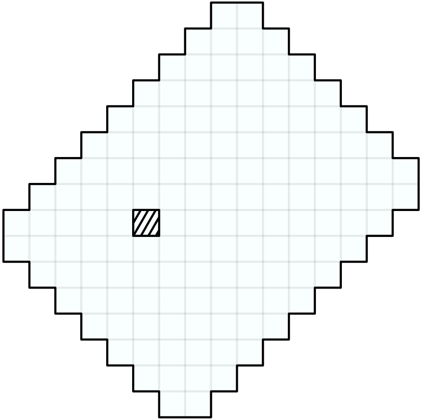
Theorem

An Aztec rectangle $\mathcal{AR}_{a,b}$ with one defect has a tiling with L-trominoes

$\iff |\mathcal{AR}_{a,b}| \equiv 1 \pmod 3$

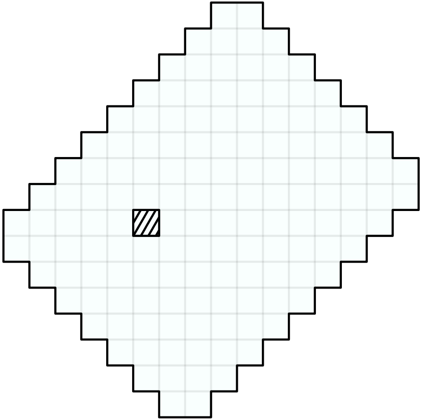
$\iff$ *a or b is equal to $3k-2$ for some $k \in \mathbb{N}$.*

Tiling Aztec Rectangle with a single defect (cont'd)



Tiling Aztec Rectangle with a single defect (cont'd)

- Place a *fringe* where it covers the defect.



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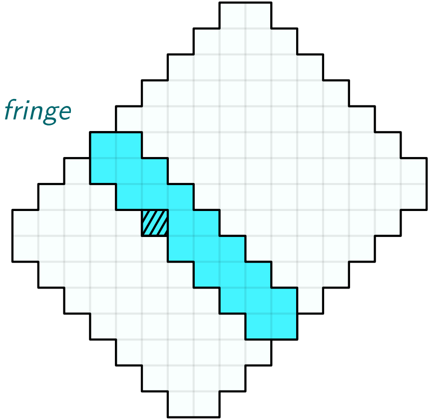

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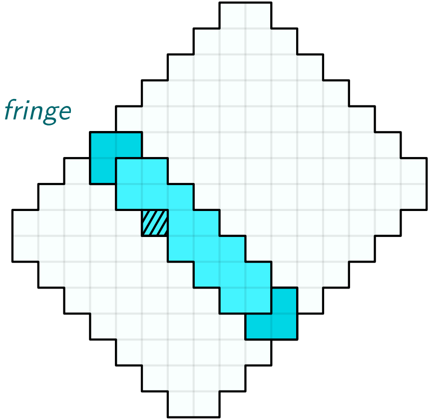

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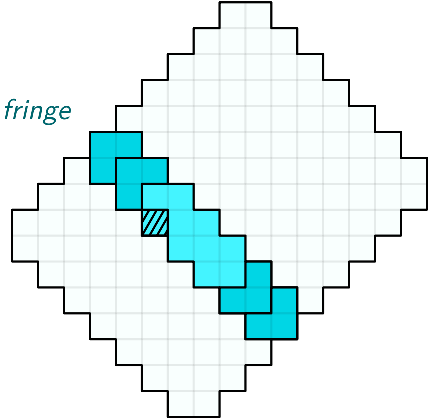

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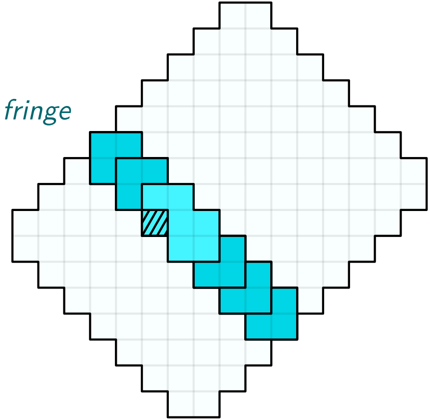

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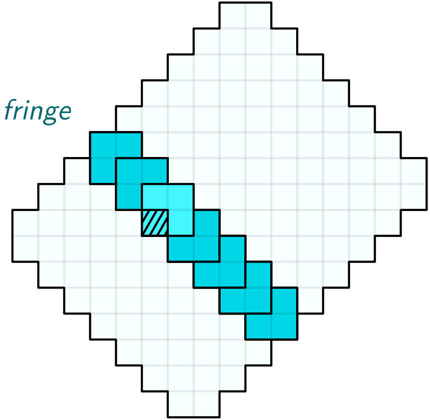

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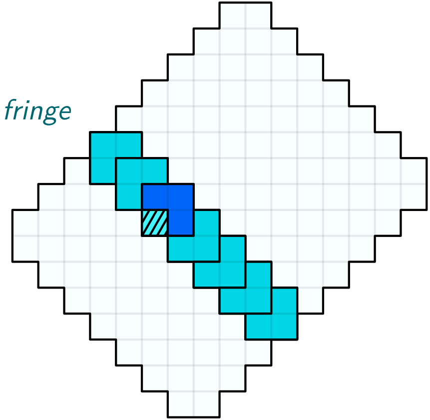

Tiling Aztec Rectangle with a single defect (cont'd)

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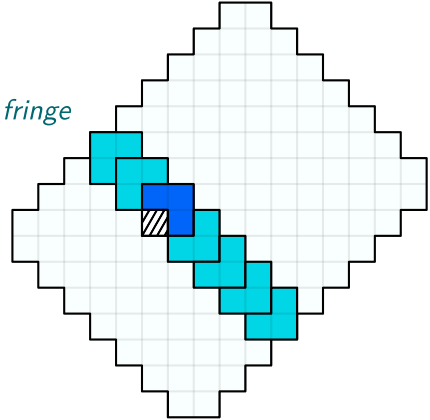

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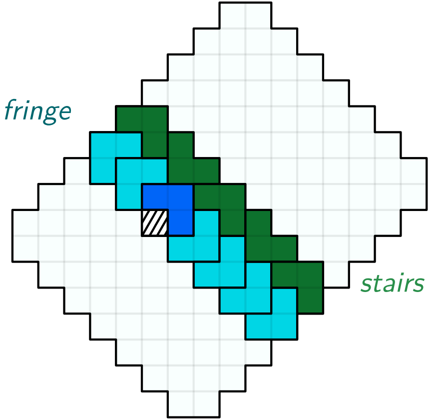

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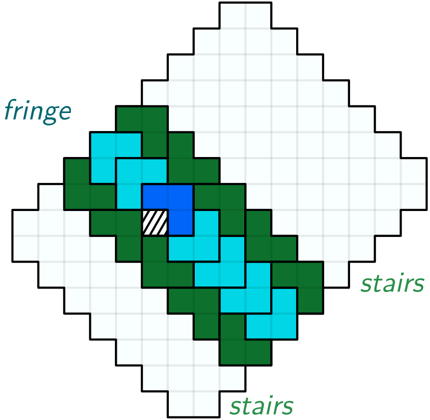

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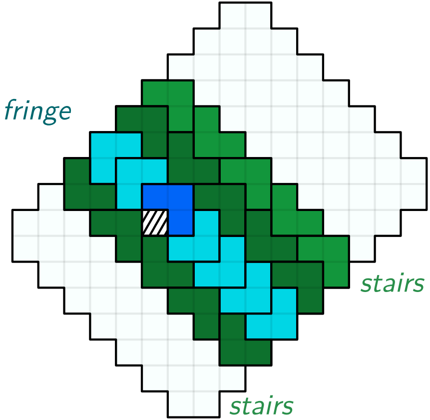

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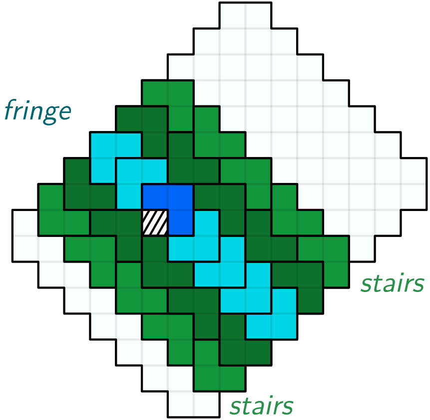

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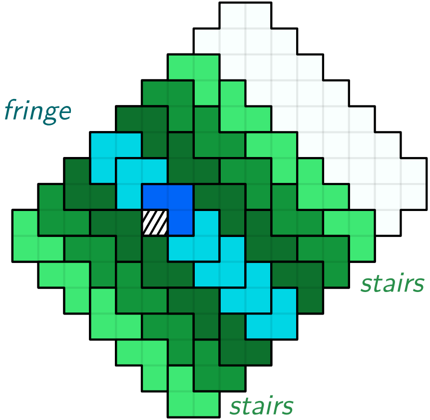

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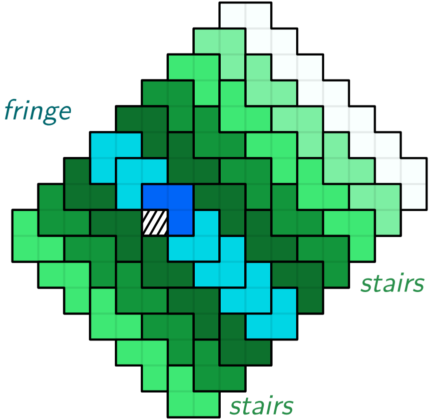

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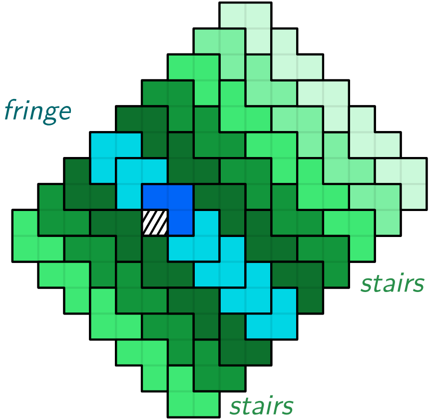

Outline

Tiling Aztec Rectangle with an unbounded number of defects

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Given a region R', we can embed R' inside a sufficiently large Aztec Rectangle $\mathcal{AR}_{a,b}$.

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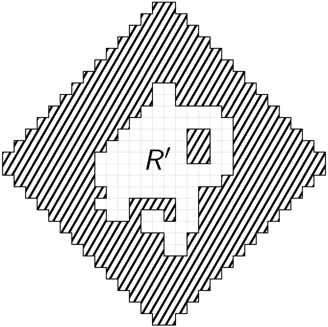

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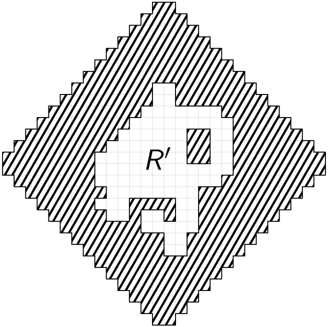

Theorem

*The problem of tiling Aztec Rectangle $\mathcal{AR}_{a,b}$ with an unbounded number of defects is **NP-complete**.*

Outline

180°L-Tromino Tiling

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Definition

The 180-tromino tiling problem only allows 180° rotations of L-trominoes, i.e., the tile set can be

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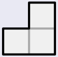
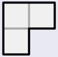
$\Sigma = \{$ **right-oriented** 180-trominoes $\} = \{$ ,  $\}$

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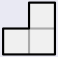
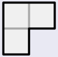
$\Sigma = \{$ **right-oriented** 180-trominoes $\} = \{$ ,  $\}$

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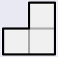
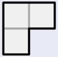
$\Sigma = \{$ **right-oriented** 180-trominoes $\} = \{$ ,  $\}$

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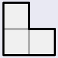
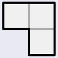
$\Sigma = \{$ **left-oriented** 180-trominoes $\} = \{$ ,  $\}$.

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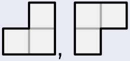
$\Sigma = \{$ **right-oriented** 180-trominoes $\} = \{$  $\}$

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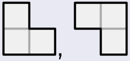
$\Sigma = \{$ **left-oriented** 180-trominoes $\} = \{$  $\}$.

With no loss of generality, we will only consider **right-oriented 180-trominoes**.

180°L-Tromino Tiling (cont'd)

Theorem

There is a one-one correspondence between 180-tromino tiling and the triangular trihex tiling [Conway and Lagarias, (1990)].

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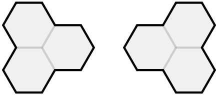

Two **triangular trihex**.

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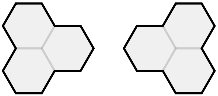

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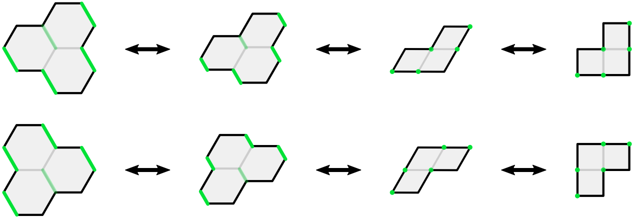

Transformation from **triangular trihex** to **180-tromino**

180°L-Tromino Tiling (cont'd)

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Definition

A **cell tetrasection** is a division of a cell into 4 equal size cells.

180°L-Tromino Tiling (cont'd)

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A **tetrasected polyomino** $P^{\boxplus}$ is obtained by tetrasecting each cell of a poylomino P.

180°L-Tromino Tiling (cont'd)

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If there is a I-tromino tiling for some R, then there is also a 180-tromino tiling for $R^{\boxplus}$.

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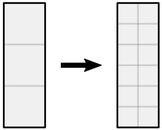

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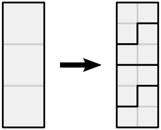

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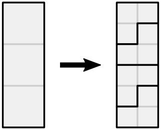

However, it is not known if the converse statement is true or false.

180°L-Tromino Tiling (cont'd)

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Horiyama et al. also proved that the I-tromino tiling problem is **NP-Complete**.

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Theorem [Horiyama, Ito, Nakatsuka, Suzuki and Uehara (2012)]

1-in-3 SAT $\leq_P$ I-tromino Tiling

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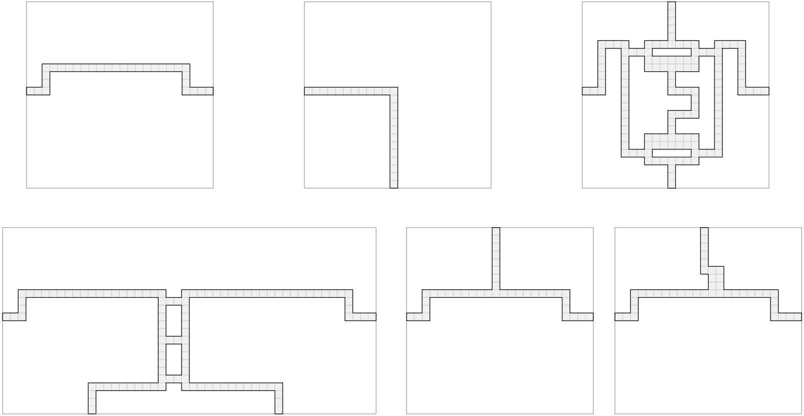

180°L-Tromino Tiling (cont'd)

In each gadget G, I-tromino tiling for G can be simulated with 180-tromino tiling for $G^{\boxplus}$.

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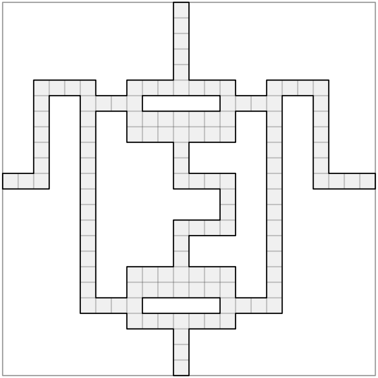

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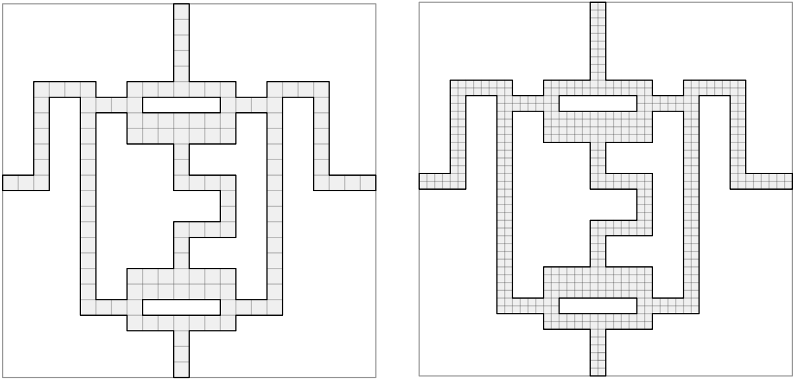

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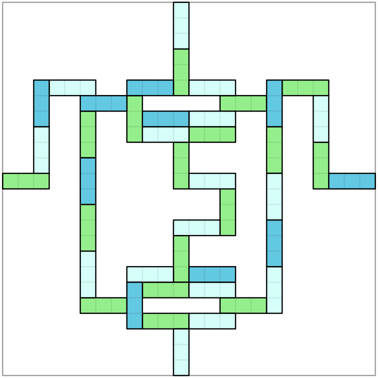

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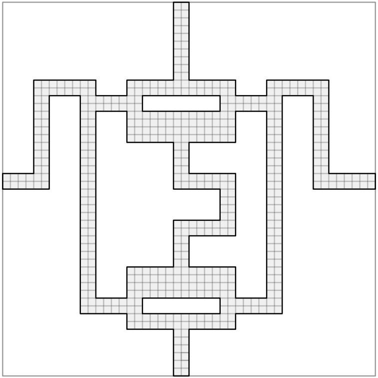

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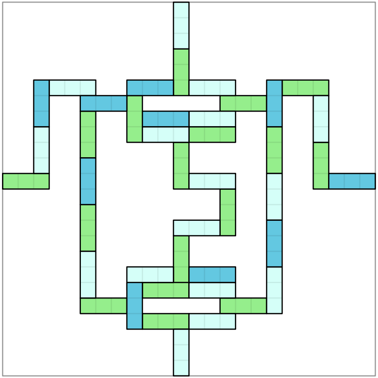

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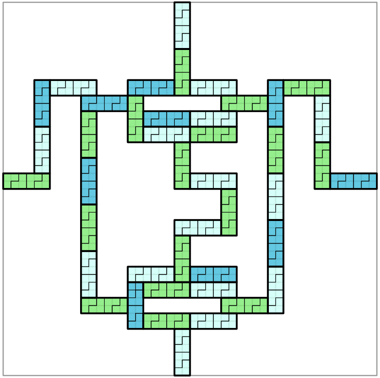

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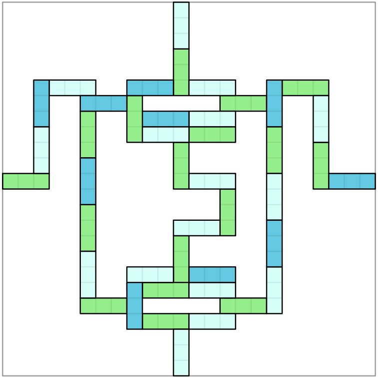

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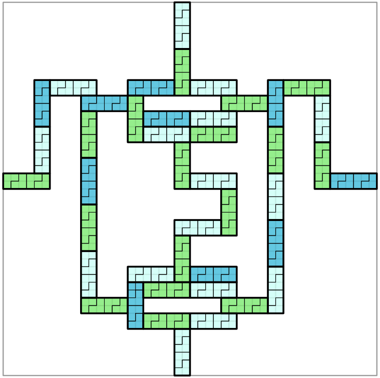

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Theorem

180-tromino tiling is ***NP-complete****.*

Outline

1. Introduction
   - Polyominoes
   - L-Tromino Tiling Problem
   - Computational Complexity
2. Tiling of the Aztec Rectangles
   - Aztec Rectangle
   - Aztec Rectangle with a single defect
   - Tiling Aztec Rectangle with unbounded number of defects
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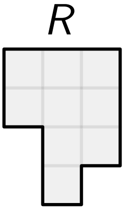

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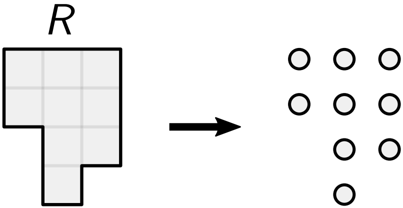
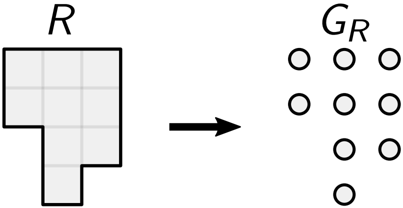


- Transformation from R to G_R:
  - Transform every cell of R to vertices of G_R.
  - Add horizontal, vertical and northeast-diagonal edges.

The 180-tromino tiling can also be reduced to the **Maximum Independent Set** problem.



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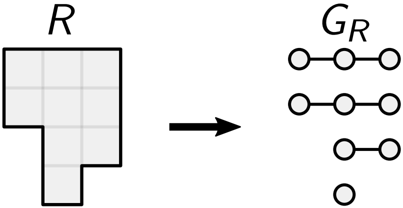
The 180-tromino tiling can also be reduced to the **Maximum Independent Set** problem.



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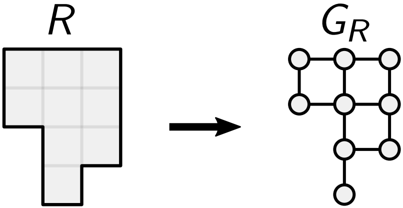

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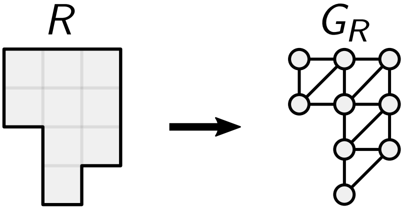

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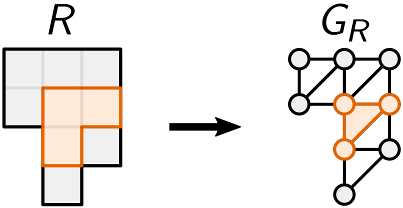

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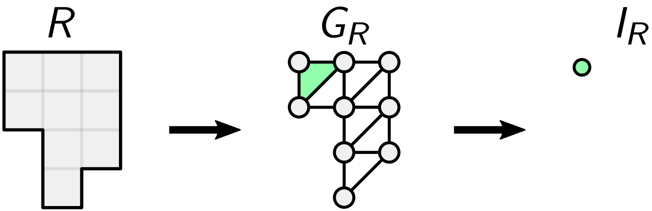

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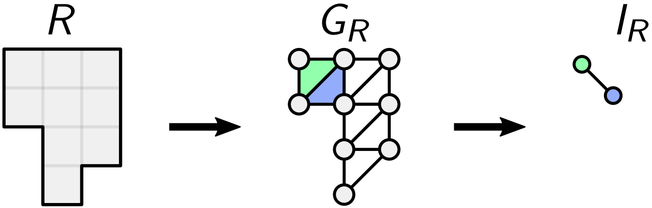

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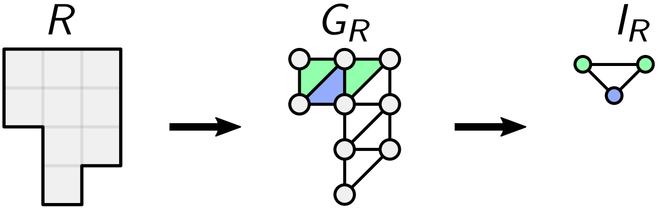

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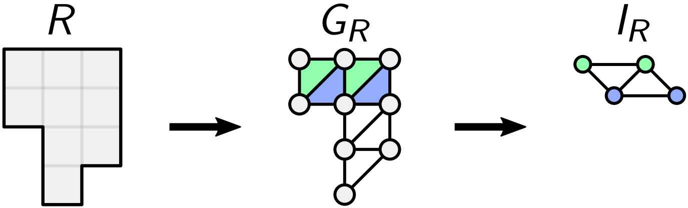

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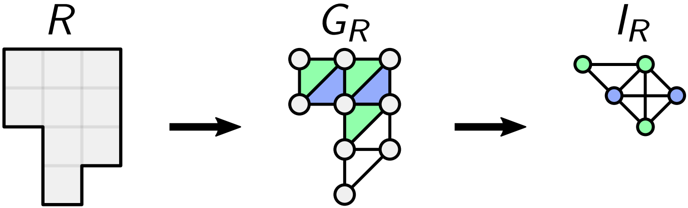

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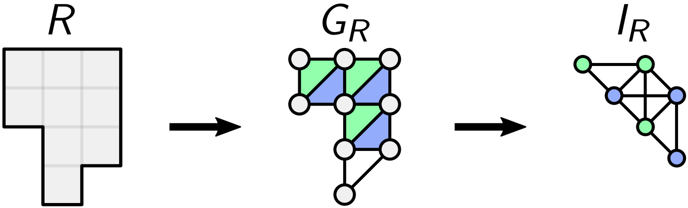

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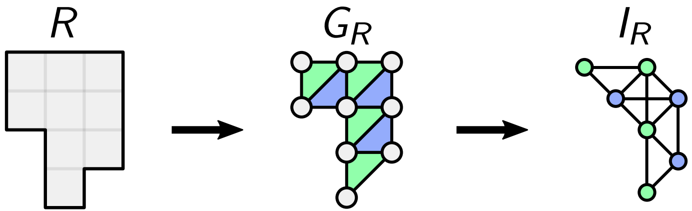

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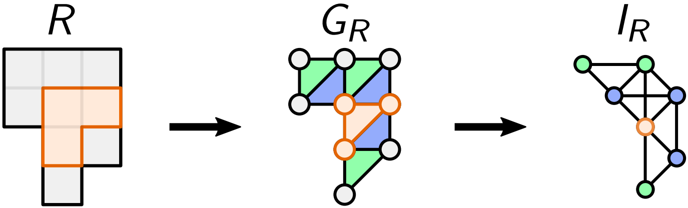

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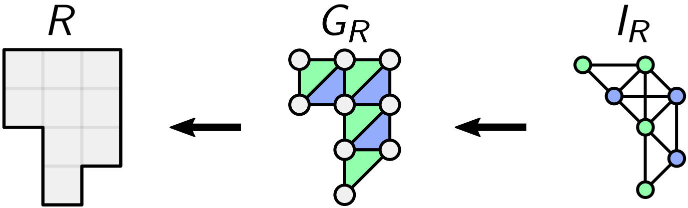

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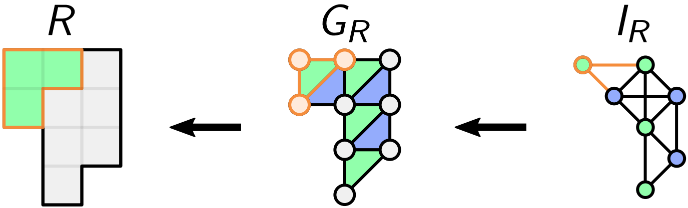

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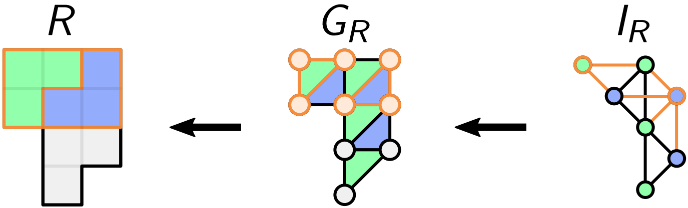

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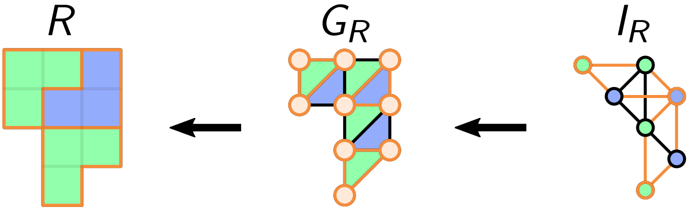

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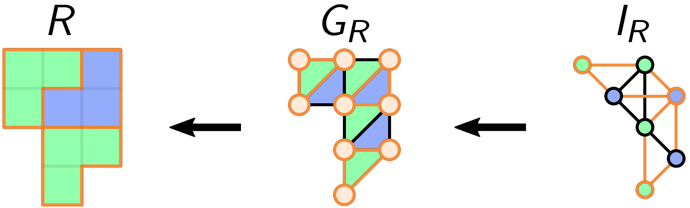

Theorem

Maximum Independent Set of I_R is equal to $\frac{|R|}{3}$
$\iff$ R has a *180-tromino tiling* .

where $|R|$ the number of cells in a region R.

Forbidden Polyominoes (cont'd)

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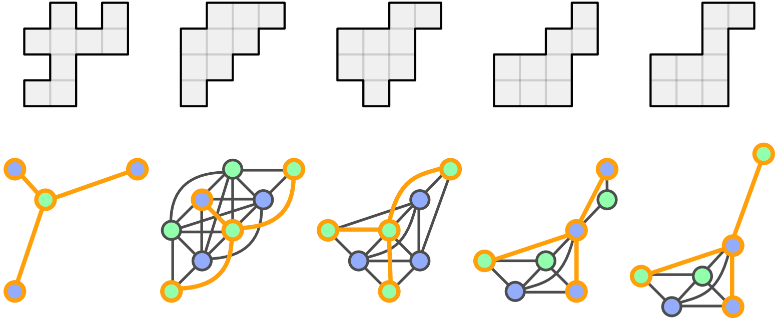

Forbidden Polyominoes (cont'd)

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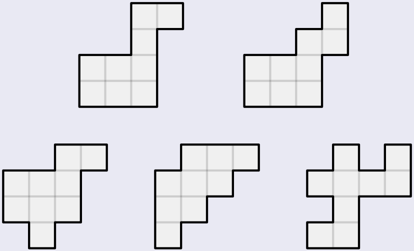
Theorem

If a region R ***doesn't*** *contains a rotated, reflected or sheared* ***forbidden polyomino****, then 180-tromino tiling can be computed in a polynomial time.*



Open Problems

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1. *Hardness of tiling the Aztec rectangle with a given number of defects.* We saw that an Aztec rectangle with 0 or 1 defects can be covered with L-trominoes in polynomial time, whereas in general the problem is NP-complete when the Aztec rectangle has an unknown number of defects. It is open if there exists a polynomial time algorithm for deciding a tiling for an Aztec rectangle with a given number of defects.

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2. *Tiling of orthogonally-convex regions.* In this work we showed several instances where a tiling can be found in polynomial time. In general, it is open if an orthogonally-convex region with no defects can be covered in polynomial time or if it is NP-complete to decide if a tiling exists.

Enumeration of tilings

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It appears that these bound can be improved substantially.

Thank you!

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You can try the tetrasected cell tiling program in your phone browser: http://bit.ly/TetrasectedTiling