Online Weekly Seminar

for Early Career Mathematicians from India

Gromov Compactness

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Plan of talk

1. Preliminaries
2. Moduli spaces
3. Energy identity
4. Application of Arzelà-Ascoli
5. Concentration of energy
6. Compactness theorem

1. Preliminaries

(X^{2n}, ω, J)

almost Kähler manifold

X : closed oriented C^∞ mfld ($\dim X = 2n$)

$\omega \in \Omega^2(X)$, $d\omega = 0$, $\omega^n > 0$

symplectic form

$J \in \mathrm{End}(TX)$ s.t. $J^2 = -1$

almost cx. str.

& $\omega(\cdot, J\cdot)$ is a metric

compatible w/ ω

(Σ^2, j)

closed Riemann surface

Σ : closed oriented C^∞ mfld

$\dim \Sigma = 2$

closed surface

$j \in \mathrm{End}(T\Sigma)$, $j^2 = -1$

hol'c structure

$u : \Sigma \longrightarrow X$ — u is a C^∞ map

J-hol'c map — du is $\mathbb{C}$-linear, i.e.,

$$J \circ du = du \circ j$$

Cauchy-Riemann eq$^{\underline{n}}$

<u>Examples</u>:

<u>A</u> Consider $(\mathbb{P}^n, \omega_{FS}, J_{std})$ and smooth algebraic curves $\Sigma \hookrightarrow \mathbb{P}^n$.

<u>B</u> Can replace $\mathbb{P}^n$ by any compact complex submanifold $X \subseteq \mathbb{P}^n$

sm. projective variety

2. Moduli spaces of J-hol'c curves

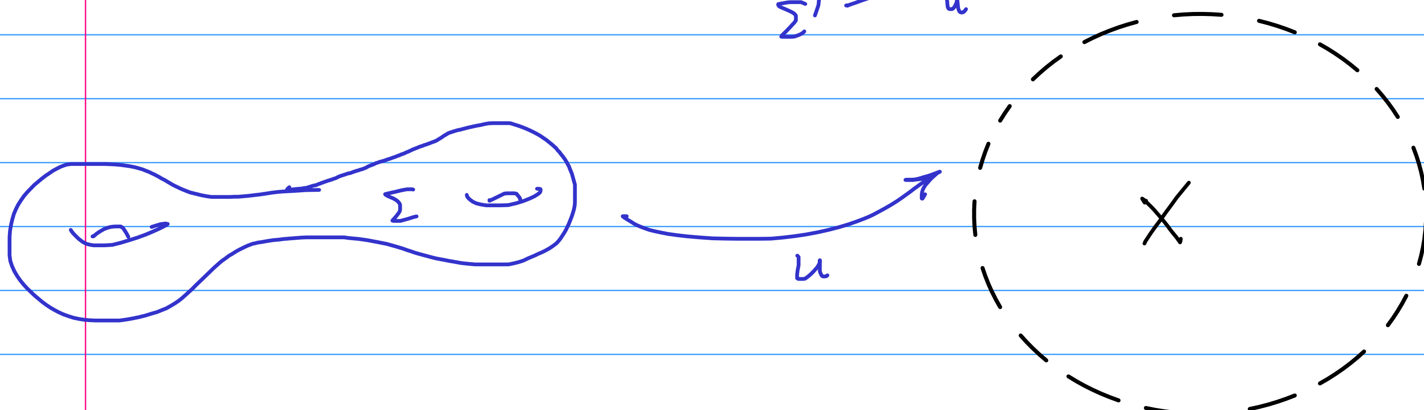
Given $g \geq 0$, $\beta \in H_2(X,\mathbb{Z})$, consider

$$\mathcal{M}_g(X,J,\beta) = \left\{ (\Sigma,j,u) \;\middle|\; \begin{array}{l} (\Sigma,j) \text{ a closed Riem. surf. of genus } g \\ u \text{ J-hol'c map, } u_*[\Sigma]=\beta \end{array} \right\} \Big/ \sim$$

"moduli space of genus g hol'c curves in class β"

where $(\Sigma, j, u) \sim (\Sigma', j', u')$ iff $\exists$ hol'c isomorphism $\varphi : (\Sigma, j) \xrightarrow{\cong} (\Sigma', j')$ s.t.

$$\begin{tikzcd} \Sigma \arrow[rr, "u"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & & X \\ \Sigma' \arrow[urr, "u'"'] & & \end{tikzcd}$$



Such moduli spaces are of interest in algebraic & symplectic geometry as they can be used to construct interesting invariants.

Examples

A Take $X = \mathbb{P}^2$, $g = 0$, $\beta = d[\mathbb{P}^1]$ with $d \geq 1$.

$d=1$: lines in $\mathbb{P}^2$ $\quad \mathcal{M}_0(\mathbb{P}^2, [\mathbb{P}^1]) \cong \mathbb{P}^2$ dual projective space

$d=2$: conics in $\mathbb{P}^2$

general d:
$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ [x:y] &\longmapsto [F_0(x,y) : F_1(x,y) : F_2(x,y)] \end{aligned}$$

w/ F_0, F_1, F_2 homogeneous of degree d

Here, $\dim_{\mathbb{C}} \mathcal{M}_0(\mathbb{P}^2, d[\mathbb{P}^1]) = 3(d+1) - 1 - 3 = 3d-1$

each F_i is in a $(d+1)$-dim'l space; overall rescaling; reparam. by $PSL_2\mathbb{C}$

Examples (contd.)

B Take $X = Bl_p \mathbb{P}^2$ where $p = [0:0:1]$

$$\begin{array}{c} X \\ \downarrow \pi \\ \mathbb{P}^2 \end{array}$$

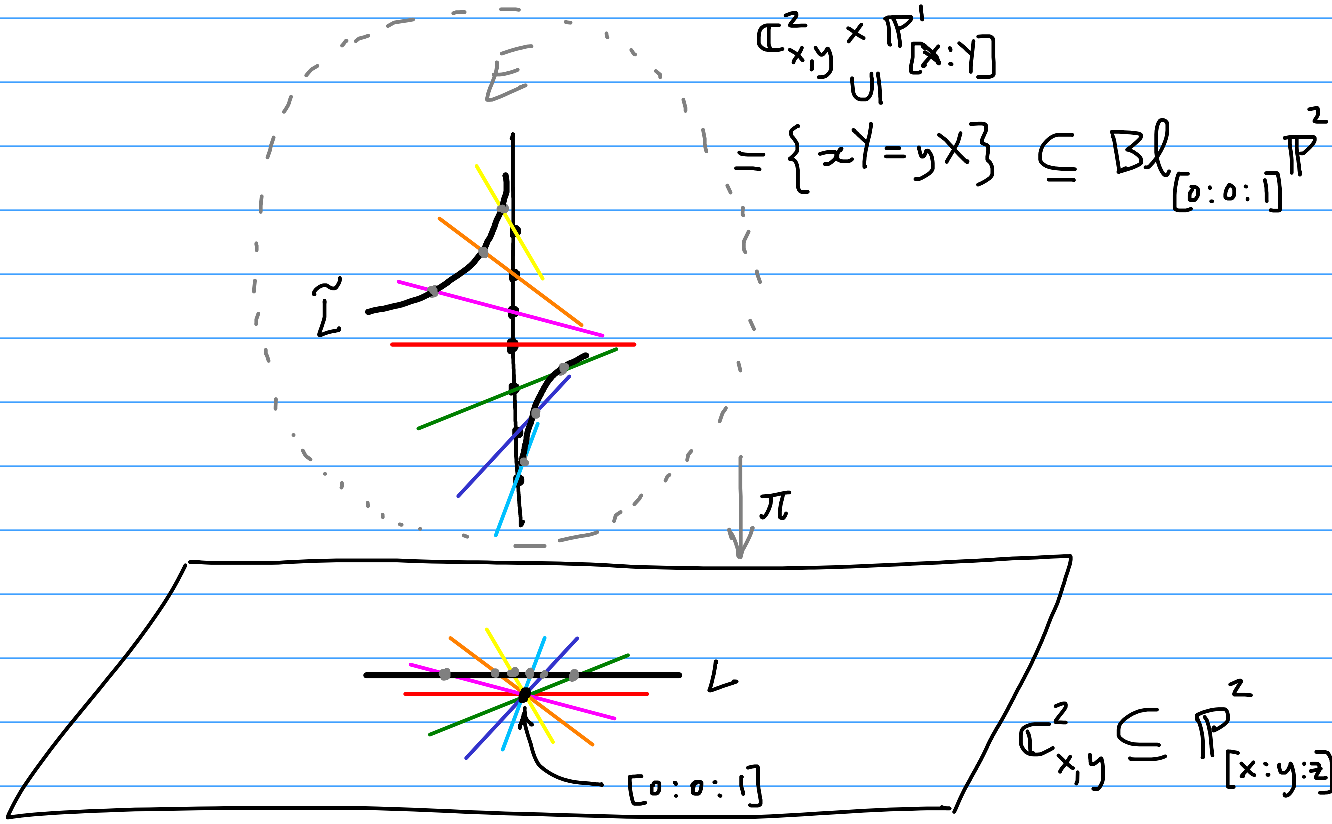
Let $L = [\mathbb{P}^1]$.

$\tilde{L}$ = class of total transform of a general line

E = class of exceptional divisor

Take $g = 0$, $\beta = \tilde{L}$.

Here, $\dim_{\mathbb{C}} \mathcal{M}_0(X, \tilde{L}) = \dim_{\mathbb{C}} \mathcal{M}_0(\mathbb{P}^2, L) = 2$



Note: As L approaches the red line (staying parallel) $\tilde{L}$ does not have a smooth limit.

$\Rightarrow \mathcal{M}_0(X, \tilde{L})$ is not compact.

Thus, we see that :

Moduli spaces of holomorphic maps with fixed genus, homology class and smooth domains are, in general, NON-COMPACT !

The preceding example suggests that :

adding certain (special) kinds of maps with singular (nodal) domains may make the moduli space COMPACT.

3. Energy identity

Given a Riemann surface (Σ, j) and C^∞ Riemannian mfld (Y, g), consider a C^∞ map $u : \Sigma \longrightarrow Y$.

Take any conformal metric h on (Σ, j)

$$z \in \Sigma \rightsquigarrow du(z) : T_z\Sigma \longrightarrow T_{u(z)}Y$$

$$|du(z)|^2_{h,g} := tr\left(du(z)^* du(z)\right)$$

$$\underbrace{e_u(z)}_{\text{energy density form of } u} := \frac{1}{2}|du(z)|^2_{h,g} \underbrace{dA_h(z)}_{\text{area form associated to } h} \in \Lambda^2 T_z^*\Sigma$$

this is independent of $h(z)$

$$\underbrace{E(u)}_{\text{energy of the map } u} := \int_\Sigma e_u \in [0, \infty]$$

energy identity

Lemma: Assume $Y = X$ and $g = \omega(\cdot, J\cdot)$.
Then, we have $u^*\omega \leq e_u$ pointwise,
with equality at $z \in \Sigma$ iff $du(z)$ is $\mathbb{C}$-linear

In particular, if u is J-holomorphic and Σ is compact, then

$$E(u) = \int_\Sigma u^*\omega = \langle [\omega], u_*[\Sigma] \rangle$$

Proof: Let $z = s + it$ be a local hol'c coord on Σ and $h = ds^2 + dt^2$

$$du = \partial_s u \otimes ds + \partial_t u \otimes dt$$

$$\begin{aligned}
|du|^2_{h,g} &= |\partial_s u|_g^2 + |\partial_t u|_g^2 \\
&= \omega(\partial_s u, J\partial_s u) + \omega(\partial_t u, J\partial_t u) \\
&= \omega(\partial_s u, \partial_t u) + \omega(\partial_s u, J\partial_s u - \partial_t u) \\
&\quad + \omega(\partial_t u, -\partial_s u) + \omega(\partial_t u, J(\partial_t u - J\partial_s u)) \\
&= \underbrace{2\omega(\partial_s u, \partial_t u)}_{(u^*\omega)(\partial_s, \partial_t)} + |\partial_t u - J\partial_s u|_g^2
\end{aligned}$$

$\square$

4. Gradient bounds & Arzelà-Ascoli

The equation $J \circ du = du \circ j$ is a perturbation of the std. Cauchy Riemann eqn (an elliptic PDE).

$$\Downarrow \text{"std. analysis"}$$

(☆) (local) C^1 bounds on u $\Rightarrow$ (local) C^∞ bounds on u

We also have the following important fact:

MEAN VALUE ESTIMATE

$$\left\{ \begin{array}{l} u : B_r(0) \longrightarrow X \\ J\text{-hol'c}, \ E(u) \leq \hbar \end{array} \right. \Rightarrow \quad r^2 |du(0)|^2 \leq C \cdot E(u)$$

$\hbar$: positive absolute constt dep. on X

C: universal constt > 0

Arzelà-Ascoli + (☆) $\Longrightarrow$

If Σ is a (not necessarily compact) Riemann surface and $u_n : \Sigma \longrightarrow X$ is a sequence of J-hol'c maps s.t.

$$\forall K \Subset \Sigma, \qquad \sup_n \sup_K |du_n| < +\infty$$

Then $\exists$ a subsequence converging to a J-hol'c $u : \Sigma \longrightarrow X$ in C^∞_{loc} (i.e. with all derivatives, unif. on cpt. sets.)

5. <u>Concentration of energy</u>

Consider a sequence $u_n : \Sigma \longrightarrow X$ of J-hol'c maps with

$$E_0 = \sup_n E(u_n) < +\infty$$

Is this sequence pre-compact?

Suppose NOT. Then, $\exists$

(1) a compact $K \subseteq \Sigma$ &

(2) a subsequence (still u_n) s.t.

$$\sup_n \sup_K |du_n| \to +\infty$$

in fact, we may pass to a further subsequence to get $K \ni z_n \to z_0 \in K$ s.t.

$$|du_n(z_n)| \to +\infty \qquad (\star)$$

energy concentration

Claim: $\lim_{\varepsilon \to 0} \liminf_{n\to\infty} E(u_n, B_\varepsilon(z_0)) \geq \hbar$

Proof: Fix any $0 < \varepsilon \ll 1$. Then $B_{2\varepsilon}(z_0) \supseteq B_\varepsilon(z_n)$ $\forall n \gg 1$

Now, $E(u_n, B_\varepsilon(z_n)) \leq \hbar$

$$\xRightarrow[\text{MEAN VAL. INEQ.}]{} \quad |du_n(z_n)|^2 \leq \frac{C\hbar}{\varepsilon^2} \quad \forall n \gg 1$$

contradicting $|du_n(z_n)| \longrightarrow +\infty$

$\boxtimes$

We now arrive at the following statement.

Convergence modulo bubbling

<u>Thm</u>. Let $u_n : \Sigma \longrightarrow X$ be a sequence of J-hol'c curves with

$$\sup_n E(u_n) = E_0 < +\infty .$$

Then, $\exists$ a finite subset $\Gamma \subseteq \Sigma$ and a subseq, still denoted u_n, s.t.

(1) $|\Gamma| \leq \lfloor E_0/\hbar \rfloor$

(2) $u_n \longrightarrow u$, a J-hol'c map, in $C^\infty_{loc}(\Sigma \setminus \Gamma)$

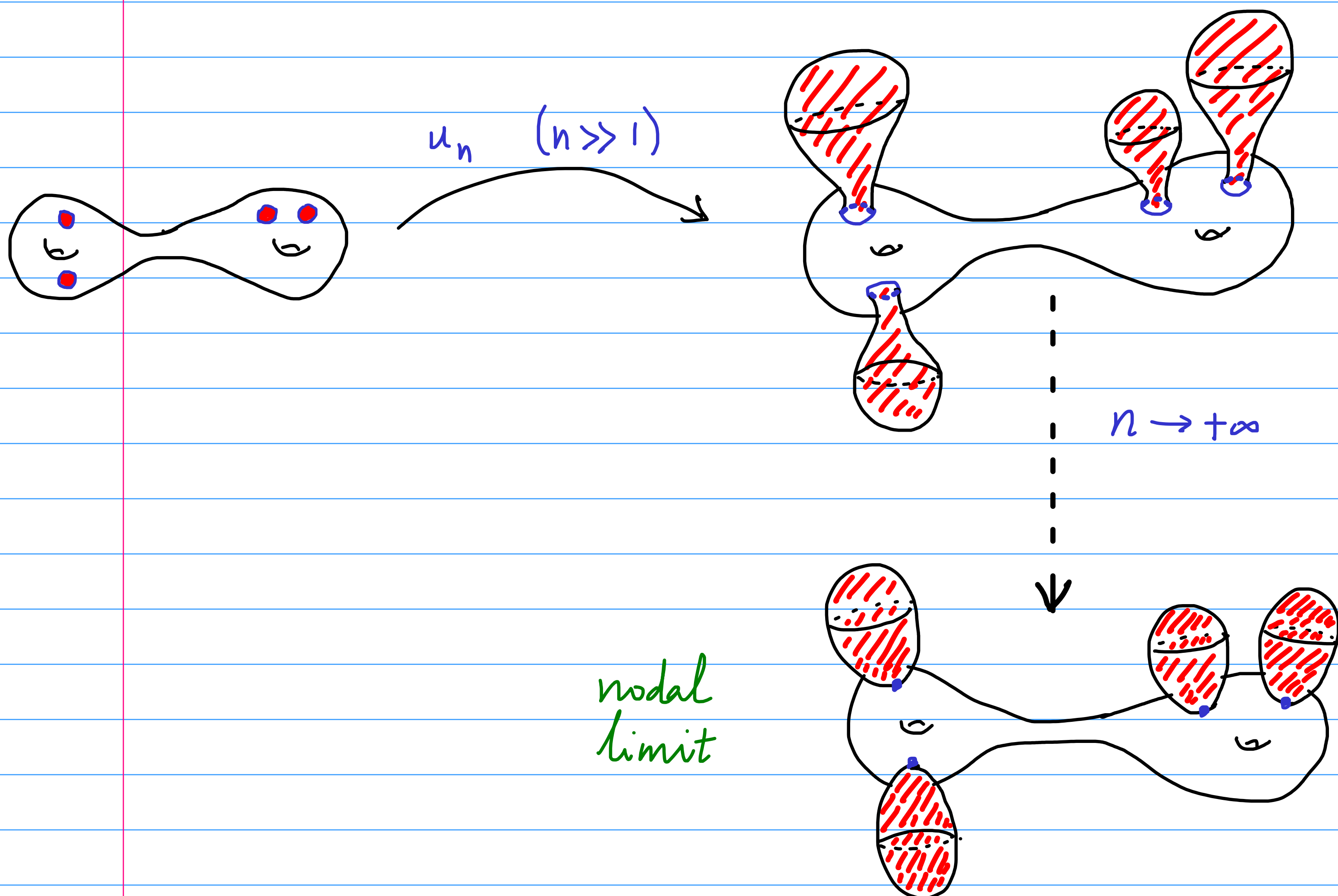
(3) $E(u) \leq E_0$

(4) $\forall z \in \Gamma$, the limit

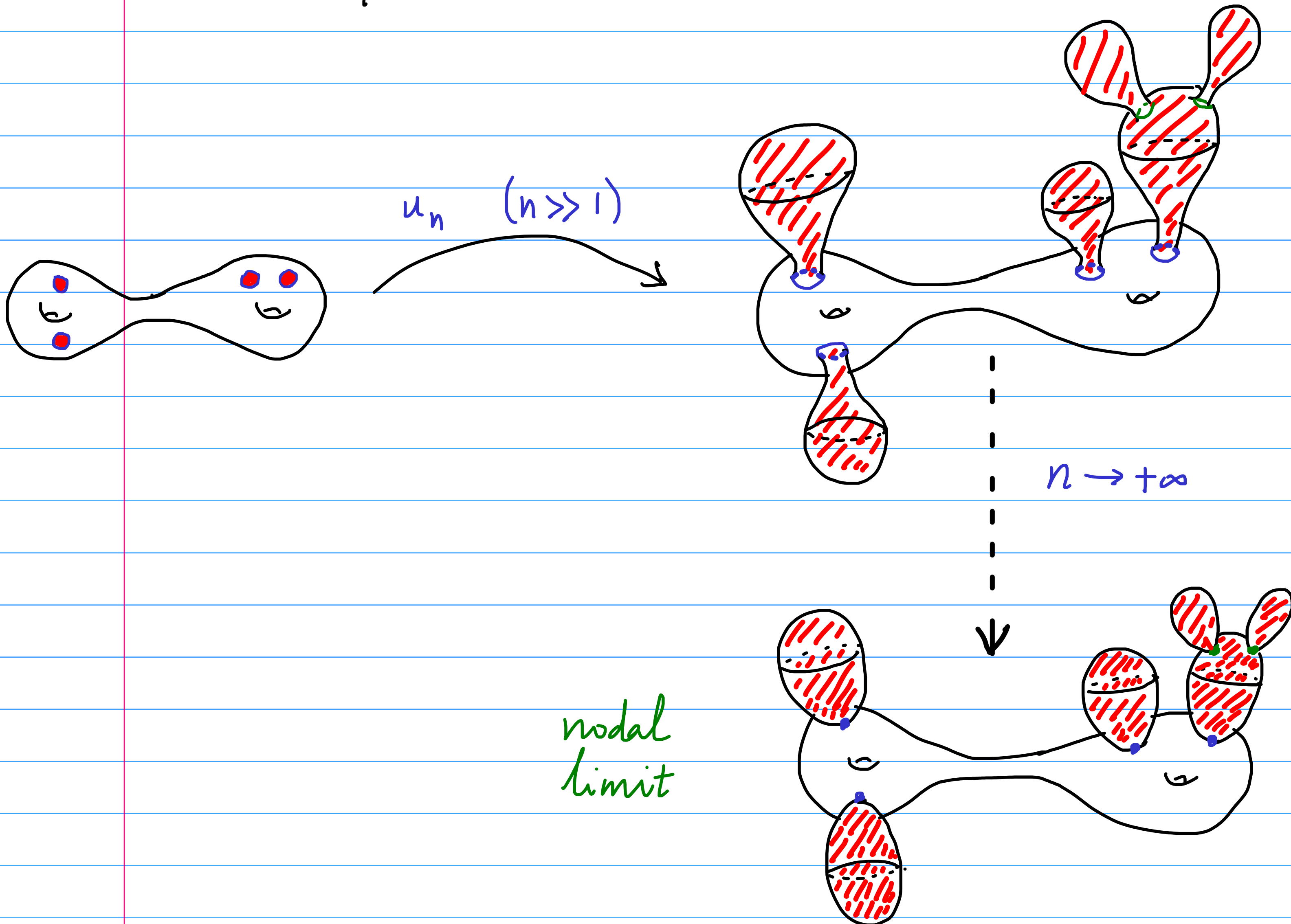
$$m_z = \lim_{\varepsilon \to 0} \lim_{n \to \infty} E(u, B_\varepsilon(z))$$

exists and is $\geq \hbar$

What happens at the points of Γ ?

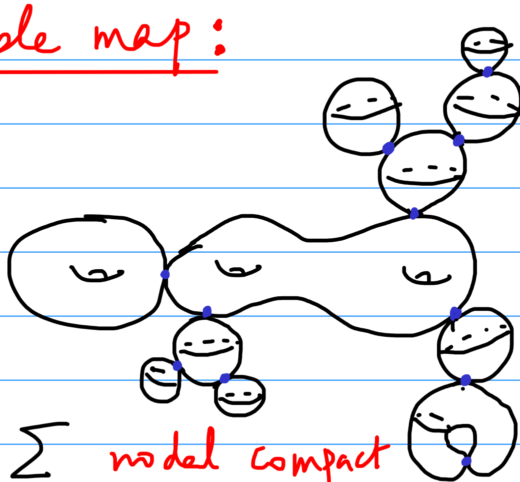


More complicated possibilities...



6. Stable maps & Gromov compactness

Stable map:



Σ nodal compact Riemann surface $\xrightarrow{u \quad C^\infty \text{ map}} X$

s.t. each for each irr comp $\Sigma' \subseteq \Sigma$, either $\int_{\Sigma'} u^*\omega > 0$

or (the normalization of) Σ' has atleast

(a) 1 (if $g_{\Sigma'} = 1$)

(b) 3 (if $g_{\Sigma'} = 0$)

geom. genus

(pre-images of) nodal points

Given $g \geqslant 0$ and $\beta \in H_2(X, \mathbb{Z})$, define

$$\overline{\mathcal{M}}_g(X, J, \beta) = \left\{ (\Sigma, j, u) \;\middle|\; \begin{array}{c} (\Sigma, j) \text{ nodal cpt Riem surf} \\ \text{of arith genus } g \text{ s.t.} \\ \text{w/ } u \text{ it's stable, } J\text{-hol'c} \\ \text{and } u_*[\Sigma] = \beta \end{array} \right\} \Big/ \text{iso}$$

"Moduli space of stable genus g J-hol'c maps to X in class β"

Thm (Gromov, Kontsevich)

$$\bar{\mathcal{M}}_g(X, J, \beta) \text{ is compact and Hausdorff.}$$

Idea: Start with a sequence (Σ_n, j_n, u_n), WLOG Σ_n smooth

$$(u_n)_*[\Sigma_n] = \beta \xRightarrow{\text{energy identity}} E(u_n) = \langle [\omega], \beta \rangle = \text{constant indep of } n \quad (\star)$$

Use Deligne-Mumford compactness (compactness of $\bar{\mathcal{M}}_g$) to extract a subsequence $(\Sigma_n, j_n) \longrightarrow (\Sigma', j')$.

Now, we are (almost) in the fixed domain case & we may proceed as before using the energy bound $(\star)$ and the energy concentration argument.

Thank You!