Online Seminar of Assamese Mathematicians

An approach to construct Mathematical model through system of ordinary differential equation

An invited talk
by

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Outline of the presentation

- Mathematical modeling
- Utility of Mathematical Model
- Types of Mathematical Model
- Basic steps in Modeling
- Basic Definitions and Tools
- Prey-predator mathematical model: An introduction
- Example: Effect of habitat complexity on rhinocers and tiger population model with additional food and Poaching on rhinoceros : An application to Kaziranga National Park, Assam
- Discussion & Conclusion
- Reference

Mathematical modeling

- It is the process of using mathematics to solve real- world problems.
- Mathematical models are basically a simplified description of a system, built to help us better understand the operation of a real system and the interactions of its main components.
- Mathematical models are collections of variables, equations, and starting values that form a cohesive representation of a process or behavior.
- The most important part of modeling is to make sure that the concerned mathematical model can exhibit the well - known system behaviors for the system under consideration.

Utility of Mathematical Model

- Interactions among the members of biological communities and components of the abiotic environment are extremely complex, mathematical models are useful for thorough understanding how ecosystems function and for making predictions about managing ecosystems.
- Provide a way to design and evaluate protocols to manage and control animal populations, natural resources (e.g., forests), wildlife resources (e.g., fisheries, deer, tiger population), and infectious diseases.
- Researchers on animal conservation can make use of the models to aid the accomplishment of field experiments, through the indication of the parameters, which should be observed.

Types of Mathematical Model

- Deterministic or Stochastic models
- Linear or Non-Linear
- Static or Dynamic
- Discrete or Continuous

Basic steps in Modeling

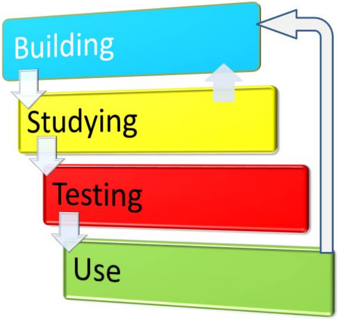

Figure 1: Basic steps in Modeling

Basic definitions and Tools

Basic Definitions

Ecological interactions: Ecological interactions are the relationships between two species in an ecosystem . The interactions can be categorized into many different classes, such as:

- **Predation** is a biological interaction in which one species feeds on another. Most of the interactions in a food web are predatory. This interaction enhances the fitness of predators, but reduces the fitness of the prey species.
- **Competition** between two species occurs when they share a limited resource and each tends to prevent the other from accessing it. This reduces the fitness of one or both species.

Functional Responses: It is the relationship between an individual's rate of consumption and food density. It describes the way a predator responds to the changing density of its prey.

Holling type II functional response:

$$f(N) = \frac{bN}{1 + cN}$$

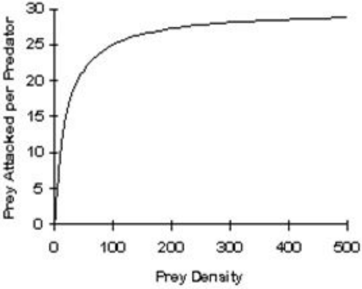

Figure 2: Holling type-II functional response

where b and c are positive constants that describe the effects of capture rate and handling time on the feeding rate of the predator. The number of prey that a predator can consume is limited and consequently the predator reaches a saturation level. Predators of this type cause maximum mortality at low prey density.

Equlibrium points: Equlibrium point is a state/ solution in which the system does not change with time, in particular the state variable remain constant.

- An equilibrium point of a system is said to be **hyperbolic** if all eigenvalues of the Jacobian matrix evaluated at the point have non-zero real parts.
- If at least one eigenvalue of the Jacobian matrix is zero or has a zero real part, then the equilibrium is said to be **non-hyperbolic**.

Stability of an Equilibrium point: Stability (of ecosystem) refers to the capability of a natural system to apply self-regulating mechanisms so as to return to a steady state after an outside disturbance.

Local stability indicates that a system is stable over small short-lived disturbances, while global stability indicates that the system is highly resistant to changes in species composition and/ or food web dynamics.

Stable/ Lyapunov stable, Unstable, Asymptotically stable: Consider an autonomous nonlinear dynamical system $\dot{x} = f(x(t))$, $x(0) = 0$, where $x(t) \epsilon D \subseteq R^n$ denotes the system state vector, D denotes an open set containing the origin and $f : D \rightarrow R^n$ continous on D. Suppose f has an equilibrium at $\bar{x}$ then the equlibrium point $x = \bar{x}$ is

- **Stable/Lyapunov stable**, if for $\epsilon > 0$, $\exists\ \delta > 0$ such that $\|x(0) - \bar{x}\| < \delta$, then for every $t \geq 0$ we have $\|x(t) - \bar{x}\| < \epsilon$.
- **Unstable**, if it is not stable.
- **Asymptotically stable** if it is Lyapunov stable and $\exists\ \delta > 0$ such that $\|x(0) - \bar{x}\| < \delta$, then $lim_{t \rightarrow \infty} x(t) = \bar{x}$.

Limit Cycle : A limit cycle is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed; they spiral towards or away from the limit cycle.

A limit cycle is

- stable, if for all x in some neighbourhood; the nearby trajectories are attracted to the limit cycle,
- unstable, if for all x in some neighbourhood; the nearby trajectories are repelled from the limit cycle,
- semistable, if it is attracted on one side and repelled on the other.

Bifurcation: The term bifurcation is commonly used in the study of nonlinear dynamics to describe any sudden change in the behavior of the system as some parameter is varied. At a point of bifurcation, stability may be gained or loss. The bifurcation then refers to the splitting of the behavior of the system into two regions: one above, the other below the particular parameter value at which the change occurs.

Andronov-Hopf bifurcation: It is the appearance or disappearance of a limit cycle from an equilibrium in dynamical systems generated by ODEs, when the equilibrium changes stability via a pair of purely imaginary eigenvalues. Hopf bifurcation is of two types:

1. Supercritical Hopf: bifurcating periodic orbit is stable.
2. Subcritical Hopf: bifurcating periodic orbit is unstable.

The method of Characteristic roots:

1. Routh- Hurwitz criteria
2. Descarte's rule of sign

Lyapunov direct method: Lyapunov's direct method consists in finding Lyapunov function. The major role in this process is played by positive or negative definite functions.

Lyapunov function: Let $D \subseteq R^n$ be an open neighbourhood of the equilibrium point x_e of a system $\dot{\bar{x}} = f(\bar{x})$, then the function $L : D \to R$, satisfying the following properties.

1. L is continously differentiable,
2. $L > 0 \ \forall \ \bar{x} \in D - \{x_e\}$ and $L(x_e) = 0$ is called Lyapunov function.

Routh-Hurwitz Criteria

Let the constant $a_1, a_2, a_3 ...., a_n$ be real numbers. The equation

$$L(\lambda) = \lambda^n + a_1\lambda^{n-1} + ------------------ + a_n = 0 \quad (1)$$

has roots with negative real parts iff the values of determinant of the following matrices

$$H_1 = (a_1), \quad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, ...,$$

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & -- & 0 \\ a_3 & a_2 & a_1 & -- & 0 \\ -- & -- & -- & -- & 0 \\ -- & -- & -- & -- & -- \\ 0 & 0 & 0 & -- & a_n \end{bmatrix} \quad \text{are all positive}$$

Here (l, m) entry in the matrix H_j is

$$\begin{cases} a_{2l-m} & \text{for} \quad 0 < 2l - m < n \\ 1 & \text{for} \quad 2l = m \\ 0 & \text{for} \quad 2l < m \text{ or } 2l > n + m \end{cases}$$

In particular, for quadratic and cubic polynomials these condition reduce to

$$\begin{aligned} &\text{i)} \quad a_1 > 0, \quad a_2 > 0 \quad \text{and} \\ &\text{ii)} \quad a_1 > 0, a_3 > 0, \quad a_1 a_2 > a_3 \end{aligned} \tag{2}$$

respectively (Kot, 2001).

Descartes' Rule of sign

The characteristic polynomial of nth order can be taken in the form

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + - - - - - - - - - - - - +a_0, \qquad (3)$$

where the coefficient a_i, $i = 0, 1, 2......n$ are all real and $a_n > 0$. Let N be the number of sign changes in the sequence of coefficient$\{a_n, a_{n-1}, ..., a_0\}$, ignoring any which is zero. Descartes's rule of signs says that there are at most N roots of (3) which are real and positive, further, that there are N, $N-2$ or $N-4$,........ real positive roots. By setting $\omega = -\lambda$ and again applying the rule, information is obtained about the possible real negative roots. Together these often give valuable information on the sign of all the roots, which from a stability point of view is usually all we require (Murray, 1989).

Sylvester Criterion

Let us consider an autonomous differential system of the form

$$\dot{x} = f(x) \tag{4}$$

where, $f \in C\left[R^n, R^n\right]$ and assuming that f is smooth enough to ensure the existence and uniqueness of the solution of (4). Let $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$ in some neighbourhood of the origin so that (4) admits the so-called zero solution ($x = 0$) and the origin is an isolated critical point of (4).

Let,

$$V(x) = x^T Bx = \sum_{i,j=1}^{n} b_{ij} x_i x_j, \tag{5}$$

be a quadratic form with the symmetric matrix $B = (b_{ij})$, that is $b_{ij} = b_{ji}$.

The necessary and sufficient condition for $V(x)$ to be positive definite is that the determinant of all the successive principal minors of the symmetric matrix $B = (b_{ij})$ be positive, that is,

$$b_{11} > 0, \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\begin{vmatrix} b_{11} & b_{12} & -- & b_{1n} \\ b_{21} & b_{22} & -- & b_{2n} \\ -- & -- & -- & -- \\ b_{n1} & b_{n2} & -- & b_{nn} \end{vmatrix} > 0.$$

Let S_ρ be a set $S_\rho = \{x \epsilon R^n : \|x\| < \rho\}$ and let $R^+ = [0, \infty)$ and $J = [t_0, \infty)$, $t_0 \geq 0$. Suppose $x(t) = x(t, t_0, x_0)$ is any solution of (4) with the initial value $x(t_0) = x_0$ such that $\|x\| < \rho$ for $t \epsilon J$. Also, since equation (4) is autonomous, we can further suppose, without any loss of generality, that $t_0 = 0$.

Theorem

If there exists a positive definite scalar function $V(x)$ such that $\dot{V}(x) \leq 0$ on S_ρ, then the zero solution of (4) is stable.

Theorem

If there exists a positive definite scalar function $V(x)$ such that $\dot{V}(x)$ is negative on S_ρ, then the zero solution of (4) is asymptotically stable.

Theorem

If there exists a scalar function $V(x)$, $V(0) = 0$, such that $V(x)$ is positive definite on S_ρ and if in every neighbourhood N of the origin, $N \subset S_\rho$, there is a point x_0 where $\dot{V}(x_0) > 0$, then the zero solution of (4) is unstable.

where $S_\rho = \{x \epsilon R^n : \|x\| < \rho\}$.

Prey-Predator Mathematical Model : An introduction

- Lotka-Volterra model is one of the oldest and the simplest model of predator-prey interactions. The model was developed independently by Lotka (1925) and Volterra (1926). The Lotka-Volterra equations, also known as the predator –prey equations, are a pair of first-order, non –linear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. It is the basis of many models used present days in the analysis of population dynamics & is one of the popular models in mathematical ecology.

- The populations change through time according to the following pair of equations:

$$\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= -cy + dxy \end{aligned} \tag{6}$$

where, x is the number of prey (for example: rabbits), y is the number of some predators (for example: foxes) $\dot{x}$ and $\dot{y}$ represent the growth rates of the two populations over time, t represent time, a is the growth rate of prey, b is the searching efficiency or attack rate, c is the predator mortality rate and d is the growth rate of predator or predator's ability of turning food into offspring.

- Predators and prey can influence one another's evolution. Traits that enhance a predator's ability to find and capture prey will be selected for in the predator, while traits that enhance the prey's ability to avoid being eaten will be selected for in the prey. The "goals" of these traits are not compatible, and it is the interaction of these selective pressures that influences the dynamics of the predator and prey populations. Predicting the outcome of species interactions is also of interest to biologists trying to understand how communities are structured and sustained.

A General Predator prey model

The general form for a mathematical model that describes the dynamics between any two species having prey-predator interaction, has the following structure.(Bairagi & Jana (2011))

$$\begin{aligned} \frac{dx}{dt} &= xf(x) - p(x,y) \\ \frac{dy}{dt} &= \theta p(x,y) - dy \end{aligned} \tag{7}$$

where $x(t)$ and $y(t)$ represent the prey and predator population density respectively at a given time t respectively. $f(x)$ is the per capita growth rate of prey in absence of predator and d is the food-independent predator mortality rate. $p(x,y)$ is the functional response of predator, which is defined as the number of prey caught per predator per unit of time. The term $\theta p(x,y)$ is known as the numerical response measuring the number of newly born predators for each captured prey and $\theta(0 < \theta < 1)$ is the conversion efficiency.

Example: Effect of habitat complexity on rhinocers and tiger population model with additional food and Poaching on rhinoceros : An application to Kaziranga National Park, Assam

Ecosystem of Kaziranga National Park – Importance and threats

Importance

Kaziranga national park is an important landmark in the world for holding around 85% of the worlds One horned rhinocerous.

- It is a home to high degrees of diversified species with great visibility.
- Compared with other protected areas in India, Kaziranga has achieved notable success in wildlife conservation.

Threats

Threats to wildlife of Kaziranga National Park can be summed up as follows:

- Poaching
- Annual flood
- Erosion
- Siltation and weeds
- Illegal fishing
- Heavy traffic
- Live stock grazing
- Breach in embankments
- Wildfires
- Insect attack and pathological problems

Field Data

Some of the data collected from PCCF Assam, regarding tiger, rhino and their interactions are shown in the figures.

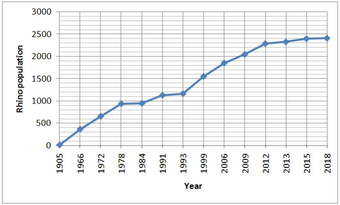

Figure 3: Rhino population in KNP since 1905

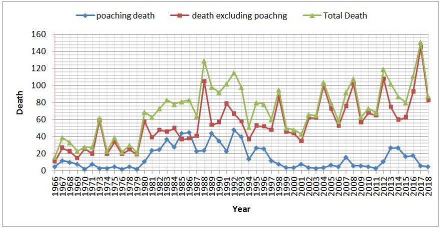

Figure 4: Total Rhino death in KNP due to poaching and natural causes

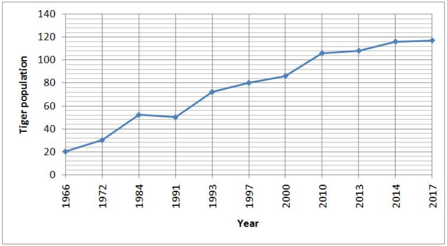

Figure 5: Tiger population in KNP

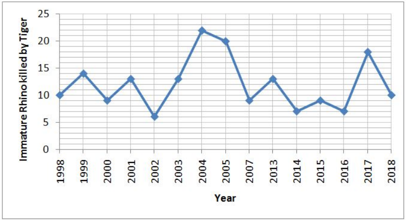

Figure 6: Immature Rhino killed by Tiger in KNP

Mathematical Model I

Assumptions

Based on the discussion on field data on KNP, some assumptions and other factors of importance for the age structured prey-predator model are given below:

- The birth rate of the immature rhino population is proportional to the existing mature rhino population with a proportionality constant a_1.
- For the immature population, the death rate and transformation rate to mature are proportional to the existing immature population with proportionality constants d_1 and r_1 respectively.
- The natural death rate of mature prey is assumed to be d_2.
- The human poaching rate on mature prey is P.
- According to the Forest Department of Assam there is no record of death due to in-fight between mature and immature rhino, so intra-specific competition between immature and mature rhino is not considered here.

- Intra specific interference among mature rhinos is ρ.
- Coefficient $K \quad (0 < K < 1)$ is the conversion efficiency, measuring the number of newly born predators for each captured prey.
- Tiger as predator population do not kill mature rhino for food easily as for tiger other herbivores like swamp deer, buffalo are also available in good number. Only 4 such cases found during 2013-2018 [Source: PCCF Wildlife, Assam]; therefore in the model tiger and mature rhino interaction is not considered.
- From the data obtained from PCCF Wildlife, Assam

$$a_1 r_1 > (d_1 + r_1)(d_2 + P) \tag{8}$$

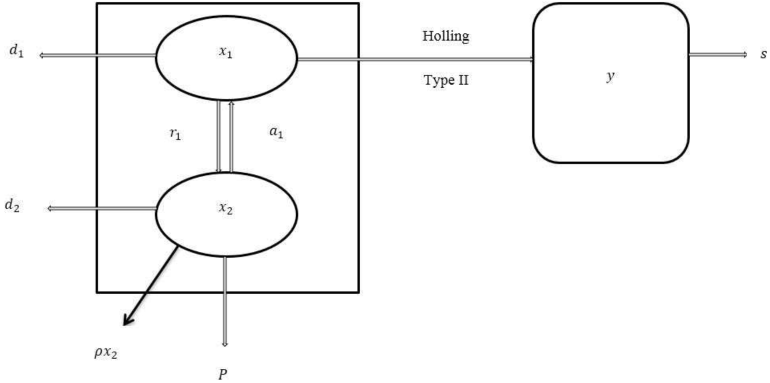

Figure 7: Schematic diagram for the model system (9)

Model Equations

The model equations are as follows:

$$\begin{aligned} \frac{dx_1}{dt} &= a_1x_2 - d_1x_1 - \frac{ax_1y}{1+ahx_1} - r_1x_1, \\ \frac{dx_2}{dt} &= r_1x_1 - d_2x_2 - Px_2 - \rho x_2^2, \\ \frac{dy}{dt} &= -sy + K\frac{ax_1y}{1+ahx_1}, \end{aligned} \tag{9}$$

with initial conditions $x_1(0) > 0$, $x_2(0) > 0$, $y(0) > 0$. Here $x_1(t)$, $x_2(t)$ and $y(t)$ denote respectively the immature prey, mature prey and predator population density at time t.

Positivity and Boundedness of solutions

Positive Invariance: The system (9) can be put in the matrix form as

$$\dot{\bar{X}} = F(\bar{X}) \text{ with } \bar{X}(0) = \bar{X}_0 \in R^3_+, \text{ where } \bar{X} = (x_1, x_2, y)^T \in R^3_+,$$

$$F(\bar{X}) = \begin{bmatrix} a_1x_2 - d_1x_1 - \frac{ax_1y}{1+ahx_1} - r_1x_1 \\ r_1x_1 - d_2x_2 - Px_2 - \rho x_2^2 \\ \frac{Kax_1y}{1+ahx_1} - sy \end{bmatrix},$$

$$F(\bar{X})\colon C_+ \to R^3 \text{ and } F \in C^\infty_+(R^3).$$

It can be seen that whenever $\bar{X}(0) \in R^3_+$ such that $X_i = 0$ then $F_i(\bar{X})\Big|_{x_i=0} \geq 0$ for (i=1,2,3). Now any solution of $\dot{\bar{X}} = F(\bar{X})$ with $\bar{X}_0 \in R^3_+$, say $\bar{X}(t) = \bar{X}(t, \bar{X}_0)$ is such that $\bar{X}(t) \in R^3_+$ for all $t > 0$ (Nagumo 1942).

Theorem

The set $\Omega_1 = \{(x_1, x_2, y)\colon 0 < x_1 \leq \frac{a_1^2}{4\rho v},\ 0 < x_2 \leq \frac{a_1^2}{4\rho v},\ \ 0 < y \leq \frac{1}{K}\frac{a_1^2}{4\rho v},\ 0 < x_1(t) + x_2(t) + \frac{y(t)}{K} \leq \frac{a_1^2}{4\rho v}\}$ *attracts all solutions in the interior of the positive orthant, where,* $v = min\{d_1, d_2 + P, \frac{s}{K}\}$.

Existence of Equilibrium points

System (9) possesses the following equilibrium points:

- $E_0(0,0,0)$ always exists.
- $E_1(\bar{x_1}, \bar{x_2}, 0)$ exists under the condition (8), where,

$$\bar{x_1} = \frac{a_1 \bar{x}_2}{d_1 + r_1},$$
$$\bar{x_2} = \frac{a_1 r_1 - (d_2 + P)(d_1 + r_1)}{(d_1 + r_1)\rho}.$$

- $E_2(x_1^*, x_2^*, y^*)$ exists, provided the conditions $a_1 x_2^* > \frac{s(d_1+r_1)}{a(K-sh)}$ and $K > sh$ are where,

$$x_1^* = \frac{s}{a(K - sh)},$$
$$x_2^* = \frac{-(d_2 + P) + \sqrt{(d_2 + P)^2 + \frac{4 r_1 s \rho}{a(K-sh)}}}{2\rho},$$
$$y^* = \frac{K}{s}\left\{a_1 x_2^* - \frac{s(d_1 + r_1)}{a(K - sh)}\right\}.$$

Local Stability Analysis

- $E_0(0,0,0)$ is always a saddle point.
-
  - $E_1(\bar{x}_1, \bar{x}_2, 0)$ is locally asymptotically stable, if, $s > \frac{Ka\bar{x}_1}{1+ah\bar{x}_1}$.
  - Otherwise $E_1(\bar{x}_1, \bar{x}_2, 0)$ unstable.
- The interior equilibrium point $E_2(x_1^*, x_2^*, y^*)$ is locally asymptotically stable iff

$$\left\{d_1 + r_1 + \frac{ay^*}{(1+ahx_1^*)^2}\right\}(d_2 + P + 2\rho x_2^*) > a_1 r_1, \qquad (10)$$

i.e. the birth rate of the immature rhino and transformation rate to mature rhino is less than a threshold value.

Hopf-bifurcation

Existence

Considering poaching effect P as a bifurcation parameter, the necessary and sufficient condition for Hopf-bifurcation to occur are

$$\begin{aligned} i) \quad & A_1(P^*) > 0,\ A_3(P^*) > 0, \\ ii) \quad & f(P^*) = A_1(P^*)A_2(P^*) - A_3(P^*) = 0 \ \text{ and} \\ iii) \quad & Re\left[\frac{d\lambda_j}{dP}\right]_{P=P^*} \neq 0,\ j = 1, 2, 3. \end{aligned} \tag{11}$$

where $P = P^*$ is the critical value of P. A_1, A_2 and A_3 are the coefficients in the characteristic equation of the variational matrix evaluated at $E_2(x_1^*, x_2^*, y^*)$ given by

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{12}$$

with

$$A_1 = d_1 + r_1 + \frac{ay^*}{(1 + ahx_1^*)^2} + d_2 + P + 2\rho x_2^* > 0,$$

$$A_2 = (d_1 + r_1 + \frac{ay^*}{(1 + ahx_1^*)^2})(d_2 + P + 2\rho x_2^*) + \frac{Ka^2 x_1^* y^*}{(1 + ahx_1^*)^3} - a_1 r_1,$$

$$A_3 = \frac{Ka^2 x_1^* y^*}{(1 + ahx_1^*)^3}(d_2 + P + 2\rho x_2^*) > 0,$$

and λ_j is the eigen value of the variational matrix associated with E_2.

Global Stability

In this section the global stability of the system of equations (9) is discussed.

Theorem

If the following inequalities hold in the region Ω_1 :

$$\begin{aligned} a) \quad & (d_1 + r_1)(1 + ahx_1^*) > a^2hx_1^*y^*, \\ b) \quad & 2\left(d_1 + r_1 - \frac{a^2hx_1^*y^*}{1 + ahx_1^*}\right)\left(\frac{4r_1x_1^*v}{x_2^*a_1^2} + 1\right) > \frac{a_1^2}{\rho}, \end{aligned} \tag{13}$$

then the positive equilibrium point E_2 is globally asymptotically stable.

Numerical Simulation

Numerical simulations are carried out in this section to validate the analytical results that were obtained in the previous sections. The parameter poaching rate 'P' is the key parameter that directly influence the dynamics of the system and ecologically balanced behavior of the park. This section is divided into two parts based on:

i. The ecological behavior of the park , and
ii. The complex dynamical behavior of the system

For this some biologically feasible parameters are chosen that satisfies various analytical conditions.

Ecological behavior of KNP

To see the ecological behavior of the park and to observe the effect of key parameter (poaching rate 'P') on ecological balance of KNP, the following set of values of parameters are chosen:

$$\begin{aligned} &a_1 = 45, \quad d_1 = 5, \quad a = 3, \quad h = 0.02, \quad r_1 = 22, \quad \rho = 2, \\ &d_2 = 0.3, \quad P = 1, \quad s = 1, \quad K = 0.05, \end{aligned} \tag{14}$$

with the initial conditions $x_1(0) = 1$, $x_2(0) = 1$ and $y(0) = 1$. These values of parameters satisfy the local stability condition (Equation 3) of the system (2). Thus the positive equilibrium point is locally asymptotically stable. Now, we will concentrate our attention to maintain the ecological balanced behavior of the park.

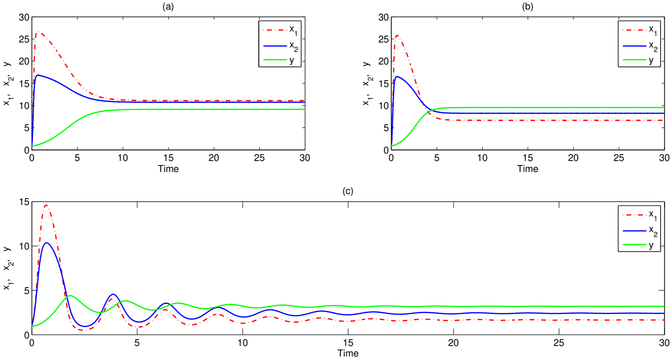

Figure 8: Ecologically balanced and imbalanced behavior of the park w.r.t P

For the set of values of parameters given in equation (14), it is observed that when attacking rate by the tiger population ($a = 3$) and poaching effect ($P = 1$) is less, then the density of rhino population is greater than tiger population (Fig. 8(a)). But if the attacking rate ($a = 5$) increases, then the density of tiger population also increases compared to rhino population (Fig. 8(b)). If the attacking rate as well as poaching effect increase ($a = 20, P = 10$), then the trajectories first oscillate and then go to the respective equilibrium level. Here the density of tiger population is larger than rhino population (Fig. 8(c)). Thus the park is in an ecologically imbalanced situation. To recover from this situation, modification in the model is required so that attacking rate can be minimized and for maintainance of the ecological balance of the park.

Complex dynamical behavior of the system

Stability of the interior equilibrium point

To see the dynamical behavior of the system, the following set of values of the parameters

$$\begin{aligned} &a_1 = 45, \quad d_1 = 5, \quad a = 20, \quad h = 0.02, \quad r_1 = 22, \quad \rho = 2, \\ &d_2 = 0.3, \quad P = 10, \quad s = 1, \quad K = 0.05, \end{aligned} \tag{15}$$

are chosen with the initial conditions (1; 1; 1) and the positive equilibrium point $E_2(1.6668, 2.4214, 3.1980)$ is obtained.

These set of parameters satisfy the local as well as global stability of conditions (10) and (13). The trajectories of x_1, x_2 and y are plotted in Fig (9) and it is observed from Fig. (9(a)) that all the trajectories starting from (1, 1, 1), first oscillate and then go to their respective equilibrium levels. Fig. (9(b)) shows that the trajectory starts from the different initial point converges to the positive equilibrium point. This proves the local and global stability behavior of the system around the positive equilibrium point. It is also observed from Fig. (9(a)) that as the attacking rate by the tiger population on immature rhino is high and due to the effect of poaching, the immature rhino and mature rhino go to the lower level and tiger population goes to higher equilibrium level.

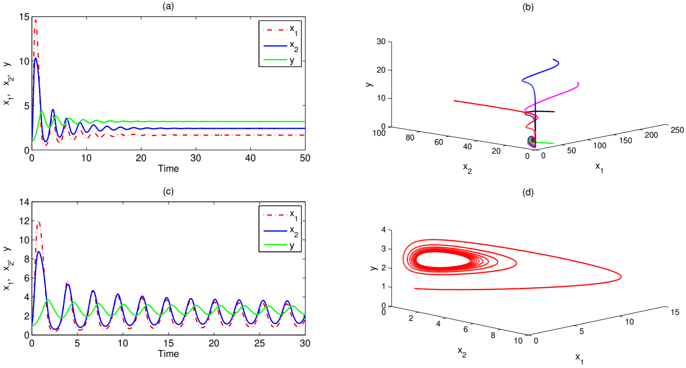

Figure 9: (a) Local (b) Global stability of unique positive equilibrium (c) Oscillation behavior and (d) Limit cycle w.r.t $P = 12.45 > P^* = 12.15$

Graph trajectory of immature rhino, mature rhino and tiger population considering poaching as a bifurcation parameter

The oscillation behavior of the rhino as well as the tiger population and also limit cycle through Hopf bifurcation are observed considering the poaching effect 'P' as a bifurcation parameter. The critical point of poaching is $P = P^* = 12.15$. Thus, considering $P = 12.45 > P^*$ and other values same as (15), it is noticed that the trajectories starting from the inital point (1,1,1), oscillate with respect to time t (Fig. (9(c)) and also limit cycle around the positive equilibrium point (Fig. (9(d)). Thus the positive equilibrium point becomes unstable now. Thus to make it stable it will be better to concentrate the attention to control the poaching effect of the mature rhino.

Again if the attacking rate on the immature rhino by the tiger population increases then the positive equilibrium point E_2 becomes unstable. For the set of values of parameters (15) with initial point (1,1,1) and $a = 25(> a^* = 23.5)$, the system shows oscillation behavior and limit cycle (Fig.10(a-b)). Thus to control the attacking rate on immature rhino by the tiger population, modification is required in the model.

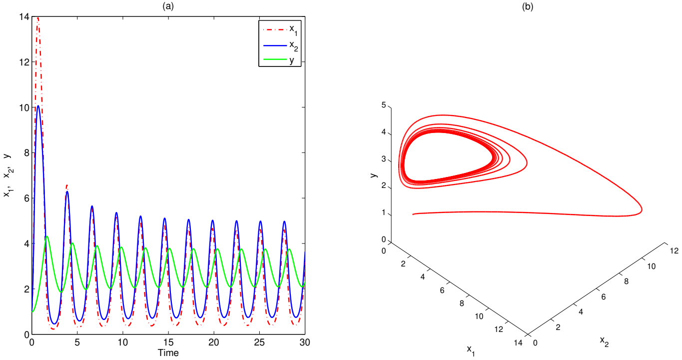

Figure 10: (a) Oscillatory behavior (b) Limit cycle of the system (3.2) w.r.t $a = 25 > 23.5 = a^*$

Discussion & Conclusion

- According to the census report and real data collected from PCCF Wildlife, Assam, on Kaziranga National Park (KNP), a mathematical models is formulated and analyzed to understand the dynamics of rhino (prey) and tiger (natural enemy, the predator) interaction. As the tiger feeds on immature rhino, stage structure is considered on the rhino population by classifying them into two sub populations: immature rhino (horn not developed) and mature rhino (horn developed). In view of ecology, the main objective of this study is to increase the size of the rhinoceros population in this park and also to maintain the ecological balance of the park. From the mathematical point of view, the stability and complex dynamical behavior of the system are analyzed for the model.

- The first part in the numerical simulation section describes the ecologically behavior of the park while the second discusses the complex dynamical behavior.The park is considered to be in a state of ecological balance if it maintains higher densities of the rhino population compared to tiger population otherwise ecologically imbalanced. It is observed that if the poaching effect and/or attacking rate are higher, then the park is in ecologically imbalanced situation.

- In the second part it is observed that if the poaching effect increases, then the positive equilibrium point becomes unstable showing its bifurcation behavior. Also, from the field data (during 1981-1996) it was observed that due to the increase of poaching effect there was a little fluctuation in the rhino population. The numerical simulation results are consistent with the real situation. Thus if the poaching rate is increased beyond a threshold value, an adverse effect on the rhino population is observed which could lead them towards extinction. Again, if the attacking rate on immature rhino by tiger population increases, the positive equilibrium point becomes unstable. To control the attacking rate and the oscillation behavior of the system, some modifications has to be introduced.

Mathematical Model-II

From the data collected from PCCF Wildlife, Assam, it was observed that the tiger population preys not only on the immature rhino but also on other herbivores of KNP. The previous system (Mathematical model-I) was formulated by considering the rhino as the main food for tiger. In this section, the other herbivores are considered as "additional food" for tiger population and accordingly, the system (9) is modified by modifying the functional response to make way for availability of additional food for the predator.

Let the predator population be provided with additional food of constant biomass M, which is distributed uniformly in the habitat. It is assumed that the additional food is non-reproducing and is supplied at a constant rate. The number of additional food encounters per predator is proportional to the density of additional food. The proportionality constant characterizes the ability of the predator to identify the additional food (Srinivasu et al. [39]).

Based on the discussion of Sahoo & Poria [30] and Samanta et al. [31], let, h_x, h_M denote tiger's handling time for rhino and additional food and e_x, e_M represent tiger's searching efficiency for rhino and additional food respectively, then

$\mu = \frac{h_M}{h_x}$ indicates how long it takes the tiger to handle additional food relative to rhino population. This parameter characterizes quality of additional food. If $\mu > 1$ (i.e. $h_M > h_x$), then the tiger can easily capture immature rhino than additional food and it implies that the additional food available is of poor quality. And if, $\mu < 1$ then the additional food available is of rich quality.

$\eta = \frac{e_M}{e_x}$ indicates the ease with which tiger detects additional food relative to rhino population.

$A = \eta M$ represents the quantity of additional food for the tiger as it is directly proportional to the biomass of the additional food (M). Thus the Holling type-II functional response with additional food M is modified into the following form:

$$g(x) = \frac{ax}{1 + ahx + \mu A}. \tag{16}$$

Thus, considering Sahoo & Poria [30], Samanta et. al. [31] and the functional response (17) the system of equations become

$$\begin{aligned}
\frac{dx_1}{dt} &= a_1x_2 - d_1x_1 - \frac{ax_1y}{1+\mu A + ahx_1} - r_1x_1, \\
\frac{dx_2}{dt} &= r_1x_1 - d_2x_2 - Px_2 - \rho x_2^2, \\
\frac{dy}{dt} &= -sy + \frac{Kax_1y}{1+\mu A + ahx_1} + \frac{K\mu y}{1+\mu A + ahx_1},
\end{aligned} \tag{17}$$

with initial conditions $x_1(0) > 0$, $x_2(0) > 0$ and $y(0) > 0$.

In the following lemma, we show that all solutions of system (17) are bounded which implies that the system is ecologically well behaved.

Lemma

The set $\Omega_2 = \Big\{(x_1, x_2, y)\colon 0 < x_1 \leq \frac{a_1^2}{4\rho d_0},\ 0 < x_2 \leq \frac{a_1^2}{4\rho d_0},$ $0 < y \leq \frac{1}{K}\frac{a_1^2}{4\rho d_0},\ 0 < x_1(t) + x_2(t) + \frac{y(t)}{K} \leq \frac{a_1^2}{4\rho d_0}\Big\}$ *attracts all solutions in the interior of the positive orthant, where* $d_0 = min\{d_1,\ d_2 + P,\ s - \frac{K\mu}{1+\mu A}\}$ *and* $s > \frac{K\mu}{(1+\mu A)}$.

In the following section we analyze the system (17).

The system (17) has three non-negative equilibria as shown below.

1. $O_0(0,0,0)$ always exists.
2. $O_1(\tilde{x}_1, \tilde{x}_2, 0)$ exists under the condition (8), where $\tilde{x}_1$, $\tilde{x}_2$ are same as $\bar{x}_1$, $\bar{x}_2$ of the system (9).
3. $O_2(\ddot{x}_1, \ddot{x}_2, \ddot{y})$ exists if

$$s(1+\mu A) > K\mu \text{ and } K > sh, \tag{18}$$

where,

$$\ddot{x}_1 = \frac{s(1+\mu A) - K\mu}{a(K - sh)},$$

$$\ddot{x}_2 = \frac{-(d_2+P) + \sqrt{(d_2+P)^2 + 4r_1\rho\left\{\frac{s(1+\mu A)-K\mu}{a(K-sh)}\right\}}}{2\rho},$$

$$\ddot{y} = \frac{K\{1+\mu(A-h)\}}{\{s(1+\mu A) - K\mu\}}[a_1\ddot{x}_2 - (d_1+r_1)\ddot{x}_1].$$

Here the local as well as global stability of the system (17) around the equilibrium points are discussed. The results are given below:

i. The equilibrium point $O_0(0,0,0)$ is always a saddle point.
ii. The equilibrium point $O_1(\tilde{x}_1, \tilde{x}_2, 0)$ is
    a) locally asymptotically stable if $s > \frac{K(a\tilde{x}_1+\mu)}{1+ah\tilde{x}_1+\mu A}$,
    b) otherwise it is a saddle point.
iii. The interior equilibrium point $O_2(\ddot{x}_1, \ddot{x}_2, \ddot{y})$ is locally asymptotically stable iff

$$\left\{d_1 + r_1 + \frac{a\ddot{y}(1+\mu A)}{(1+\mu A+ah\ddot{x}_1)^2}\right\}(d_2 + P + 2\rho\ddot{x}_2) > a_1 r_1. \quad (19)$$

i.e. the birth rate of the immature rhino and transformation rate to mature rhino is less than a modified threshold value.

1. The positive equilibrium point O_2 is globally asymptotically stable in the region Ω_2 under the following conditions:

$$\begin{aligned} &a) \quad (d_1 + r_1)(1 + \mu A)(1 + ah\ddot{x}_1 + \mu A) > a^2 h\ddot{x}_1\ddot{y}, \\ &b) \quad 2\Big\{d_1 + r_1 - \frac{a^2 h\ddot{x}_1\ddot{y}}{(1 + \mu A)(1 + ah\ddot{x}_1 + \mu A)}\Big\}\Big\{1 + \frac{4d_0\ddot{x}_1 r_1}{\ddot{x}_2 a_1^2}\Big\} > \frac{a_1^2}{\rho}. \end{aligned} \tag{20}$$

Proof.

The positive definite function about O_2 is

$$V_2(x_1, x_2, y) = (x_1 - \ddot{x}_1)^2 + (x_2 - \ddot{x}_2 - \ln \frac{x_2}{\ddot{x}_2}) + l_1(y - \ddot{y} - \ln \frac{y}{\ddot{y}}),$$

where, l_1 is a suitable constant to be determined. □

Numerical simulation for Model II

Behavior of the park for the system (17)

To see the ecological behavior of the park and to observe the effect of key parameters (poaching rate 'P', the quality 'μ', quantity 'A' of the additional food on ecological balance of KNP, the following set of values of parameters are chosen:

$$\begin{aligned} &a_1 = 45, \quad d_1 = 5, \quad a = 3, \quad h = 0.02, \quad r_1 = 22, \quad \rho = 2, \\ &d_2 = 0.3, \quad P = 1, \quad s = 1, \quad K = 0.05, \quad \mu = 0, \quad A = 0 \end{aligned} \tag{21}$$

with the initial conditions $x_1(0) = 1,\ x_2(0) = 1$ and $y(0) = 1$.

As mentioned earlier, μ indicates the quality of the additional food. Thus μ is divided into two parts : $\mu > 1$ i.e. additional food is of poor quality and $\mu < 1$, i.e. the additional food is of high quality. Then for different values of A, the behavior of immature rhino, mature rhino and tiger population with respect to time t are given in Figs (11) and (12). Thus this section is divided into two subsections:

1. $\mu > 1$ i.e. when the additional food is of poor quality. For the values of parameters (21) with $a = 20,\ P = 10, \mu = 10,\ A = 0.2$, it is observed from Fig. (11(a)) that the tiger population settles in the higher equilibrium level whereas the rhino go to the lower equilibrium level. Thus the park is not showing ecologically balanced behavior. But if the value of A increases i.e. $A = 0.5$ (Fig. 11(b)), the density of rhino population increases and goes to a higher equilibrium level compared to the density of tiger population. So, the park is showing ecologically balanced behavior.

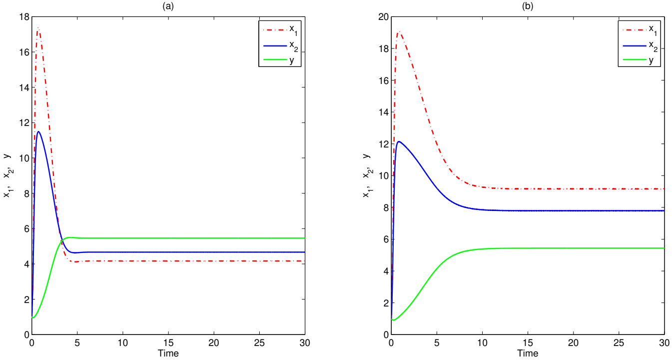

Figure 11: Ecologically imbalanced and balanced behavior of the system (17) w.r.t A for $\mu > 1$

ii. $\mu < 1$ i.e. when the additional food is of high quality: Choosing the same set of values of parameters (21) with $a = 20,\ P = 10,\ \mu = .4,\ A = 3$, it is observed from Fig. (12(a)) that the density of the tiger population is higher than densities of rhino population. Thus the park is in an ecologically imbalanced situation. But the behavior changes if the value of A increases (i.e. for $A = 5$, Fig. (12(b)).

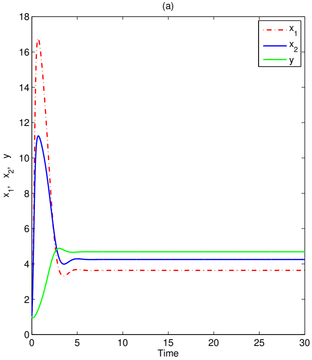

Figure 12: Ecologically imbalanced and balanced behavior of the system (17) w.r.t A for $\mu < 1$

Thus the balanced behavior of the park can be maintained by increasing the quantity of the additional food.

But to maintain the balanced behavior of the park in the presence of poor or high quality with poor quantity of the additional food, high attacking rate, poaching effect, some effort is needed to recover this situation.

Dynamical behavior of the system

To see the dynamical behavior of the system, the following set of values of the parameters

$$\begin{aligned} &a_1 = 45, \quad d_1 = 5, \quad a = 20, \quad h = 0.02, \quad r_1 = 22, \quad \rho = 2, \\ &d_2 = 0.3, \quad P = 10, \quad s = 1, \quad K = 0.05, \end{aligned} \tag{22}$$

are chosen with the initial conditions (1; 1; 1) and the positive equilibrium point $E_2(1.6668, 2.4214, 3.1980)$ is obtained.

Graph trajectory of immature rhino, mature rhino and tiger population for different values of additional food quantity A

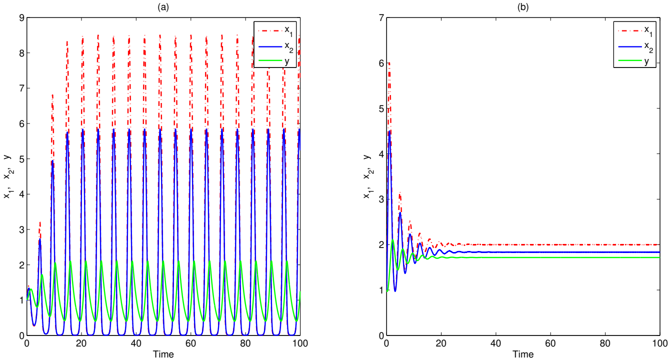

Figure 13: (a) Oscillatory behavior (b) Stable behavior of the positive equilibrium point w.r.t a

In this section the system (17) has been studied with poaching on the mature rhino and additional food for the tiger population. We have chosen the set of values of parameters (22) with high attacking rate and poaching effect (i.e. $a = 25, P = 20$) i.e. the system is showing oscillation behavior. Now the quality and quantity of the additional food i.e. ($\mu = 10$, $A = 0.01$) is added to the system. Here it is observed that the equilibrium point remains unstable (Fig.13(a)). But if the value of the quantity of the additional food increases (i.e. $A = 0.1$), then the positive equilibrium point becomes stable (Fig.13(b)).

Mathematical Model-III

In the previous section, presence of additional food was considered to save the immature rhino from tiger population. At the same time, to increase the immature rhino population specific effort is needed. If, the rhino is kept in a reserved region where the rhino tiger interaction is less and poaching is greatly reduced then, the rhino population may eventually increase. Keeping all these in mind, the habitat complexity is introduced in the system and accordingly, the Holling type-II functional response followed by Dubey et. al. [4], Sahoo& Poria [30] is modified.

Again, the existence of habitat complexity reduces the interaction rate between prey and the predator thereby reducing the probability of capturing prey by reducing the searching efficiency of the predator. Winfield [44] proved that the habitat complexity is more likely to affect the attack coefficient than the handling time. Thus the attack rate a is replaced by $a(1-c)$, where c is a dimensional less quantity that measures the degree of habitat complexity with $0 < c < 1$ and it reduces the predation rate. Again based on the discussion of Dubey et. al. [4] and Sahoo & Poria [30], the Holling type-II functional response with habitat complexity is modified into the following form:

$$g(x) = \frac{a(1-c)x}{1 + a(1-c)hx + \mu A}. \tag{23}$$

It is to be noted, when $c = 0$ i.e. when there is no complexity and $\mu = 0$, we get back the Holling Type II response function.

Thus considering the modified Holling type-II functional response (i.e. Equation (24)), the system of equations become:

$$\begin{aligned}
\frac{dx_1}{dt} &= a_1 x_2 - d_1 x_1 - \frac{a(1-c)x_1 y}{1+\mu A + a(1-c)hx_1} - r_1 x_1, \\
\frac{dx_2}{dt} &= r_1 x_1 - d_2 x_2 - P x_2 - \rho x_2^2, \\
\frac{dy}{dt} &= -sy + \frac{Ka(1-c)x_1 y}{1+\mu A + a(1-c)hx_1} + \frac{K\mu y}{1+\mu A + a(1-c)hx_1},
\end{aligned} \tag{24}$$

with initial conditions $x_1(0) > 0$, $x_2(0) > 0$ and $y(0) > 0$, where c $(0 < c < 1)$ is a dimensionless constant representing the degree of habitat complexity.

In the following Theorem, we show that all solutions of system (24) are bounded.

Lemma

The set $\Omega_3 = \Big\{(x_1, x_2, y)\colon 0 < x_1 \leq \frac{a_1^2}{4\rho\gamma},\ 0 < x_2 \leq \frac{a_1^2}{4\rho\gamma},$
$0 < y \leq \frac{1}{K}\frac{a_1^2}{4\rho\gamma}$ *and* $0 < x_1(t) + x_2(t) + \frac{y(t)}{K} \leq \frac{a_1^2}{4\rho\gamma}\Big\}$ *attracts all solutions in the interior of the positive orthant, where* $\gamma = min\{d_1,\ d_2 + P,\ s - \frac{K\mu}{1+\mu A}\}$ *and* $s > \frac{K\mu}{1+\mu A}$.

In the following section we analyze the system (24).

The system (24) has three non-negative equilibria as shown below.

1. $N_0(0,0,0)$ always exists.
2. $N_1(\check{x}_1, \check{x}_2, 0)$, exists under the condition of (8).
3. $N_2(\widehat{x}_1, \widehat{x}_2, \widehat{y})$ exists under the condition same as (18) where,

$$\begin{aligned}
\widehat{x}_1 &= \frac{s(1+\mu A) - K\mu}{a(K - sh)(1-c)}, \\
\widehat{x}_2 &= \frac{-(d_2 + P) + \sqrt{(d_2+P)^2 + 4\widehat{x}_1 \rho r_1}}{2\rho}, \\
\widehat{y} &= \frac{K\{1 + \mu(A - h)\}}{\{s(1+\mu A) - K\mu\}}[a_1 \widehat{x}_2 - (d_1 + r_1)\widehat{x}_1].
\end{aligned} \tag{25}$$

Here the local as well as global stability of the system (24) around the equilibrium points are discussed. The results are given below:

1. The equilibrium point $N_0(0,0,0)$ is always saddle.
2. The equilibrium point $N_1(\breve{x}_1, \breve{x}_2, 0)$ is
   a) stable if $s > \frac{K[a(1-c)\breve{x}_1+\mu]}{1+ah(1-c)\breve{x}_1+\mu A}$,
   b) otherwise it is unstable.
3. The interior equilibrium point $N_2(\widehat{x}_1, \widehat{x}_2, \widehat{y})$ is locally asymptotically stable iff

$$a_1 r_1 < \left\{ d_1 + r_1 + \frac{a(1-c)(1+\mu A)\widehat{y}}{(1+\mu A + a(1-c)h\widehat{x}_1)^2} \right\} (d_2 + P + 2\rho \widehat{x}_2). \tag{26}$$

1. If the following conditions hold true

$$a)\quad (d_1 + r_1)(1 + \mu A)\{1 + ah(1-c)\widehat{x}_1 + \mu A\} > a^2(1-c)^2 h\widehat{x}_1\widehat{y},$$

$$b)\quad 2\Big\{d_1 + r_1 - \frac{a^2(1-c)^2 h\widehat{x}_1\widehat{y}}{(1+\mu A)(1+ah(1-c)\widehat{x}_1 + \mu A}\Big\}\Big(\frac{4r_1\widehat{x}_1\gamma}{\widehat{x}_2 a_1^2} + 1\Big) > \frac{a_1^2}{\rho} \qquad (27)$$

then N_2 is globally asymptotically stable in the region Ω_3 .

Proof.

The positive definite function about N_2 is

$$V_3(x_1, x_2, y) = (x_1 - \widehat{x}_1)^2 + (x_2 - \widehat{x}_2 - \ln \frac{x_2}{\widehat{x}_2}) + l_2(y - \widehat{y} - \ln \frac{y}{\widehat{y}}),$$

where, l_2 is a suitable constant to be determined. □

Numerical simulation for Model III

For the following set of values of parameters are chosen:

$$\begin{aligned} &a_1 = 45, \quad d_1 = 5, \quad a = 3, \quad h = 0.02, \quad r_1 = 22, \quad \rho = 2, \\ &d_2 = 0.3, \quad P = 1, \quad s = 1, \quad K = 0.05, \quad \mu = 0, \quad A = 0 \end{aligned} \tag{28}$$

with the initial conditions $x_1(0) = 1,\ x_2(0) = 1$ and $y(0) = 1$.

Behavior of the park for the system (24)

This section is also divided into two subsections.

1. When $\mu > 1$ i.e. For set of values of parameters (28) with $a = 20$, $P = 10$, $\mu = 10$, $A = 0.2$ i.e. when the park is in imbalanced situation, if the habitat complexity is introduced but in a little amount (i.e. $c = 0.2$), it is seen from Fig. (14(a)) that the density of tiger population is little higher than rhino population. That is the park is still in imbalanced situation. But if the value of habitat complexity increases i.e. $c = 0.5$, then the situation reverse (Fig. 14(b)) i.e. density of rhino is greater than density of tiger population. Thus the balanced behavior of the park is maintained by increasing the habitat complexity.

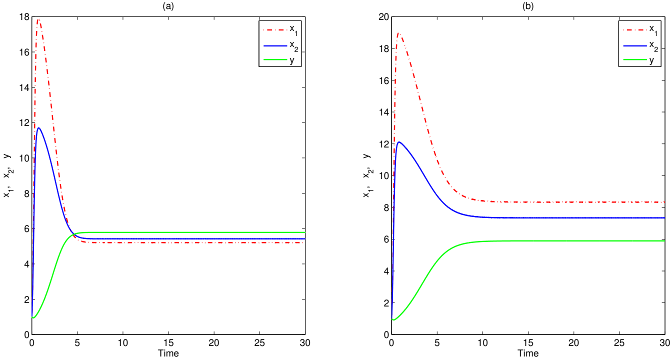

Figure 14: Ecologically imbalanced and balanced behavior of the system (24) w.r.t c for $\mu > 1$

1. When $\mu < 1$
   Similarly, introducing the habitat complexity (c) in the system (24) but in a small amount (i.e. $c = 0.2$), the park is showing ecologically imbalanced situation for the set of values of parameters (28) with $a = 20,\ P = 10,\ \mu = .4,\ A = 3$ (Fig. 15(a)). But if c increases (i.e. $c = 0.5$), the park maintains the balanced behavior (15(b)).

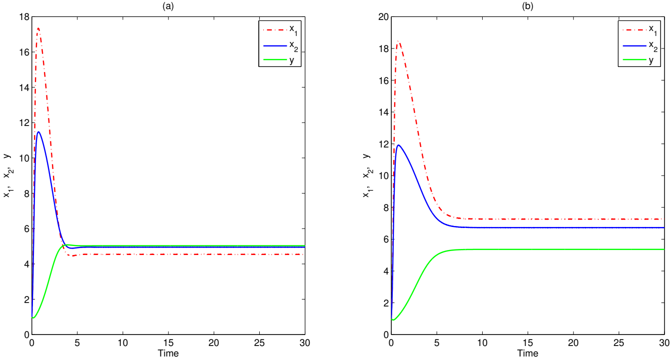

Figure 15: Ecologically imbalanced and balanced behavior of the system (24) w.r.t c for $\mu < 1$

Thus it is observed that if the attacking rate and poaching effect increase, the park is in an ecologically imbalanced situation. But, by introducing good quantity of additional food for the tiger population and/or habitat complexity on immature rhino population, ecologically balanced behavior can be maintained.

Dynamical behavior of the system

Graph trajectory of immature rhino, mature rhino and tiger population for different values of habitat complexity c: When the quantity of the additional food is less (i.e. Fig. 13(a)), to control the oscillation behavior, the habitat complexity is used. From Fig. (16(a)), it is observed that when it is small i.e. $c = 0.1$, then the equilibrium point still unstable but if c increases (i.e. $c = 0.4$), then stable behavior of the system can be seen (Fig.16(b)). If the value of c increases further (i.e. $c = 0.8$), the balanced behavior of the park is maintained.

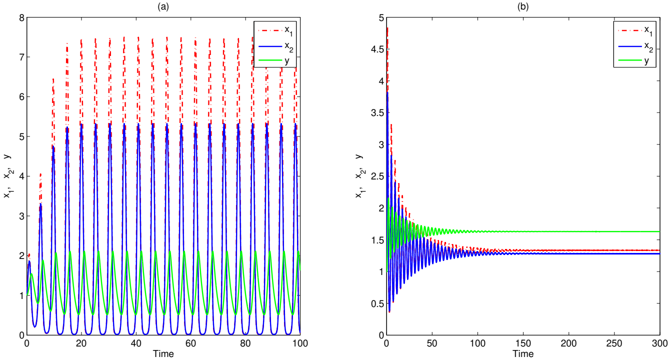

Figure 16: (a) Oscillatory behavior (b) Stable behavior of the positive equilibrium point w.r.t c

Discussion & Conclusion

- Though Kaziranga is famous all over the world for the Greater one-horned rhinoceros, many other herbivores are also present in really good number and the tiger preys on them too. Thus, the previous system is modified by modifying the Holling type-II functional response incorporating the effect of additional food availability.The analytical results describe the effect of additional food on the dynamics of the previous system.
- As the natural land-form provides certain degree of protection to the immature rhino from the tiger population. So to increase the number of rhino and to maintain the ecological balance of KNP, the second system is further extended by introducing habitat complexity in the Holling type-II functional response. The previous analytical results are modified with the factor habitat complexity for this system.

Discussion & Conclusion

- It is observed that if the poaching effect and/or attacking rate are higher, then the park is in ecologically imbalanced situation. This imbalanced situation can be controlled by introducing appropriate amount of additional food and habitat complexity.
- In the first system it is observed that if the poaching rate is increased beyond a threshold value, an adverse effect on the rhino population is observed which could lead them towards extinction. Again, if the attacking rate on immature rhino by tiger population increases, the positive equilibrium point becomes unstable. To control the attacking rate and the oscillation behavior of the system, the additional food and habitat complexity become important parameters.

References

1. Bairagi N, Jana D (2012) Age-structured predator-prey model with habitat complexity: oscillations and control, Dynamical systems 27(4):475-499.https://doi.org /10.1080/14689367.2012.723678
2. B. Dubey, A. Kumar, A.P. Maiti, Global stability and Hopf-bifurcation of prey-predator system with two discrete delays including habitat complexity and prey refuge, Comm. Nonlin. Sci. Numer. Simul. 67(2019) 528-554 (doi.org/10.1016/j.cnsns.2
3. Liu J, Zhang Z (2016) Dynamics of a predator-prey system with stage structure and two delays, J Nonlinear sci. appl 9:3074-3089
4. Mathur VB, Verma A, Dudley N, Stolton S, Hockings M, James R (0000) Kaziranga National Park and world heritage site, India: Taking the long view, UNF-UNESCO Enhancing Our Heritage Project Team :111-115
5. Medhi A, Saha AK (1998) Land Cover Change and Rhino Habitat Mapping of Kaziranga National Park, Assam. Climate change and biodiversity, Advances in Geographical and Environmental Sciences, Springer, Tokyo Chapter 10: 125-138
6. Misra OP, Sinha P, Singh C (2013) Stability and bifurcation analysis of a prey-predator model with age based predation, Applied Mathematical Modelling 37: 6519-6529
7. Mortoja SG, Panja P, Monda SK (2018) Dynamics of a predator-prey model with stage- structure on both species and anti-predator behavior, Informatics in medicene unlocked 10:50-57
8. Nagumo M (1942) Uber die Lage der Intergralkurven gew onlicher Differentialgle ichungen, Proceedings Physico-Mathematical Society of Japan 24:551-559

References

9. Patra A, Tushar J, Dubey B (2017) Modeling and simulation of a wetland park: An application to Keoladeo National Park, India, Journal of Mathematics and computers in simulation, 134:54-78
10. B. Sahoo, S. Poria, Effects of additional food in a delayed predator-prey model, Mathematical biosciences. 261(2015) 62-73.
11. S. Samanta, R. Dhar, I.M. Elmojtaba, J. Chattopadhyay, The role of additional food in a predator-prey model with a prey refuge, Journal of biological systems. 24 (2016):1-21
12. Sih A, Crowley P, McPeek M, Petrankaa J, Strohmeier K (1985) Predation, competition, and prey communities, a review of held experiments, Ann. Rev. Ecol.Syst 16: 269-311
13. P.D.N. Srinivasu, B.S.R.V. Prasad, Erratum to: Time optimal control of an additional food provided predator–prey system with applications to pest management and biological conservation, Journal of Mathematical Biology 61(2)(2010) 319-321
14. Talukdar BK (2002) Tiger predation of rhino calves at Kaziranga National Park, Assam, Tiger paper 29(4):18-20
15. I.J. Windfield, The influence of simulated aquatic macrophytes on the zooplankton consumption rate of juvenile roach, rutilus rutilus, rudd, scardinius erythrophthalmus, and perch, perch fluviatilis, Journal of Fish biology. 29(1986) 37-48 (doi.org/10.1111/j.1095-8649.1986.tb04997.x).

Thank You