Application of the Rogers-Ramanujan continued fraction to partition functions

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Contents

Partitions

A partition $\pi = (\pi_0, \pi_1, \ldots, \pi_{k-1})$ of a nonnegative integer n is a finite sequence of non-increasing positive integers (called *parts*) $\pi_0, \pi_1, \ldots, \pi_{k-1}$ such that $\pi_0 + \pi_1 + \cdots + \pi_{k-1} = n$.
The partition function $p(n)$ is defined as the number of partitions of n. For example, $p(5)$=7, since there are seven partitions of 5, namely,

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1,1), \text{ and } (1,1,1,1,1).$$

By convention, $p(0) = 1$.

The generating function for $p(n)$, due to Euler, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, for any complex number a and $|q| < 1$, we define

$$(a;q)_0 := 1,$$
$$(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k), \quad n \geq 1,$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

In the sequel, for any positive integer j, we use

$$E_j := (q^j; q^j)_\infty$$

Ramanujan's partition congruences

In 1919, Ramanujan [*Proc. Camb. Philos. Soc.* **19** (1919) 207–210] found nice congruence properties for $p(n)$ modulo 5, 7, and 11, namely, for any nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$
$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

He also found the exact generating functions of $p(5n+4)$ and $p(7n+5)$ as given below:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\,\frac{E_5^5}{E_1^6}, \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\,\frac{E_7^3}{E_1^4} + 49q\frac{E_7^7}{E_1^8}, \tag{1.4}$$

which immediately implies (1.1) and (1.2) respectively. It can also be shown from the above generating functions that

$$p(25n+24) \equiv 0 \pmod{25} \text{ and } p(49n+47) \equiv 0 \pmod{49}.$$

In 1939, Zuckerman [*Duke Math. J.* **5(1)** (1939) 88–110] found the generating functions of $p(25n+24)$, $p(49n+47)$ and $p(13n+6)$ analogous to (1.3) and (1.4). In particular, he showed that

$$\sum_{n=0}^{\infty} p(25n+24)q^n = 63\times 5^2\frac{E_5^6}{E_1^7} + 52\times 5^5 q\frac{E_5^{12}}{E_1^{13}} + 63\times 5^7 q^2\frac{E_5^{18}}{E_1^{19}} + 6\times 5^{10} q^3\frac{E_5^{24}}{E_1^{25}} + 5^{12} q^4\frac{E_5^{30}}{E_1^{31}}.$$

Ramanujan [*Proc. Camb. Philos. Soc.* **19** (1919) 207–210] also offered a more general conjecture for congruences of $p(n)$ modulo arbitrary powers of 5, 7 and 11. In particular, if $\alpha \geq 1$ and if δ_α is the reciprocal modulo 5^α of 24, then

$$p(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^\alpha}.$$

In his unpublished manuscript [*The Lost Notebook and Other Unpublished Papers*], [*Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary*], Ramanujan gave a proof of the above. Hirschhorn and Hunt [*J. Reine Angew. Math.*, **326** (1981) 1–17] gave the elementary proof of the above by finding the generating function of $p(5^\alpha n + \delta_\alpha)$.

Ramanujan's theta function

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1.$$

We have the following two useful cases:

$$\varphi(-q) := f(-q,-q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} = (q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{E_1^2}{E_2},$$

and

$$\psi(q) := f(q,q^3) = \sum_{j=0}^{\infty} q^{j(j+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{E_2^2}{E_1}.$$

The product representations of the special cases arise from Jacobi's famous triple product identity [*Number Theory in the Spirit of Ramanujan*, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

We refer to Berndt's books *Ramanujan's Notebooks, Part III* or *Number Theory in the Spirit of Ramanujan*, for various properties satisfied by $f(a,b)$.

Rogers-Ramanujan continued fraction

For $|q| < 1$, the famous Rogers-Ramanujan continued fraction $\mathcal{R}(q)$, is defined by

$$\mathcal{R}(q) := \frac{q^{1/5}}{1} \, {}_{+} \, \frac{q}{1} \, {}_{+} \, \frac{q^2}{1} \, {}_{+} \, \frac{q^3}{1} \, {}_{+} \cdots = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

We refer to Andrews and Berndt's book *Ramanujan's Lost Notebook, Part I*, for many diversified results on $\mathcal{R}(q)$.

t-dissection

If $P(q)$ denotes a power series in q, then a t-dissection of $P(q)$ is given by

$$[P(q)]_{t-\text{dissection}} = \sum_{k=0}^{t-1} q^k P_k(q^t),$$

where P_k's are power series in q^t.

For example, if $R(q) = q^{1/5}/\mathcal{R}(q)$, then the 5-dissections of E_1 and $1/E_1$ are given by

$$E_1 = E_{25}\left(R(q^5) - q - \frac{q^2}{R(q^5)}\right)$$

and

$$\frac{1}{E_1} = \frac{E_{25}^5}{E_5^6}\left(R(q^5)^4 + qR(q^5)^3 + 2q^2R(q^5)^2 + 3q^3R(q^5) + 5q^4 - \frac{3q^5}{R(q^5)} + \frac{2q^6}{R(q^5)^2} - \frac{q^7}{R(q^5)^3} + \frac{q^8}{R(q^5)^4}\right).$$

For a proof of the above, we refer to Berndt's books *Ramanujan's Notebooks, Part III* or *Number Theory in the Spirit of Ramanujan*.

Partitions into distinct parts

Let $Q(n)$ denote the number of partitions of n into distinct parts. For example, $Q(5) = 3$ since there are three partitions of 5 into distinct parts, namely, $(5), (4, 1),$ and $(3, 2)$.
One of Euler's famous results on partitions is that the number of partitions of n into distinct parts is equinumerous to the number of partitions of n into odd parts. Note that there are also three number of partitions of 5 into odd parts, namely, $(5), (3, 1, 1),$ and $(1, 1, 1, 1, 1)$.

The generating function of $Q(n)$ is given by

$$\sum_{n=0}^{\infty} Q(n)q^n = (-q;q)_\infty.$$

Equivalently, by Euler's result,

$$\sum_{n=0}^{\infty} Q(n)q^n = \frac{1}{(q;q^2)_\infty}.$$

Partition with distinct parts

In our work, we find the exact generating functions of $Q(5n+1)$, $Q(25n+1)$ and $Q(125n+26)$ that are analogous to (1.3) and (1.4), and some new congruences for $Q(n)$, the number of partitions of n into distinct parts.

Rödseth [*Arbok Univ. Bergen Mat.-Natur. Ser.* **13** (1969) 13–27] found the following infinite family of congruences modulo powers of 5 for $Q(n)$, the number of partitions of n into distinct parts:

If $\gamma_j = \dfrac{25^{[(j+1)/2]} - 1}{24}$, then for any nonnegative integer n,

$$Q\left(5^{2j+1}n + \gamma_{2j+1}\right) \equiv 0 \pmod{5^j}. \tag{2.5}$$

By using the theory of modular forms and Hecke operators, Lovejoy ([*Adv. Math.* **158** (2001) 253–263], [*Bull. Lond. Math. Soc.* **35** (2003) 41–46]) found some more infinite families of congruences modulo powers of 5 for $Q(n)$. In particular, we recall that, for $r = 1, 3, 4$, and for all nonnegative integers n,

$$Q\left(5^{2j+1}n + \gamma_{2j} + r5^{2j}\right) \equiv 0 \pmod{5^j}.$$

This is (2.5) when $r = 1$.

We find the following generating function representations of $Q(5n+1)$, $Q(25n+1)$ and $Q(125n+26)$.

Theorem

For any nonnegative integer n, we have

$$\sum_{n=0}^{\infty} Q(5n+1)q^n = \frac{E_2^2 E_5^3}{E_1^4 E_{10}},$$

$$\sum_{n=0}^{\infty} Q(25n+1)q^n = \frac{E_2^3 E_5^4}{E_1^5 E_{10}^2} + 160q\,\frac{E_2^4 E_{10} E_5^3}{E_1^8} + 2800q^2 \frac{E_2^5 E_5^2 E_{10}^4}{E_1^{11}} + 16000q^3 \frac{E_2^6 E_5 E_{10}^7}{E_1^{14}} + 32000q^4 \frac{E_2^7 E_{10}^{10}}{E_1^{17}},$$

$$\sum_{n=0}^{\infty} Q(125n+26)q^n$$

$$= 33 \times 5 \frac{E_2^2 E_5^3}{E_1^4 E_{10}} + 1039573 \times 2^2 \times 5 \; q \; \frac{E_2^3 E_5^2 E_{10}^2}{E_1^7}$$

$$+ 84358511 \times 2^4 \times 5^2 q^2 \frac{E_2^4 E_5 E_{10}^5}{E_1^{10}} + 1519417629 \times 2^6 \times 5^3 q^3 \frac{E_2^5 E_{10}^8}{E_1^{13}}$$

$$+ 57468885219 \times 2^8 \times 5^3 q^4 \frac{E_2^6 E_{10}^{11}}{E_1^{16} E_5}$$

$$+ 239126250621 \times 2^{10} \times 5^4 q^5 \frac{E_2^7 E_{10}^{14}}{E_1^{19} E_5^2} + 493702983 \times 2^{20} \times 5^6 q^6 \frac{E_2^8 E_{10}^{17}}{E_1^{22} E_5^3}$$

$$+ 57851635449 \times 2^{16} \times 5^7 q^7 \frac{E_2^9 E_{10}^{20}}{E_1^{25} E_5^4}$$

$$+\,155363323153\times 2^{17}\times 5^{8}q^{8}\frac{E_2^{10}E_{10}^{23}}{E_1^{28}E_5^{5}}$$

$$+\,99443868167\times 2^{22}\times 5^{8}q^{9}\frac{E_2^{11}E_{10}^{26}}{E_1^{31}E_5^{6}}$$

$$+\,1277863945093\times 2^{20}\times 5^{9}q^{10}\frac{E_2^{12}E_{10}^{29}}{E_1^{34}E_5^{7}}$$

$$+\,82117001559\times 2^{23}\times 5^{11}q^{11}\frac{E_2^{13}E_{10}^{32}}{E_1^{37}E_5^{8}}$$

$$+\,85675198911\times 2^{24}\times 5^{12}q^{12}\frac{E_2^{14}E_{10}^{35}}{E_1^{40}E_5^{9}}$$

$$+\,916288433\times 2^{29}\times 5^{14}q^{13}\frac{E_2^{15}E_{10}^{38}}{E_1^{43}E_5^{10}}$$

$$+32357578059\times 2^{29}\times 5^{13}q^{14}\frac{E_2^{16}E_{10}^{41}}{E_1^{46}E_5^{11}}$$
$$+2366343709\times 2^{33}\times 5^{14}q^{15}\frac{E_2^{17}E_{10}^{44}}{E_1^{49}E_5^{12}}$$
$$+57370733\times 2^{36}\times 5^{16}q^{16}\frac{E_2^{18}E_{10}^{47}}{E_1^{52}E_5^{13}}+22998577\times 2^{37}\times 5^{17}q^{17}\frac{E_2^{19}E_{10}^{50}}{E_1^{55}E_5^{14}}$$
$$+30309607\times 2^{36}\times 5^{18}q^{18}\frac{E_2^{20}E_{10}^{53}}{E_1^{58}E_5^{15}}+20313321\times 2^{38}\times 5^{18}q^{19}\frac{E_2^{21}E_{10}^{56}}{E_1^{61}E_5^{16}}$$
$$+2181069\times 2^{40}\times 5^{19}q^{20}\frac{E_2^{22}E_{10}^{59}}{E_1^{64}E_5^{17}}+18319\times 2^{43}\times 5^{21}q^{21}\frac{E_2^{23}E_{10}^{62}}{E_1^{67}E_5^{18}}$$
$$+29\times 2^{48}\times 5^{23}q^{22}\frac{E_2^{24}E_{10}^{65}}{E_1^{70}E_5^{19}}+521\times 2^{46}\times 5^{22}q^{23}\frac{E_2^{25}E_{10}^{68}}{E_1^{73}E_5^{20}}$$
$$+37\times 2^{49}\times 5^{22}q^{24}\frac{E_2^{26}E_{10}^{71}}{E_1^{76}E_5^{21}}+2^{50}\times 5^{23}q^{25}\frac{E_2^{27}E_{10}^{74}}{E_1^{79}E_5^{22}}.$$

With the aid of the Theorem, we deduce the cases $j = 1$ and $j = 2$ of Rödseth's congruence and the following congruences:

$$\begin{aligned} Q(125n + 76) &\equiv 0 \pmod{5}, \\ Q(125n + 101) &\equiv 0 \pmod{5}, \\ Q(625n + 276) &\equiv 0 \pmod{25}, \\ Q(625n + 401) &\equiv 0 \pmod{25}. \end{aligned}$$

Brief outline of the proof

Employing the 5-dissections of E_2 and $\frac{1}{E_1}$ in

$$\sum_{n=0}^{\infty} Q(n)q^n = \frac{1}{(q;q^2)_\infty} = \frac{E_2}{E_1},$$

and extracting the terms involving q^{5n+1}, dividing both sides of the resulting identity by q, and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} Q(5n+1)q^n = \frac{E_5^5 E_{10}}{E_1^6}\left(\left(R(q)^3R(q^2) + \frac{q^2}{R(q)^3R(q^2)}\right) + q\left(-5 - 2\left(\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2}\right)\right)\right). \quad (2.6)$$

Now using certain modular relations between $R(q)$ and $R(q^2)$, we can find some useful relations. For example, we have the following lemma.

Lemma

We have

$$\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2} = 4q\frac{E_1E_{10}^5}{E_2E_5^5},$$

and

$$R(q)^3R(q^2) + \frac{q^2}{R(q)^3R(q^2)} = \frac{E_2E_5^5}{E_1E_{10}^5} + 4q^2\frac{E_1E_{10}^5}{E_2E_5^5} + 2q.$$

Using the above lemma in (2.6) and then employing the identities

$$\frac{E_5^5}{E_1^4 E_{10}^3} = \frac{E_5}{E_2^2 E_{10}} + 4q\frac{E_{10}^2}{E_1^3 E_2}$$

and

$$\frac{E_2^3 E_5^2}{E_1^2 E_{10}^2} = \frac{E_5^5}{E_1 E_{10}^3} + q\frac{E_{10}^2}{E_2},$$

we arrive at the generating function of $Q(5n+1)$.

Next, we use identities involving $R(q)$, $R(q^3)$ and $R(q^4)$ to find generating functions and congruences modulo 5 for some partition functions.

2-color partitions

Let $p_3(n)$ denote the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of 3. For example, $p_3(6) = 16$, where the relevant partitions are (6), $(6')$, $(5,1)$, $(4,2)$, $(4,1,1)$, $(3,3)$, $(3,3')$, $(3',3')$, $(3,2,1)$, $(3',2,1)$, $(3,1,1,1)$, $(3',1,1,1)$, $(2,2,2)$, $(2,2,1,1)$, $(2,1,1,1,1)$, and $(1,1,1,1,1,1)$. Clearly, the generating function for $p_3(n)$ is given by

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{E_1 E_3}.$$

Ahmed, Baruah and Dastidar [*J. Number Theory*, **157** (2015) 184–198] proved that

$$p_3(25n+21) \equiv 0 \pmod{5}.$$

We find the following exact generating function for $p_3(5n+1)$.

Theorem

For any nonnegative integer n, we have

$$\sum_{n=0}^{\infty} p_3(5n+1)q^n = \frac{E_5^5}{E_1^6 E_{15}} + 10q\frac{E_5^{10}}{E_1^7 E_3^5} + q^2\frac{E_{15}^5}{E_3^6 E_5} + 45q^3\frac{E_5^5 E_{15}^5}{E_1^6 E_3^6} - 90q^5\frac{E_{15}^{10}}{E_1^5 E_3^7}.$$

We also deduce the congruences

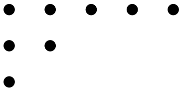
$$p_3(25n+21) \equiv 0 \pmod 5,$$

$$\sum_{n=0}^{\infty} p_3(25n+21)q^n \equiv 10\left(\frac{E_{25}}{E_1^2 E_3} + q^2\frac{E_{75}}{E_1 E_3^2}\right) \pmod{25}.$$

t-core partitions

The Ferrers-Young diagram of a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ of n is an array of left-aligned nodes with π_i nodes in the i^{th} row. Let π'_j denote the number of nodes in column j in the Ferrers-Young diagram of π. The hook number of the (i, j) node in the Ferrers-Young diagram of π is denoted by
$H(i, j) := \pi_i + \pi'_j - i - j + 1.$
A partition of n is called a t-core partition (or simply a t-core) if none of the hook numbers is a multiple of t.

For example, the Ferrers-Young diagram of the partition $\pi = (5, 2, 1)$ is given by:



The nodes $(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2)$ and $(3, 1)$ have hook numbers $7, 5, 3, 2, 1, 3, 1$ and 1 respectively. Therefore π is a 4-core. Obviously, it is a t-core for $t \geq 8$.

If $a_t(n)$ denotes the number of partitions of n that are t-cores, then the generating function for $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{E_t^t}{E_1}.$$

In particular, if $a_4(n)$ denotes the number 4-core partitions of n, then

$$\sum_{n=0}^{\infty} a_4(n)q^n = \frac{E_4^4}{E_1}.$$

By employing relations involving $R(q)$ and $R(q^4)$, we find the following generating function involving $a_4(n)$.

Theorem

$$\sum_{n=0}^{\infty} a_4(5n)q^n$$

$$= \frac{E_4^4 E_{10}^{40}}{E_1^2 E_2^8 E_5^{15} E_{20}^{16}} - 3q\frac{E_4^2 E_{10}^{15}}{E_1^5 E_2^3 E_{20}^6} + 4q\frac{E_4^3 E_{10}^{30}}{E_1^3 E_2^6 E_5^{10} E_{20}^{11}} - 20q^2\frac{E_4 E_5^5 E_{10}^5}{E_1^6 E_2 E_{20}}$$
$$- 12q^2\frac{E_4^2 E_{10}^{20}}{E_1^4 E_2^4 E_5^5 E_{20}^6} + 24q^2\frac{E_4^3 E_{10}^{35}}{E_1^2 E_2^7 E_5^{15} E_{20}^{11}} - 27q^3\frac{E_2 E_5^{10} E_{20}^4}{E_1^7}$$
$$- 60q^3\frac{E_4 E_{10}^{10}}{E_1^5 E_2^2 E_{20}} + 196q^3\frac{E_4^2 E_{10}^{25}}{E_1^3 E_2^5 E_5^{10} E_{20}^6} - 83q^4\frac{E_5^5 E_{20}^4}{E_1^6}$$
$$+ 456q^4\frac{E_4 E_{10}^{15}}{E_1^4 E_2^3 E_5^5 E_{20}} + 296q^5\frac{E_{10}^5 E_{20}^4}{E_1^5 E_2} + 96q^5\frac{E_4 E_{10}^{20}}{E_1^3 E_2^4 E_5^{10} E_{20}}$$
$$+ 128q^6\frac{E_2 E_5^5 E_{20}^9}{E_1^6 E_4 E_{10}^5} + 592q^6\frac{E_{10}^{10} E_{20}^4}{E_1^4 E_2^2 E_5^5} + 512q^7\frac{E_{20}^9}{E_1^5 E_4}.$$

Lemma

If $x = R(q)$ *and* $y = R(q^2)$*, then*

$$xy^2 - \frac{q^2}{xy^2} = K,$$

$$\frac{x^2}{y} - \frac{y}{x^2} = \frac{4q}{K},$$

$$\frac{y^3}{x} + q^2\frac{x}{y^3} = K + \frac{4q^2}{K} - 2q,$$

$$x^3y + \frac{q^2}{x^3y} = K + \frac{4q^2}{K} + 2q,$$

where $K = (E_2E_5^5)/(E_1E_{10}^5)$.

Some relations among $R(q)$, $R(q^3)$, and E_n are stated in the following lemma.

Lemma

We have

$$\frac{R(q)^3}{R(q^3)} + \frac{R(q^3)}{R(q)^3} = 2 + 9q^2 \frac{E_1 E_{15}^5}{E_3 E_5^5},$$

$$R(q)R(q^3)^3 + \frac{q^4}{R(q)R(q^3)^3} = \frac{E_3 E_5^5}{E_1 E_{15}^5} - 2q^2$$

and

$$R(q)^2 R(q^3) - \frac{R(q^3)^2}{R(q)} + q^2 \frac{R(q)}{R(q^3)^2} - \frac{q^2}{R(q)^2 R(q^3)} = 3q.$$

A relation among $R(q)$, $R(q^4)$, and E_n.

Lemma

We have

$$R(q)R(q^4) + \frac{q^2}{R(q)R(q^4)} = 2q + \frac{E_1 E_4 E_{10}^{10}}{E_2^2 E_5^5 E_{20}^5}.$$

A relation among $R(q)$, $R(q^2)$, $R(q^4)$, and E_n.

Lemma

We have

$$\frac{R(q)^2R(q^2)}{R(q^4)}+\frac{R(q^4)}{R(q)^2R(q^2)}=2+4q^2\frac{E_2E_{20}^5}{E_4E_{10}^5}.$$

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THANK YOU