What is the probability that an automorphism fixes a group element

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Let G be a finite group acting on a set Ω. Sherman [Amer. Math. Monthly, 82:261–264, 1975] introduced the probability (denoted by $\Pr(G, \Omega)$) that a randomly chosen element of Ω fixes a randomly chosen element of G.

$$\Pr(G, \Omega) = \frac{|\{(g, x) \in G \times \Omega : gx = x\}|}{|G||\Omega|}.$$

- $\Pr(G, \Omega)$ generalizes well-known notion of commutativity degree of a finite group.
- Commutativity degree of a finite group is nothing but the probability that any a random pair of elements of the group commutes.

Definition [Sherman]

The probability that an automorphism of a group fixes a random element of the group is given by

$$\Pr(G, \operatorname{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \operatorname{Aut}(G) : \alpha(x) = x\}|}{|G||\operatorname{Aut}(G)|}$$

where $\operatorname{Aut}(G)$ is the automorphism group of G.

- Sherman studied $\Pr(G, \operatorname{Aut}(G))$ for some finite abelian groups.
- Arora and Karan [Communications in Algebra, 45(3):1141–1150, 2017] studied $\Pr(G, \operatorname{Aut}(G))$ for some finite non-abelian groups.
- Dutta and Nath [Communications in Algebra, 46(3):961-969, 2018] studied $\Pr(G, \operatorname{Aut}(G))$ through a generalization.

Autocommuting probability of G

- $[x, \alpha] = x^{-1}\alpha(x)$ is called the autocommutator of x and α.
- Rismanchian and Sepehrizadeh [Hacet. J. Math. Stat., 44(4):893–899, 2015] observed that

$$\Pr(G, \operatorname{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \operatorname{Aut}(G) : [x, \alpha] = 1\}|}{|G||\operatorname{Aut}(G)|}.$$

- $\Pr(G, \operatorname{Aut}(G))$ is called autocommuting probability of G.
- If we replace $\operatorname{Aut}(G)$ by $\operatorname{Inn}(G)$ then $\Pr(G, \operatorname{Aut}(G))$ is noting but the commutativity degree fo G.

Example

Let $G = \langle a, b : a^2 = b^2 = 1, ab = ba \rangle$ be the non-cyclic group of order 4 and $\mathrm{Aut}(G) = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be its automorphism group where α_i 's are given by

$$
\begin{aligned}
\alpha_0 &: a \mapsto a & \alpha_1 &: a \mapsto a & \alpha_2 &: a \mapsto a & \alpha_3 &: a \mapsto b \\
&\quad b \mapsto b, & &\quad b \mapsto ab, & &\quad b \mapsto a, & &\quad b \mapsto ab, \\
\alpha_4 &: a \mapsto ab & \alpha_5 &: a \mapsto ab \\
&\quad b \mapsto a, & &\quad b \mapsto b.
\end{aligned}
$$

Total number of pairs (x, α) such that $\alpha(x) = x$ is 12.
Hence, $\Pr(G, \mathrm{Aut}(G)) = \frac{1}{2}$.

Some notations

We write

$$K(G) = \langle \{[x, \alpha] : x \in G \text{ and } \alpha \in \mathrm{Aut}(G)\} \rangle$$

and

$$L(G) = \{x : [x, \alpha] = 1 \text{ for all } \alpha \in \mathrm{Aut}(G)\}.$$

- $K(G)$ and $L(G)$ are called autocommutator subgroup and absolute center of G respectively.
- These notions were introduced by Hegarty [J. Algebra, 169(3), 929–935, 1994].

A generalization of autocommuting probability

Dutta and Nath [Communications in Algebra, 46(3):961-969, 2018] further generalized the notion of autocommuting probability as follows:

$$\Pr_g(G, \operatorname{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \operatorname{Aut}(G) : [x, \alpha] = g\}|}{|G||\operatorname{Aut}(G)|}.$$

- $\Pr_g(G, \operatorname{Aut}(G))$ is called g-autocommuting probability of G.
- If $g = 1$ then $\Pr_g(G, \operatorname{Aut}(G)) = \Pr(G, \operatorname{Aut}(G))$.

Outline of the talk

- Certain computing formulae
- Bounds of the ratio
- An invarience property of the ratio
- Certain characterization of groups through the ratio
- Value of the ratio for certain classes of finite groups
- A character theoretic approach

Computing formulae

We write

- $C_{\mathrm{Aut}(G)}(x) = \{\alpha \in \mathrm{Aut}(G) : \alpha(x) = x\}$
- $\mathrm{orb}(x) = \{\alpha(x) : \alpha \in \mathrm{Aut}(G)\}$

Theorem [Dutta and Nath]

Let G be a finite group. If $g \in G$ then

$$\begin{aligned} \Pr_g(G, \mathrm{Aut}(G)) &= \frac{1}{|G||\mathrm{Aut}(G)|} \sum_{\substack{x \in G \\ xg \in \mathrm{orb}(x)}} |C_{\mathrm{Aut}(G)}(x)| \\ &= \frac{1}{|G|} \sum_{\substack{x \in G \\ xg \in \mathrm{orb}(x)}} \frac{1}{|\mathrm{orb}(x)|}. \end{aligned}$$

Computing fromulae

Theorem [Dutta and Nath]

Let G be a finite group. Then

$$\Pr(G, \mathrm{Aut}(G)) = \frac{1}{|G||\mathrm{Aut}(G)|}\sum_{x\in G}|C_{\mathrm{Aut}(G)}(x)| = \frac{|\mathrm{orb}_G(G)|}{|G|},$$

where $\mathrm{orb}_G(G) = \{\mathrm{orb}_G(x) : x \in G\}$.

We have $|\{(x,\alpha) \in G \times \mathrm{Aut}(G) : [x,\alpha] = 1\}| = \sum_{\alpha\in\mathrm{Aut}(G)} |C_G(\alpha)|$ and hence

$$\Pr(G, \mathrm{Aut}(G)) = \frac{1}{|G||\mathrm{Aut}(G)|}\sum_{\alpha\in\mathrm{Aut}(G)}|C_G(\alpha)|.$$

Some bounds of $\Pr_g(G, \text{Aut}(G))$

Arora and Karana and Richmanshain and Saparizadeh [Hacettepe Journal of Mathematics and Statistics, 44(4):893-899, 2015] gave the following bounds

Theorem

Let G be a finite group. If p is the smallest prime dividing $|G|$ then

$$\frac{1}{|K(G)|}\left(\frac{|[G, \text{Aut}(G)]| - 1}{|G : L(G)|} + 1\right) \leq \Pr(G, \text{Aut}(G)) \leq \frac{p-1}{p|\text{Aut}(G)|} + \frac{1}{p}.$$

Some bounds of $\Pr_g(G, \mathrm{Aut}(G))$

Theorem [Dutta and Nath]

Let G be a finite group. Then

1. $\Pr_g(G, \mathrm{Aut}(G)) \geq \frac{|L(G)|}{|G|} + \frac{|G|-|L(G)|}{|G||\mathrm{Aut}(G)|}$ if $g = 1$.
2. $\Pr_g(G, \mathrm{Aut}(G)) \geq \frac{|L(G)|}{|G||\mathrm{Aut}(G)|}$ if $g \neq 1$.

Theorem [Dutta and Nath]

Let G be a finite group. Then

$$\Pr_g(G, \mathrm{Aut}(G)) \leq \Pr(G, \mathrm{Aut}(G)).$$

The equality holds if and only if $g = 1$.

Some bounds of $\Pr_g(G, \mathrm{Aut}(G))$

Theorem [Dutta and Nath]

Let G be a finite group. If p and q are the smallest primes dividing $|\mathrm{Aut}(G)|$ and $|G|$ respectively, then

$$\Pr(G, \mathrm{Aut}(G)) \leq \frac{p+q-1}{pq}.$$

In particular, if $p = q$ then $\Pr(G, \mathrm{Aut}(G)) \leq \frac{2p-1}{p^2} \leq \frac{3}{4}$.

Theorem [Dutta and Nath]

Let G be a finite group and let p, q be the smallest primes dividing $|\mathrm{Aut}(G)|$ and $|G|$ respectively. If G is non-abelian then

$$\Pr(G, \mathrm{Aut}(G)) \leq \frac{q^2+p-1}{pq^2}.$$

In particular, if $p = q$ then $\Pr(G, \mathrm{Aut}(G)) \leq \frac{p^2+p-1}{p^3} \leq \frac{5}{8}$.

Autoisoclinism between groups

Definition [Moghaddam et al. [Fifth International group theory conference, Islamic Azad University, Mashhad, Iran, 2013]]

Two groups G_1 and G_2 are said to be autoisoclinic if there exist isomorphisms $\psi : \frac{G_1}{L(G_1)} \to \frac{G_2}{L(G_2)}$, $\beta : K(G_1) \to K(G_2)$ and $\gamma : \mathrm{Aut}(G_1) \to \mathrm{Aut}(G_2)$ such that the following diagram commutes

$$\begin{array}{ccc} \frac{G_1}{L(G_1)} \times \mathrm{Aut}(G_1) & \xrightarrow{\psi\times\gamma} & \frac{G_2}{L(G_2)} \times \mathrm{Aut}(G_2) \\ \downarrow & & \downarrow \\ K(G_1) & \xrightarrow{\beta} & K(G_2) \end{array}$$

Rismanchain and Sepehrizadeh [Hacettepe Journal of Mathematics and Statistics, 44(4):893-899, 2015] have shown the following result.

Theorem

Let G_1 and G_2 be two autoisoclinic finite groups. Then

$$\Pr(G_1, \mathrm{Aut}(G_1)) = \Pr(G_2, \mathrm{Aut}(G_2)).$$

Theorem [Dutta and Nath]

Let G and H be two finite groups and let $(\psi \times \gamma, \beta)$ be an autoisoclinism between them. Then

$$\Pr_g(G_1, \mathrm{Aut}(G_1)) = \Pr_{\beta(g)}(G_2, \mathrm{Aut}(G_2)).$$

Certain characterizations

Theorem [Sherman]

Let G be a finite abelian group. Then $\Pr(G, \text{Aut}(G)) = \frac{2}{p}$ if and only if

$$G = \begin{cases} \mathbb{Z}_p, & \text{if p is any prime} \\ \mathbb{Z}_2 \times \mathbb{Z}_p, & \text{if p is an odd prime.} \end{cases}$$

Theorem [Dutta and Nath]

Let G be a finite group with $\Pr(G, \text{Aut}(G)) = \frac{p+q-1}{pq}$, where p and q are the smallest primes dividing $|\text{Aut}(G)|$ and $|G|$ respectively. Then

$$G \cong \mathbb{Z}_3 \text{ or } \mathbb{Z}_4.$$

Theorem [Dutta and Nath]

There is no finite non-abelian group G such that

$$\Pr(G, \text{Aut}(G)) = \frac{q^2 + p - 1}{pq^2},$$

where p and q are the smallest primes dividing $|\text{Aut}(G)|$ and $|G|$ respectively.

Arora and Karan obtained the following values.

Theorem

Let p be a prime and G be a group of order p^2. Then $\Pr(G, \mathrm{Aut}(G)) = \frac{k}{p^2}$*, where k is either* 2 *or* 3.

Theorem

Let p be an odd prime and G be a group of order p^3. Then $\Pr(G, \mathrm{Aut}(G)) = \frac{k}{p^3}$*, where* $k \in \{2, 3, 4, p + 2\}$.

Theorem

Let p be an odd prime and G be a abelian group of order p^4. Then $\Pr(G, \mathrm{Aut}(G)) = \frac{k}{p^4}$*, where* $k \in \{2, 3, 4, 5, 6\}$.

Dutta et. al. [Preprint] also obtained the value of g-autocommuting probabilities for some classes of finite groups.

Theorem

For any prime p if G is the cyclic group of order p, then

$$\Pr_g(G, \mathrm{Aut}(G)) = \begin{cases} \frac{2}{p}, & \text{if } g = 1 \\ \frac{p-2}{p(p-1)}, & \text{if } g \neq 1. \end{cases}$$

Theorem

If G is the non-cyclic group of order 4, then

$$\Pr_g(G, \mathrm{Aut}(G)) = \begin{cases} \frac{1}{2}, & \text{if } g = 1 \\ \frac{1}{6}, & \text{if } g \neq 1. \end{cases}$$

Theorem

Let G be the dihedral group of order $2p$ (where p is any prime) presented by $\langle a, b : a^p = b^2 = 1, bab^{-1} = a^{-1}\rangle$. Then

$$\Pr_g(G, \mathrm{Aut}(G)) = \begin{cases} \frac{3}{2p}, & \text{if } g = 1 \\ \frac{2p-3}{2p(p-1)}, & \text{if } g = a, a^2, \ldots, a^{p-1} \\ 0, & \text{otherwise.} \end{cases}$$

Arora and Karan also obtained the following values of the ratio.

Theorem

Let G be the dihedral group of order 2^n presented by $\langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{-1}\rangle$. Then

$$\Pr(G, \mathrm{Aut}(G)) = \frac{n+1}{2^n}$$

Theorem

Let G be the quaternion group of order 2^n presented by $\langle a, b : a^{2n-2} = b^2, bab^{-1} = a^{-1}\rangle$. Then

$$\Pr(G, \mathrm{Aut}(G)) = \begin{cases} \frac{n+1}{2^n}, & \text{for } n \geq 4 \\ \frac{n}{2^n}, & \text{for } n = 3. \end{cases}$$

A character theoretic approach

Pournaki et. al. [Journal of Pure and Applied Algebra, 212:727-734, 2008] introduced a generalization of the commuting probability as

$$\Pr_g(G) := \frac{|\{(x, y) \in G \times G : [x, y] = g\}|}{|G \times G|}.$$

Consider the map $\zeta : G \to \mathbb{C}$ given by $\zeta(g) = |\{(x, y) \in G \times G : [x, y] = g\}|$ for all $g \in G$.

A character theoretic approach

Frobenius [*Über Gruppencharaktere, Gesammelte Abhandlungen Band III, p. 1–37 (J. P. Serre, ed.), Springer-Verlag, Berlin 1968*] proved that

[Frobenius, 1968]

ζ is a character of G and

$$\zeta = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|}{\chi(1)} \chi,$$

where $\mathrm{Irr}(G)$ is the set of all irreducible characters of G.

[Pournaki et. al., 2008]

Let G is a finite group and let $g \in G'$. Then we have

$$\mathrm{Pr}_g(G) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)}$$

where $\mathrm{Irr}(G)$ is the set of all irreducible characters of G.

A character theoretic approach

- We write $N(g) = \{(x, \alpha) \in G \times \text{Aut}(G) : [x, \alpha] = g\}$.
- Define a map $\xi : G \to \mathbb{C}$ given by $\xi(g) = |N(g)|$ for all $g \in G$. Then

$$\Pr_g(G, \text{Aut}(G)) = \frac{\xi(g)}{|G||\text{Aut}(G)|}.$$

- Note that if $\text{Aut}(G) = \text{Inn}(G)$ then $\xi(g) = \zeta(g)$.

A character theoretic approach

Dutta et. al. [Preprint] proved that

Theorem

ξ is a class function on G.

Theorem

If $G = \mathbb{Z}_p, \mathbb{Z}_2 \times \mathbb{Z}_2, D_6, D_8, D_{10}$ and Q_8 then ξ is a character.

Expression for ξ

We also express ξ in terms of irreducible characters for some groups.

- If $G = \mathbb{Z}_p$ then $\xi = (p-1)\chi_1 + \chi_2 + \chi_3 + \cdots + \chi_p$, where the χ_i's are the irreducible characters of $\mathbb{Z}_p$ given in the following table. Note that $\omega = e^{\frac{2\pi i}{n}}$.

	1	a	a^2	$\dots$	a^{p-1}
χ_1	1	1	1	$\dots$	1
χ_2	1	ω	ω^2	$\dots$	ω^{p-1}
χ_3	1	ω^2	ω^4	$\dots$	$\omega^{2(p-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_p	1	ω^{p-1}	$\omega^{2(p-1)}$	$\dots$	$\omega^{(p-1)^2}$

Expression for ξ

- ξ is a character on $D_6 = \langle a, b : a^3 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ and $\xi = 6\chi_1 + 6\chi_2 + 3\chi_3$, where χ_1, χ_2 are linear irreducible characters with principal character χ_1 and χ_3 is the unique non-linear irreducible character of D_6.

Question

Whether ξ is a character on all finite groups G.

ξ is a character

Theorem

Let G be a finite group and $\mathrm{Aut}(G) = \bigsqcup_{i=1}^{m} \mathrm{Inn}(G)\beta_i$, *where β_1 is the identity automorphism of G. Then*

$$\xi(g) = \sum_{\chi \in \mathrm{Irr}(G)} \Big(\frac{|G : Z(G)|}{\chi(1)} \sum_{i=1}^{m} \langle \chi, \chi^{\beta_i} \rangle \Big) \chi(g),$$

where $\chi^{\beta_i}(x) = \chi(\beta_i(x))$ for all $x \in G$.

Corollary

ξ is a character on any finite group G.

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THANK YOU