Solution Concepts in Transferable Utility Games

Presented by
Parishmita Boruah

Department of Mathematics
Dibrugarh University

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Introduction

- What is a game?
- What is Game theory?
  It is the study of mathematical models of conflict and cooperation between intelligent and rational decision makers.

Two broad classifications:

- Non-Cooperative game theory
  Players adopt their strategies independently to maximize their utilities without making any binding agreements.
- Cooperative game theory
  Players make binding agreements and as a result of this, they form coalitions. Even though the players' individual goal is to maximize their utility, they achieve that, by cooperation.

What are Transferable Utility Games?

Definition

A cooperative game with transferable utility (TU game) consists of a non-empty set of finite players N *and a coalition function or characteristic function* $v \in \{f | f : 2^N \to \mathbb{R}, f(\emptyset) = 0\}$ *which describes the profit (or cost)* $v(S)$ *(we use the term "worth" to represent both profit and cost) generated due to the cooperation among the individuals involved in the coalition* $S \subseteq N$*. The whole set* N *is the called grand coalition.*

- In TU games players work together for a common goal under binding agreements.
- The contributions by the players to achieve this common goal fetch some worth.
- How fair is the distribution of the total worth $v(N)$?

Solution Concepts

Definition

A value or solution is a function $\Phi : G(N) \to \mathbb{R}^n$*, which assigns to each game* $v \in G(N)$ *an n-tuple of real numbers. Thus formally, for each* $v \in G(N)$, $\Phi(v) = (\Phi_1(v), ..., \Phi_n(v))$ *where* $\Phi_i(v) \in \mathbb{R}$.

- A solution of a game is an *n*-vector which assigns payoffs to each player in N.
- **Classification of Solutions**
  - Single valued
  - Set valued

Basic properties of solutions/values

Axiom

Efficiency: *A value* Φ *is called efficient if* $\sum_{i \in N} \Phi_i(v) = v(N)$ *for all games* $v \in G(N)$.

Efficiency demands that the sum of the payoffs received by all the players is equal to the worth of the grand coalition. This property guarantees that no part of the worth is being wasted in the process of division of the payoffs among the players.

Basic properties of solutions/values

Axiom

Symmetry: *Two players* $i, j \in N$ *are called symmetric if for all* $S \subseteq N \setminus \{i, j\}$, $v(S \cup i) = v(S \cup j)$. *A value* Φ *is symmetric if for each pair of symmetric players* $\Phi_i(v) = \Phi_j(v)$.

Basic properties of solutions/values

Axiom

Linearity: *A value* Φ *is called linear if* $\Phi(\alpha v + \beta w) = \alpha\Phi(v) + \beta\Phi(w)$ *for any game* $v, w \in G(N)$ *and scalars* $\alpha, \beta \in \mathbb{R}$. *If* $\Phi_i(v + w) = \Phi_i(v) + \Phi_i(w)$ *for all games* $v, w \in G(N)$, *then the value is called additive.*

Linearity implies that whether we play two games separately or combine them to have a single game, the payoffs to the players will remain same in either case.

- The values that satisfy these three properties are called ESL-values.

Marginalism and Egalitarianism

The two most prominent approaches of defining fairness in a value are marginalism and egalitarianism.

- The marginal contribution of a player is defined as $v(S \cup \{i\}) - v(S)$ for all $S \subseteq N \setminus \{i\}$. Marginalism awards only to the productive players.
- Egalitarianism does not distinguish between the productive and non-productive players, i.e., egalitarian solutions express solidarity to the non-productive players as well.
- A productive player is here referred to one who has non-negative marginal contributions from at least one coalition. The Shapley value is an extreme case of marginalism while the equal division rule (ED) is the other extreme of egalitarianism.

The Equal Division Rule

- The Equal Division Rule Φ^{ED} is a value on $G(N)$ which divides the worth $v(N)$ of the grand coalition N among all the players equally. Mathematically we have,

$$\Phi_i^{ED}(v) = \frac{v(N)}{n}, \ \forall i \in N. \tag{1}$$

Example

Let us consider a game v with $N = \{1, 2, 3\}$ such that
$v(\{1\}) = 0$, $v(\{2\}) = 0$, $v(\{3\}) = 0$,
$v(\{1, 2\}) = v(\{1, 3\}) = 0$, $v(\{2, 3\}) = 3$ and $v(\{1, 2, 3\}) = 0$.

Then, $\Phi_1^{ED}(v) = \Phi_2^{ED}(v) = \Phi_3^{ED}(v) = 0/3 = 0$.

Characterization of the Equal Division Rule

Definition

Player $i \in N$ *is a nullifying player in* $v \in G(N)$ *if* $v(S) = 0$ *for all* $S \subseteq N$ *with* $i \in S$.

Definition

A solution ϕ *satisfies the nullifying player property if* $\phi_i(v) = 0$ *whenever* i *is a nullifying player in* v.

Theorem

A value $\Phi : G(N) \to \mathbb{R}^n$ *is the equal division rule if and only if it satisfies efficiency, symmetry, additivity and the nullifying player property.*

The Shapley Value

The Shapley value was introduced by Shapley in 1953. This value assigns an expected marginal contribution to each player in the game with respect to a uniform distribution over the set of all permutations on the set of players. Specifically, let π be a permutation (or an order) on the set of players, i.e., a one-to-one function from N onto N, and let us imagine the players appearing one by one to collect their payoff according to the order π. For each player i we can denote by $p_\pi^i = \{j : \pi(i) > \pi(j)\}$ the set of players preceding player i in the order π. The marginal contribution of player i with respect to that order π is given by $v(p_\pi^i \cup i) - v(p_\pi^i)$. Now, if permutations are randomly chosen from the set Π of all permutations, with equal probability for each one of the $n!$ permutations, then the average marginal contribution of player i in the game v is given by

$$\Phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} [v(p_\pi^i \cup i) - v(p_\pi^i)] \qquad (2)$$

The Shapley Value

- Eq.(2) has the following equivalent form which we use more often.

$$\Phi_i^{Sh}(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!}[v(S \cup i) - v(S)], \ \ i \in N \qquad (3)$$

The Shapley Value Example

S	$\emptyset$	1	2	3	12	13	23	123
v(S)	0	0	0	30	0	40	30	40

Table 1: A three person buyer-seller game

Here player 3 is the seller of an object who values the object at 30; player 1 and 2 are buyers of the object who value the object at 40 and 30 respectively. Now using formula (3):

$$\Phi_1^{Sh}(v) = \underbrace{\frac{1}{3}}_{S\cup\{1\}=\{1\}}(0) + \underbrace{\frac{1}{6}}_{S\cup\{1\}=\{1,2\}}(0) + \underbrace{\frac{1}{6}}_{S\cup\{1\}=\{1,3\}}(10) + \underbrace{\frac{1}{3}}_{S\cup\{1\}=\{1,2,3\}}(10) = 5$$

$$\Phi_2^{Sh}(v) = \underbrace{\frac{1}{3}}_{S\cup\{2\}=\{2\}}(0) + \underbrace{\frac{1}{6}}_{S\cup\{2\}=\{1,2\}}(0) + \underbrace{\frac{1}{6}}_{S\cup\{2\}=\{2,3\}}(0) + \underbrace{\frac{1}{3}}_{S\cup\{2\}=\{1,2,3\}}(0) = 0$$

$$\Phi_3^{Sh}(v) = \underbrace{\frac{1}{3}}_{S\cup\{3\}=\{3\}}(30) + \underbrace{\frac{1}{6}}_{S\cup\{3\}=\{1,3\}}(40) + \underbrace{\frac{1}{6}}_{S\cup\{3\}=\{2,3\}}(30) + \underbrace{\frac{1}{3}}_{S\cup\{3\}=\{1,2,3\}}(40) = 35$$

The Shapley value of the game $(\{1,2,3\}, v)$ is $\Phi^{Sh}(v) = (5, 0, 35)$.

Characterization of the Shapley Value

Definition

Player $i \in N$ *is called a null player in* $v \in V$ *if* $v(S \cup i) - v(S) = 0$ *for all* $S \subseteq N \setminus i$. *i.e., A player* $i \in N$ *is a null player if its marginal contribution to any coalition* $S \subseteq N \setminus \{i\}$ *is zero.*

Definition

A solution ϕ *satisfies the null player property if* $\phi_i(v) = 0$ *whenever* i *is a null player in* v.

Theorem

A solution $\phi : G(N) \to \mathbb{R}^n$ *is the Shapley value if and only if it satisfies Efficiency, Symmetry, Additivity and the Null player property.*

The Egalitarian Shapley Value

The Egalitarian Shapley value is the convex combination of the Shapley value and the Equal division rule and is given by

$$\Phi_i^{\alpha}(v) = \alpha\Phi_i^{Sh}(v) + (1-\alpha)\Phi_i^{ED}(v) \quad \forall i \in N, v \in G(N), \text{ for any } \alpha \in [0,1] \tag{4}$$

Characterization of the Egalitarian Shapley Value

Definition

(Null player in a productive environment) If i is a null player in v and $v(N) \geq 0$, then for all $v \in V$ and $i \in N$, it holds that $\Phi_i(v) \geq 0$.

Definition

(Desirability property) For all $v \in G(N)$ and all $i, j \in N$, if $v(S \cup i) \geq v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, then $\Phi_i(v) \geq \Phi_j(v)$.

Theorem

A solution $\Phi : G(N) \rightarrow \mathbb{R}^n$ is equal to the Egalitarian Shapley value if and only if it satisfies efficiency, desirability, additivity and the null player in a productive environment property.

The Discounted Shapley Values

The δ-discounted Shapley value $\Phi^{Sh^{\delta}}$ introduced by Joosten is given as follows:

$$\Phi_i^{Sh^{\delta}}(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!}[\delta^{n-s-1}v(S \cup i) - \delta^{n-s}v(S)],\ i \in N,\ \delta \in [0,1]. \tag{5}$$

Note that the class δ-discounted Shapley values contain the equal division rule for $\delta = 0$ and the Shapley value for $\delta = 1$. This value is similar to the egalitarian Shapley value as both these values are flexible in their own way to choose the level of marginalism and egalitarianism as the situation demands. Thus, these values provide a trade-off between marginalism and egalitarianism.

Characterization of the Discounted Shapley Values

Definition

For $\delta \in [0,1]$, the player $i \in N$ is called a δ-reducing player in $v \in G(N)$ if for all $S \subseteq N \setminus i$, we have $v(S \cup i) = \delta v(S)$.

Definition

A solution Φ satisfies the δ-reducing player property namely, $\Phi_i(v) = 0$ whenever i is a δ-reducing player in v.

Theorem

A solution $\Phi : G(N) \to \mathbb{R}^n$ is equal to the δ-discounted Shapley value if and only if it satisfies efficiency, symmetry, additivity and the δ-reducing player property.

The Solidarity Value

Definition

The average marginal contribution of a player in a coalition S denoted by $A^v(S)$ as follows:

$$A^v(S) = \frac{1}{s}\sum_{k\in S}[v(S) - v(S \setminus k)]$$

The solidarity value to a player is the expectation of her average marginal contributions in all the coalitions where she belongs to and is given by

$$\Phi_i^{Sol}(v) = \sum_{i\in S} \frac{(n-s)!(s-1)!}{n!} A^v(S). \tag{6}$$

The Solidarity Value Vs The Shapley Value

Example(Three Brothers): Players 1, 2 and 3 are brothers and they live together. Player 1 and 2 can make together a profit of one unit, that is, $v\{1,2\} = 1$. Player 3 is a disabled person and can contribute nothing to any coalition. Therefore, $v\{1,2,3\} = 1$. Further, we have $v\{1,3\} = v\{2,3\} = 0$. Finally, we assume that $v\{i\} = 0$ for every player i.

- The Shapley Value of this game is $\Phi^{Sh}(v) = (1/2, 1/2, 0)$.
- Should the disabled brother leave his family?
- If players 1 and 2 take the responsibility for their brother (player 3), then the solidarity value $\Phi^{Sol}(v) = (7/18, 7/18, 4/18)$ seems to be a "better" solution for the game v than its Shapley value.

Characterization of the Solidarity Value

Definition

A player $i \in N$ is an A-Null player in a game v, if $A^v(S) = 0$ for every coalition S containing player i.

The A-Null player axiom demands that if a player i has the average marginal contribution $A^v(S) = 0$, then she receives nothing from the game.

Definition

(A-Null player axiom) A value Φ satisfies the A-Null player axiom, namely, $\Phi_i(v) = 0$ whenever $i \in N$ is an A-Null player in v.

Theorem

A solution $\Phi : G(N) \to \mathbb{R}^n$ satisfies efficiency, symmetry, additivity and the A-null player property if and only if it is the Solidarity value.

References

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