A generalized modified Bessel function and explicit transformations of certain Lambert series

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Joint work with Atul Dixit and Aashita Kesarwani

Outline of the talk

- The series $\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny}$, its historical background and its applications
- Generalized modified Bessel function
- Our master identity
- Special cases of the master identity
- A new transformation formula involving $r_k(n)$

Definition: modular forms

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A modular form $f : \mathbb{H} \to \mathbb{C}$ of weight k for $SL_2(\mathbb{Z})$ is a function on $\mathbb{H}$ with the following properties:

- *f is a holomorphic function on $\mathbb{H}$.*
- $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z), \ \forall\ z \in \mathbb{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$
- *f is holomorphic at $i\infty$.*

• The classical example of modular forms is Eisenstein series $E_k(z)$, k is an even positive integer greater than 2, given by

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}.$$

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Object of interest

Let $a \in \mathbb{C}$, $\mathrm{Re}(y) > 0$. Consider the series

$$\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny} = \sum_{n=1}^{\infty} \frac{n^a}{e^{ny} - 1}, \tag{1.1}$$

where $\sigma_a(n) = \sum_{d|n} d^a$.

• When $a = 2m+1, m \in \mathbb{N}$ and $y = -2\pi i z$ (so that $z \in \mathbb{H}$), either of the above series essentially represents the Eisenstein series of weight $2m+2$.

• When $a = -2m-1, m \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} \sigma_a(n) e^{2\pi i n z}$ represents Eichler integral corresponding to the Eisenstein series $E_{2m+2}(z)$.

• Moreover, $\sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}$ is essentially the quasi-modular form $E_2(z)$, and the series $\sum_{n=1}^{\infty} \sigma_{-1}(n) e^{2\pi i n z}$ is what appears in the transformation formula of logarithm of the Dedekind eta function $\eta(z)$.

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Transformation formula for a an odd integer: Ramanujan's famous formula for $\zeta(2m+1)$

- For a odd integer, we have the famous formula of Ramanujan for $\zeta(2m+1)$:
  Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Then for $m \in \mathbb{Z}\backslash\{0\}$, we have

$$\alpha^{-m}\left\{\frac{1}{2}\zeta(2m+1)+\sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2n\alpha}-1}\right\}$$
$$=(-\beta)^{-m}\left\{\frac{1}{2}\zeta(2m+1)+\sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2n\beta}-1}\right\}$$
$$-2^{2m}\sum_{k=0}^{m+1}\frac{(-1)^k B_{2k}B_{2m+2-2k}}{(2k)!(2m+2-2k)!}\alpha^{m+1-k}\beta^k.$$

- The history, implications and modern interpretation of this formula are nicely discussed by Berndt and Straub[1].

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- In all of the above cases, that is, when a an odd integer, the series $\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny}$ satisfies a transformation formula, and hence plays a fundamental role in the theory of modular forms.
- However, an explicit transformation for the series $\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny}$ when a is an *even integer* is conspicuously absent from the literature except in the special case $a = 0$.
- Guinand[2] studied the series $\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny}$ for $a = 2k$, $k \in \mathbb{R}$, such that $k + \frac{1}{2} \notin \mathbb{Z}$. Guinand obtained a transformation for the series which, for real positive y, shows how the modularity is obstructed by means of an appearance of an extra integral.

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Transformation formula for a an even integer

- For $a = 0$, Wigert-Bellman[3] gave the following result: For $\mathrm{Re}(y) > 0$, we have

$$\sum_{n=1}^{\infty} d(n)e^{-ny} - \frac{1}{4} - \frac{\gamma - \log(y)}{y}$$
$$= \frac{2}{y}\sum_{n=1}^{\infty} d(n)\left\{U\left(1,1,\frac{4\pi^2 n}{y}\right) + U\left(1,1,-\frac{4\pi^2 n}{y}\right)\right\},$$

where $U(a;c;z)$ is Tricomi confluent hypergeometric function defined by

$$U(a;c;z) := \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,{}_1F_1(a;c;z) + \frac{\Gamma(c-1)}{\Gamma(a)z^{c-1}}\,{}_1F_1(a-c+1;2-c;z),$$

with ${}_1F_1(b;c;z)$ being confluent hypergeometric function given by

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Moments of Riemann zeta function

- An important problem in analytic Number Theory is to understand the moments

$$L_{2k}(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} \, dt.$$

For positive real number k, it is believed that

$$L_{2k}(T) \sim C_k T (\log T)^{k^2}$$

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Application in the theory of moments of Riemann zeta function

- If we define

$$\Phi(z) := \sum_{m=1}^{\infty} d(m)e^{-mz} - \frac{\gamma - \log z}{z}.$$

Then, for $z = ixe^{-i\theta}, 0 < \theta < \pi/2$ and $\alpha > 0$, we have the transformation formula[4]

$$\Phi(z^{-1}) = 2\pi i z \Phi(4\pi^2 z) + O(x^\alpha).$$

- By the application of the above transformation formula, following asymptotic formula for the second moment of the Riemann zeta function due to Atkinson, is proved by M. Lukkarinen[5] , as $T \to \infty$,

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Application in the theory of smoothly weighted moments

- Smoothly weighted moment of the Riemann zeta function

$$L_{2k}(\delta) := \int_0^\infty \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} e^{-\delta t}\, dt.$$

- For $k = 1$, the best formula for $L_2(\delta)$ was a main term plus an asymptotic, but not convergent, series of powers of δ as $\delta \to 0^+$.
- In an interesting paper, Bettin and Conrey[6] studied the series $\sum_{n=1}^{\infty} \sigma_a(n) e^{2\pi i n z}$, where $a \in \mathbb{C}$ and $z \in \mathbb{H}$, showing that it can be analytically continued to $|\arg(z)| < \pi$, and as an application of their result they proved an exact formula for the second moments of the Riemann zeta function. Their formula gives an asymptotic series that is also convergent.
- They also gave a simple proof of the Voronoï summation formula.

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Our objective

- However, in the above results, either an asymptotic estimate for the series $\sum_{n=1}^{\infty} \sigma_a(n) e^{2\pi i n z}$ is given or a transformation involving a line integral, and hence an explicit transformation is missing.

- We fill this gap and obtain an explicit transformation for the series $\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny}$ first, for any $a \in \mathbb{C}$ such that $\mathrm{Re}(a) > -1$, and then for $\mathrm{Re}(a) > -2m-3$, where $m \in \mathbb{N} \cup \{0\}$, by analytic continuation.

- We then obtain, as corollaries, not only the well-known results in the theory of modular forms but also new transformations for $\sum_{n=1}^{\infty} \sigma_{2m}(n) e^{-ny}$, $m \in \mathbb{Z}$.

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Some definitions

- The Generalized hypergeometric series is defined as

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix} \middle| z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{1.2} $$

  The above series converges absolutely for all z if $p \leq q$ and for $|z| < 1$ if $p = q + 1$, and it diverges for all $z \neq 0$ if $p > q + 1$ and the series does not terminate.
- $(a)_n := a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$.

Definitions: Bessel functions

- The Bessel functions of the first kind and the second kind of order ν are defined by

$$J_\nu(z) := \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)} \qquad (z, \nu \in \mathbb{C}),$$

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin \pi\nu} \qquad (z \in \mathbb{C}, \nu \notin \mathbb{Z}),$$

along with $Y_n(z) = \lim_{\nu \to n} Y_\nu(z)$ for $n \in \mathbb{Z}$. Here $\Gamma(s)$ denotes Euler's gamma function.

- The modified Bessel functions of the first and second kinds are defined by

$$I_\nu(z) := \begin{cases} e^{-\frac{1}{2}\pi\nu i} J_\nu(e^{\frac{1}{2}\pi i} z), & \text{if } -\pi < \arg z \leq \frac{\pi}{2}, \\ e^{\frac{3}{2}\pi\nu i} J_\nu(e^{-\frac{3}{2}\pi i} z), & \text{if } \frac{\pi}{2} < \arg z \leq \pi, \end{cases}$$

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Paper of Koshliakov

- Koshliakov[7] proved the following remarkable result, that is, for $-\frac{1}{2} < \nu < \frac{1}{2}$,

$$\int_0^\infty K_\nu(t)\left(\cos(\pi\nu)M_{2\nu}(2\sqrt{xt}) - \sin(\pi\nu)J_{2\nu}(2\sqrt{xt})\right)\,dt = K_\nu(x), \tag{1.3}$$

where $M_\nu(x) := \frac{2}{\pi}K_\nu(x) - Y_\nu(x)$.

- It is easy to see though that this identity is valid for complex ν such that $-\frac{1}{2} < \mathrm{Re}(\nu) < \frac{1}{2}$.
- The kernel

$$\cos(\pi\nu)M_{2\nu}(2\sqrt{xt}) - \sin(\pi\nu)J_{2\nu}(2\sqrt{xt})$$

appears in the Voronoi summation formula.

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- Koshliakov gives a more general result of which (1.3) is a special case, that is, for[8] $\mu > -1/2$ and $\nu > -\frac{1}{2} + |\mu|$,

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$$= \frac{\pi 2^{\mu+\nu-1}}{\sin(\nu\pi)}\Bigg\{\left(\frac{x}{2}\right)^{-\nu}\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(1-\nu)\Gamma(\frac{1}{2}-\nu)}\,{}_1F_2\left(\begin{matrix}\mu+\frac{1}{2}\\ \frac{1}{2}-\nu, 1-\nu\end{matrix}\middle|\frac{x^2}{4}\right)$$
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- Upon letting $\mu = -\nu$ in (1.4) gives (1.3) as the right-hand side reduces to $K_\nu(x)$.
- Therefore, the right-hand side of (1.4) can be conceived to be a one-variable generalization of $K_\nu(x)$.

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Our generalization of Koshliakov's generalized modified Bessel function is defined for $\nu \in \mathbb{C}\backslash(\mathbb{Z}\backslash\{0\})$, and $z, \mu, w \in \mathbb{C}$ such that $\mu + w \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$, by

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{}_{\mu}K_\nu(z,w) := \frac{\pi z^w 2^{\mu+\nu-1}}{\sin(\nu\pi)}\left\{\frac{\left(\frac{z}{2}\right)^{-\nu}\Gamma(\mu+w+\frac{1}{2})}{\Gamma(1-\nu)\Gamma(w+\frac{1}{2}-\nu)}\,{}_1F_2\left(\begin{matrix}\mu+w+\frac{1}{2}\\ w+\frac{1}{2}-\nu, 1-\nu\end{matrix}\middle|\frac{z^2}{4}\right)\right.
$$
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with ${}_{\mu}K_0(z,w) = \lim_{\nu\to 0}{}_{\mu}K_\nu(z,w)$.

- When $w = 0$, ${}_{\mu}K_\nu(z,w)$ reduces to *Koshliakov's generalized modified Bessel function of the second kind.*
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Main result

Theorem (Dixit-Kesarwani-K.)

Let $\mathrm{Re}(y) > 0$. *For* $\mathrm{Re}(a) > -1$, *the following transformation holds:*

$$\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny} + \frac{1}{2}\left(\left(\frac{2\pi}{y}\right)^{1+a} \operatorname{cosec}\left(\frac{\pi a}{2}\right) + 1\right)\zeta(-a) - \frac{\zeta(1-a)}{y}$$
$$= \frac{2\pi}{y \sin\left(\frac{\pi a}{2}\right)} \sum_{n=1}^{\infty} \sigma_a(n)\left(\frac{(2\pi n)^{-a}}{\Gamma(1-a)}\, {}_1F_2\left(1; \frac{1-a}{2}, 1-\frac{a}{2}; \frac{4\pi^4 n^2}{y^2}\right) - \left(\frac{2\pi}{y}\right)^{a} \cosh\left(\frac{4\pi^2 n}{y}\right)\right).$$

• Note that the right-hand side of the above transformation is nothing but

$$\frac{2\sqrt{2\pi}}{y^{1+\frac{a}{2}}} \sum_{n=1}^{\infty} \sigma_a(n) n^{-\frac{a}{2}}\, {}_{\frac{1}{2}}K_{\frac{a}{2}}\left(\frac{4\pi^2 n}{y}, 0\right).$$

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Master identity

Theorem (Dixit-Kesarwani-K.)

Let $\mathrm{Re}(y) > 0$ *and* $m \in \mathbb{N} \cup \{0\}$. *Then for* $\mathrm{Re}(a) > -2m-3$, *the following identity holds:*

$$\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny} + \frac{1}{2}\left(\left(\frac{2\pi}{y}\right)^{1+a} \operatorname{cosec}\left(\frac{\pi a}{2}\right) + 1\right)\zeta(-a) - \frac{\zeta(1-a)}{y}$$

$$= \frac{2\sqrt{2\pi}}{y^{1+\frac{a}{2}}} \sum_{n=1}^{\infty} \sigma_a(n) n^{-\frac{a}{2}} \left\{ \frac{1}{2} K_{\frac{a}{2}}\left(\frac{4\pi^2 n}{y}, 0\right) - \frac{\pi 2^{\frac{3}{2}+a}}{\sin\left(\frac{\pi a}{2}\right)} \left(\frac{4\pi^2 n}{y}\right)^{-\frac{a}{2}-2} \right.$$

$$\left. \times A_m\left(\frac{1}{2}, \frac{a}{2}, 0; \frac{4\pi^2 n}{y}\right)\right\} - \frac{y(2\pi)^{-a-3}}{\sin\left(\frac{\pi a}{2}\right)} \sum_{k=0}^{m} \frac{\zeta(a+2k+2)\zeta(2k+2)}{\Gamma(-a-1-2k)} \left(\frac{4\pi^2}{y}\right)^{-2k},$$

where

$$A_m(\mu, \nu, w; z) = \sum_{k=0}^{m} \frac{(-1)^{-\mu-w-\frac{1}{2}}\Gamma\left(\mu + w + \frac{1}{2} + k\right)}{k!\Gamma\left(-\nu-\mu-k\right)\Gamma\left(\frac{1}{2}-\nu-\mu-w-k\right)} \left(\frac{z}{2}\right)^{-2k}.$$

Outline of the proof

- Koshliakov's result is:

$$\int_0^\infty K_\mu(t)t^{\mu+\nu}\left(\cos(\pi\nu)M_{2\nu}(2\sqrt{xt})-\sin(\pi\nu)J_{2\nu}(2\sqrt{xt})\right)\,dt$$
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- Let $\mu=\frac{1}{2},\nu=\frac{a}{2}$ and replace x by $4\pi^2x/y$ in the above equation and then use $K_{\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2x}}e^{-x}$ to obtain

$$\int_0^\infty e^{-ty}t^{\frac{a}{2}}\left(\cos\left(\frac{\pi a}{2}\right)M_a(4\pi\sqrt{xt})-\sin\left(\frac{\pi a}{2}\right)J_a(4\pi\sqrt{xt})\right)\,dt$$
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Proof continued

- The following analogue of Voronoï summation formula for $\sigma_a(n)$ due to Guinand is one of the main tool for the proof.
- Let $-\frac{1}{2} < \mathrm{Re}(a) < \frac{1}{2}$. If $f(x)$ and $f'(x)$ are integrals, f tends to zero as $x \to \infty$, $f(x), xf'(x)$ and $x^2 f''(x)$ belong to $L^2(0,\infty)$, and

$$g(x) = 2\pi \int_0^\infty f(t) \left(\cos\left(\frac{\pi a}{2}\right) M_a(4\pi\sqrt{xt}) - \sin\left(\frac{\pi a}{2}\right) J_a(4\pi\sqrt{xt}) \right) dt,$$

then the following transformation holds:

$$\sum_{n=1}^\infty \sigma_{-a}(n) n^{\frac{a}{2}} f(n) - \zeta(1+a)\int_0^\infty x^{\frac{a}{2}} f(x)\,dx - \zeta(1-a)\int_0^\infty x^{-\frac{a}{2}} f(x)\,dx$$

$$= \sum_{n=1}^\infty \sigma_{-a}(n) n^{\frac{a}{2}} g(n) - \zeta(1+a)\int_0^\infty x^{\frac{a}{2}} g(x)\,dx - \zeta(1-a)\int_0^\infty x^{-\frac{a}{2}} g(x)$$

- The phrase '$f(x)$ is an integral' means f is an integral of some function, that is, f can be written in the form $f(x) = \int_a^x h(t)\,dt$ for some function h and $-\infty \le a < x$.
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$$g(x) = \frac{2\pi}{y\sin\left(\frac{\pi a}{2}\right)}\left\{\frac{x^{-\frac{a}{2}}(2\pi)^{-a}}{\Gamma(1-a)}\,{}_1F_2\left(1;\frac{1-a}{2},1-\frac{a}{2};\frac{4\pi^4x^2}{y^2}\right) - \left(\frac{2\pi\sqrt{x}}{y}\right)^a\cosh\left(\frac{4\pi^2x}{y}\right)\right\}.$$

- It is easy to see

$$\int_0^\infty x^{\frac{a}{2}}f(x)\,dx = y^{-a-1}\Gamma(a+1),$$

$$\int_0^\infty x^{-\frac{a}{2}}f(x)\,dx = \frac{1}{y}.$$

- For $0 < a < 1/2$ and $y > 0$,

$$\int_0^\infty x^{\frac{a}{2}}g(x)\;dx = \frac{2^{-2-a}\pi^{-a}\sec\left(\frac{\pi a}{2}\right)}{\Gamma(-a)},$$

$$\int_0^\infty x^{-\frac{a}{2}}g(x)\;dx = 0.$$

Proof continued

- Let $f(x) = e^{-xy} x^{\frac{a}{2}}$. Then

$$g(x) = \frac{2\pi}{y \sin\left(\frac{\pi a}{2}\right)} \left\{ \frac{x^{-\frac{a}{2}} (2\pi)^{-a}}{\Gamma(1-a)} {}_1F_2\left(1; \frac{1-a}{2}, 1-\frac{a}{2}; \frac{4\pi^4 x^2}{y^2}\right) - \left(\frac{2\pi\sqrt{x}}{y}\right)^a \cosh\left(\frac{4\pi^2 x}{y}\right) \right\}.$$

- It is easy to see

$$\int_0^\infty x^{\frac{a}{2}} f(x)\, dx = y^{-a-1}\Gamma(a+1),$$
$$\int_0^\infty x^{-\frac{a}{2}} f(x)\, dx = \frac{1}{y}.$$

- For $0 < a < 1/2$ and $y > 0$,

$$\int_0^\infty x^{\frac{a}{2}} g(x)\ dx = \frac{2^{-2-a}\pi^{-a}\sec\left(\frac{\pi a}{2}\right)}{\Gamma(-a)},$$
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Proof continued

- After some simplification, for $0 < a < \frac{1}{2}$ and $y > 0$, we proved:

$$\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny} + \frac{1}{2}\left(\left(\frac{2\pi}{y}\right)^{1+a} \operatorname{cosec}\left(\frac{\pi a}{2}\right) + 1\right)\zeta(-a) - \frac{\zeta(1-a)}{y}$$

$$= \frac{2\pi}{y \sin\left(\frac{\pi a}{2}\right)} \sum_{n=1}^{\infty} \sigma_a(n) \left(\frac{(2\pi n)^{-a}}{\Gamma(1-a)}\, {}_1F_2\left(1; \frac{1-a}{2}, 1-\frac{a}{2}; \frac{4\pi^4 n^2}{y^2}\right) - \left(\frac{2\pi}{y}\right)^{a} \cosh\left(\frac{4\pi^2 n}{y}\right)\right)$$

$$= \frac{2\sqrt{2\pi}}{y^{1+\frac{a}{2}}} \sum_{n=1}^{\infty} \sigma_a(n) n^{-\frac{a}{2}}\, {}_{\frac{1}{2}}K_{\frac{a}{2}}\left(\frac{4\pi^2 n}{y}, 0\right).$$

- Note that, as $n \to \infty$, ${}_{\frac{1}{2}}K_{\frac{a}{2}}\left(\frac{4\pi^2 n}{y}, 0\right) = O(n^{-\frac{1}{2}\mathrm{Re}(a)-2})$.
- This shows that the right-hand side converges uniformly for $\mathrm{Re}(a) > -1$ therefore it is analytic in this region. It is also easy to see that left-hand side is also analytic in this region, therefore transformation holds for $\mathrm{Re}(a) > -1$.

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- After some simplification, for $0 < a < \frac{1}{2}$ and $y > 0$, we proved:

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$$\left. - \left(\frac{2\pi}{y}\right)^{a} \cosh\left(\frac{4\pi^2 n}{y}\right)\right)$$

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Proof continued

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Lemma (Dixit-Kesarwani-K.)

Let $|\arg(-z)| \leq \pi$. *As* $z \to \infty$,

$$ {}_{\mu}K_{\nu}\left(z,w\right) = \frac{\pi 2^{3\mu+2\nu+2w}}{\sin(\pi\nu) z^{w+2\mu+\nu+1}} \Big\{ A_m(\mu,\nu,w;z) + B_m(\mu,\nu,w;z) + O_{\mu,\nu,w}\left(|z|^{-2m-2}\right) \Big\}, \tag{1.5} $$

where

$$ A_m(\mu,\nu,w;z) = \sum_{k=0}^{m} \frac{(-1)^{-\mu-w-\frac{1}{2}}\Gamma\left(\mu+w+\frac{1}{2}+k\right)}{k!\Gamma\left(-\nu-\mu-k\right)\Gamma\left(\frac{1}{2}-\nu-\mu-w-k\right)} \left(\frac{z}{2}\right)^{-2k} $$

$$ B_m(\mu,\nu,w;z) = \sum_{k=0}^{m} \frac{(-1)^{-\mu-\nu-w-\frac{1}{2}}\Gamma\left(\mu+\nu+w+\frac{1}{2}+k\right)}{k!\Gamma\left(-\mu-\nu-k\right)\Gamma\left(\frac{1}{2}-\mu-w-k\right)} \left(\frac{z}{2}\right)^{-2k}. $$

- If we rewrite the right-hand side then

$$\sum_{n=1}^{\infty}\sigma_a(n)e^{-ny}+\frac{1}{2}\left(\left(\frac{2\pi}{y}\right)^{1+a}\operatorname{cosec}\left(\frac{\pi a}{2}\right)+1\right)\zeta(-a)-\frac{\zeta(1-a)}{y}$$
$$=\frac{2\sqrt{2\pi}}{y^{1+\frac{a}{2}}}\sum_{n=1}^{\infty}\sigma_a(n)n^{-\frac{a}{2}}\left\{{}_{\frac{1}{2}}K_{\frac{a}{2}}\left(\frac{4\pi^2 n}{y},0\right)-\frac{\pi 2^{\frac{3}{2}+a}}{\sin\left(\frac{\pi a}{2}\right)}\left(\frac{4\pi^2 n}{y}\right)^{-\frac{a}{2}-2}\right.$$
$$\left.\times A_m\left(\frac{1}{2},\frac{a}{2},0;\frac{4\pi^2 n}{y}\right)\right\}+\frac{y\pi^{-a-5/2}}{2\sin\left(\frac{\pi a}{2}\right)}\sum_{n=1}^{\infty}\frac{\sigma_a(n)}{n^{a+2}}A_m\left(\frac{1}{2},\frac{a}{2},0;\frac{4\pi^2 n}{y}\right).$$

- Note that

$$\frac{\sigma_a(n)}{n^{\frac{a}{2}}}\left\{{}_{\frac{1}{2}}K_{\frac{a}{2}}\left(\frac{4\pi^2 n}{y},0\right)-\frac{\pi 2^{\frac{3}{2}+a}}{\sin\left(\frac{\pi a}{2}\right)}\left(\frac{4\pi^2 n}{y}\right)^{-\frac{a}{2}-2}A_m\left(\frac{1}{2},\frac{a}{2},0;\frac{4\pi^2 n}{y}\right)\right\}$$
$$=O_{a,y}\left(n^{-2m-4-\frac{1}{2}\mathrm{Re}(a)+\frac{1}{2}|\mathrm{Re}(a)|+\epsilon}\right).$$

- This implies that the series on the right-hand side converges uniformly in $\mathrm{Re}(a)>-2m-3+\epsilon$ for any $\epsilon>0$. Since the summand is analytic, we see by Weierstrass' theorem that this series represents an analytic function of a for $\mathrm{Re}(a)>-2m-3$.

Analytic continuation of the Main result

- If we rewrite the right-hand side then

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$$=\frac{2\sqrt{2\pi}}{y^{1+\frac{a}{2}}}\sum_{n=1}^{\infty}\sigma_a(n)n^{-\frac{a}{2}}\left\{{}_{\frac{1}{2}}K_{\frac{a}{2}}\left(\frac{4\pi^2n}{y},0\right)-\frac{\pi 2^{\frac{3}{2}+a}}{\sin\left(\frac{\pi a}{2}\right)}\left(\frac{4\pi^2n}{y}\right)^{-\frac{a}{2}-2}\right.$$

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- Use the well-known fact

$$\sum_{n=1}^{\infty} \frac{\sigma_z(n)}{n^s} = \zeta(s)\zeta(s-z), \tag{1.6}$$

valid for $\mathrm{Re}(s) > \max\{1,\ 1+\mathrm{Re}(z)\}$ to simplify

$$\frac{y\pi^{-a-5/2}}{2\sin\left(\frac{\pi a}{2}\right)} \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^{a+2}} A_m\left(\frac{1}{2}, \frac{a}{2}, 0; \frac{4\pi^2 n}{y}\right)$$

as

$$-\frac{y(2\pi)^{-a-3}}{\sin\left(\frac{\pi a}{2}\right)} \sum_{k=0}^{m} \frac{\zeta(a+2k+2)\zeta(2k+2)}{\Gamma(-a-1-2k)} \left(\frac{4\pi^2}{y}\right)^{-2k}.$$

Theorem (Dixit-Kesarwani-K.)

Let $m \in \mathbb{N}$. Then for $\mathrm{Re}(y)>0$, *we have*

$$\sum_{n=1}^{\infty}\sigma_{2m}(n)e^{-ny}-\frac{(2m)!}{y^{2m+1}}\zeta(2m+1)+\frac{B_{2m}}{2my}$$
$$=(-1)^m\frac{2}{\pi}\left(\frac{2\pi}{y}\right)^{2m+1}\sum_{n=1}^{\infty}\sigma_{2m}(n)\Bigg\{\sinh\left(\frac{4\pi^2n}{y}\right)\mathrm{Shi}\left(\frac{4\pi^2n}{y}\right)$$
$$-\cosh\left(\frac{4\pi^2n}{y}\right)\mathrm{Chi}\left(\frac{4\pi^2n}{y}\right)+\sum_{j=1}^{m}(2j-1)!\left(\frac{4\pi^2n}{y}\right)^{-2j}\Bigg\}.$$

- The functions $\mathrm{Shi}(z)$ and $\mathrm{Chi}(z)$ are the *hyperbolic sine and cosine integrals* defined by

$$\mathrm{Shi}(z):=\int_0^z\frac{\sinh(t)}{t}\,dt,$$

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The transformation for $\sum_{n=1}^{\infty} \sigma_{2m}(n) e^{-ny}, m > 0$

Theorem (Dixit-Kesarwani-K.)

Let $m \in \mathbb{N}$. *Then for* $\mathrm{Re}(y) > 0$, *we have*

$$\sum_{n=1}^{\infty} \sigma_{2m}(n) e^{-ny} - \frac{(2m)!}{y^{2m+1}} \zeta(2m+1) + \frac{B_{2m}}{2my}$$

$$= (-1)^m \frac{2}{\pi} \left(\frac{2\pi}{y}\right)^{2m+1} \sum_{n=1}^{\infty} \sigma_{2m}(n) \left\{ \sinh\left(\frac{4\pi^2 n}{y}\right) \mathrm{Shi}\left(\frac{4\pi^2 n}{y}\right) \right.$$

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The series in modular v/s in non-modular transformations

- The series $\sum_{n=1}^{\infty} \sigma_{2m}(n) e^{-ny}$ as well as $\sum_{n=1}^{\infty} \sigma_{-2m}(n) e^{-ny}$ and $\sum_{n=1}^{\infty} d(n) e^{-ny}$ gets transformed to

$$\sum_{n=1}^{\infty} \sigma_{2m}(n) \Bigg\{ \sinh\left(\frac{4\pi^2 n}{y}\right) \mathrm{Shi}\left(\frac{4\pi^2 n}{y}\right) - \cosh\left(\frac{4\pi^2 n}{y}\right) \mathrm{Chi}\left(\frac{4\pi^2 n}{y}\right) + \sum_{j=1}^{m} (2j-1)! \left(\frac{4\pi^2 n}{y}\right)^{-2j} \Bigg\}.$$

- This is a natural analogue of the series

$$\sum_{n=1}^{\infty} \sigma_{2m+1}(n) \left\{ \sinh\left(\frac{4\pi^2 n}{y}\right) - \cosh\left(\frac{4\pi^2 n}{y}\right) \right\} = -\sum_{n=1}^{\infty} \sigma_{2m+1}(n) e^{-\frac{4\pi^2 n}{y}}.$$

which appears in the corresponding transformation satisfied by $\sum_{n=1}^{\infty} \sigma_{2m+1}(n) e^{-ny}$.

Proof

- Our transformation is:

$$\sum_{n=1}^{\infty}\sigma_a(n)e^{-ny}+\frac{1}{2}\left(\left(\frac{2\pi}{y}\right)^{1+a}\csc\left(\frac{\pi a}{2}\right)+1\right)\zeta(-a)-\frac{\zeta(1-a)}{y}$$
$$=\frac{2\pi}{y\sin\left(\frac{\pi a}{2}\right)}\sum_{n=1}^{\infty}\sigma_a(n)\left(\frac{(2\pi n)^{-a}}{\Gamma(1-a)}{}_1F_2\left(1;\frac{1-a}{2},1-\frac{a}{2};\frac{4\pi^4n^2}{y^2}\right)\right.$$
$$\left.-\left(\frac{2\pi}{y}\right)^{a}\cosh\left(\frac{4\pi^2 n}{y}\right)\right).$$

- Observe that we have $\frac{0}{0}$ form on the both sides of the above transformation.
- Therefore we need to find the derivative of ${}_1F_2$. That is also new:

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Lemma (Dixit-Kesarwani-K.)

Let $n \in \mathbb{N}$*,* $m \in \mathbb{N} \cup \{0\}$ *and* $y \in \mathbb{C}$*. Then*

$$\frac{d}{da}\left(\frac{1}{\Gamma(1-a)}\,{}_1F_2\left(1; 1-\frac{a}{2}, \frac{1-a}{2}; \frac{4\pi^4 n^2}{y^2}\right)\right)\Bigg|_{a=2m}$$
$$= \left(\frac{4\pi^2 n}{y}\right)^{2m}\left\{\sinh\left(\frac{4\pi^2 n}{y}\right)\mathrm{Shi}\left(\frac{4\pi^2 n}{y}\right) - \cosh\left(\frac{4\pi^2 n}{y}\right)\mathrm{Chi}\left(\frac{4\pi^2 n}{y}\right)\right.$$
$$\left. + \log\left(\frac{4\pi^2 n}{y}\right)\cosh\left(\frac{4\pi^2 n}{y}\right) + \sum_{j=1}^{m}(2j-1)!\left(\frac{4\pi^2 n}{y}\right)^{-2j}\right\}.$$

• As a special case $a = -2m-1,\ m > 0$ of our Theorem, we get the famous formula of Ramanujan for $\zeta(2m+1)$:

Theorem (Ramanujan)

Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then for $m \in \mathbb{Z}\backslash\{0\}$, we have

$$\alpha^{-m}\left\{\frac{1}{2}\zeta(2m+1) + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2n\alpha}-1}\right\}$$
$$= (-\beta)^{-m}\left\{\frac{1}{2}\zeta(2m+1) + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2n\beta}-1}\right\}$$
$$- 2^{2m}\sum_{k=0}^{m+1}\frac{(-1)^k B_{2k}B_{2m+2-2k}}{(2k)!(2m+2-2k)!}\alpha^{m+1-k}\beta^k.$$

Theorem (Dixit-Kesarwani-K.)

Let $m \in \mathbb{N}$. If α and β are complex numbers such that $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, *and* $\alpha\beta = \pi^2$, *then*

$$\alpha^{-\left(m-\frac{1}{2}\right)} \left\{ \frac{1}{2}\zeta(2m) + \sum_{n=1}^{\infty} \frac{n^{-2m}}{e^{2n\alpha}-1} \right\}$$

$$= (-1)^{m+1}\beta^{-\left(m-\frac{1}{2}\right)} \left\{ \frac{\gamma}{\pi}\zeta(2m) + \frac{1}{2\pi}\sum_{n=1}^{\infty} n^{-2m} \left(\psi\left(\frac{in\beta}{\pi}\right) + \psi\left(-\frac{in\beta}{\pi}\right)\right)\right\}$$

$$+ \sum_{k=0}^{m-1} \frac{2^{2k-1}B_{2k}}{(2k)!}\zeta(2m-2k+1)\alpha^{2k-m-\frac{1}{2}}.$$

- Here $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Other transformations from the literature

If $g : \mathbb{Z} \to \mathbb{C}$ has period q then g has mean value $M(g) = \frac{1}{q}\sum_{n=0}^{q-1} g(n)$, and the Dirichlet series

$$L(s,g) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

converges absolutely in the half-plane $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$.

Theorem (Bradley)

Let m and q be positive integers, $\omega = e^{2\pi i/m}$, $g : \mathbb{Z} \to \mathbb{C}$ periodic of period q. Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. If g is odd then

$$\alpha^{-m+\frac{1}{2}}\left\{\frac{1}{2}L(2m,g) + \sum_{n=1}^{\infty}\frac{n^{-2m}g(n)}{e^{2n\alpha}-1}\right\}$$
$$= (-1)^m \beta^{-m+\frac{1}{2}} i q^{-1} \sum_{k=1}^{q-1} g(k) \sum_{n=1}^{\infty} \frac{n^{-2m}}{e^{2n\beta/q}\omega^k - 1}$$
$$+ \sum_{j=0}^{m} (-1)^{j+1} \alpha^{j-\frac{1}{2}-m} \beta^{-j} \zeta(2j) L(2m-2j+1, g).$$

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Let m and q be positive integers, $\omega = e^{2\pi i/m}$, $g : \mathbb{Z} \to \mathbb{C}$ periodic of period q. Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. If g is odd then

$$\alpha^{-m+\frac{1}{2}} \left\{ \frac{1}{2} L(2m, g) + \sum_{n=1}^{\infty} \frac{n^{-2m} g(n)}{e^{2n\alpha} - 1} \right\}$$

$$= (-1)^m \beta^{-m+\frac{1}{2}} i q^{-1} \sum_{k=1}^{q-1} g(k) \sum_{n=1}^{\infty} \frac{n^{-2m}}{e^{2n\beta/q}\omega^k - 1}$$

$$+ \sum_{j=0}^{m} (-1)^{j+1} \alpha^{j-\frac{1}{2}-m} \beta^{-j} \zeta(2j) L(2m - 2j + 1, g).$$

Theorem

If g is even then

$$\alpha^{-m}\left\{\frac{1}{2}L(2m+1,g)+\sum_{n=1}^{\infty}\frac{n^{-2m-1}g(n)}{e^{2n\alpha}-1}\right\}$$
$$=(-1)^m\beta^{-m}\left\{\frac{1}{2}M(g)\zeta(2m+1)+q^{-1}\sum_{k=1}^{q-1}g(k)\sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2n\beta/q}\omega^k-1}\right\}$$
$$+\sum_{j=0}^{m+1}(-1)^{j+1}\alpha^{j-m-1}\beta^{-j}\zeta(2j)L(2m-2j+2,g).$$

- The above result is due to Bradley[9].
- Berndt also obtained similar results[10].

[9] D. M. Bradley, *Series acceleration formulas for Dirichlet series with periodic coefficients*, Ramanujan J. **6** (2002), 331–346.

[10] B.C. Berndt, Periodic Bernoulli numbers, summation formulas and applications, in: Theory and Application of Special Functions, R.A. Askey, ed., Academic Press, New York, 1975, pp. 143–189.

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Theorem: Let $m \in \mathbb{N}$. Then for $\mathrm{Re}(y) > 0$, we have

$$\sum_{n=1}^{\infty} \sigma_{-2m}(n) e^{-ny} + \left(\frac{1}{2} + (-1)^m \frac{\gamma}{\pi} \left(\frac{y}{2\pi} \right)^{2m-1} \right) \zeta(2m) = \frac{2(-1)^{m+1}}{\pi} \left(\frac{y}{2\pi} \right)^{2m-1}$$

$$\times \left[\sum_{n=1}^{\infty} \sigma_{-2m}(n) \left\{ \sinh\left(\frac{4\pi^2 n}{y} \right) \mathrm{Shi}\left(\frac{4\pi^2 n}{y} \right) - \cosh\left(\frac{4\pi^2 n}{y} \right) \mathrm{Chi}\left(\frac{4\pi^2 n}{y} \right) \right\} \right.$$

$$\left. - \log\left(\frac{2\pi}{y} \right) \zeta(2m) + \frac{1}{2} \zeta'(2m) + \frac{2\pi^2}{y^2} \sum_{k=0}^{m-1} \frac{(-1)^{k+1}}{(2\pi/y)^{-2k}} \zeta(2k+3) \zeta(2m-2k-2) \right]$$

- We make use of the following beautiful recent result[11]:

Theorem (Dixit, Gupta, K.-, Maji)

Let $\mathrm{Re}(u) > 0$. *Then*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{t \cos(t)}{t^2 + n^2 u^2} \, dt = \frac{1}{2} \left\{ \log\left(\frac{u}{2\pi} \right) - \frac{1}{2} \left(\psi\left(\frac{iu}{2\pi} \right) + \psi\left(-\frac{iu}{2\pi} \right) \right) \right\}.$$

[11] A. Dixit, R. Gupta, R. Kumar and B. Maji, *Generalized Lambert series, Raabe's cosine transform and a generalization of Ramanujan's formula for* $\zeta(2m+1)$, Nagoya Math. J. **239** (2020), 232–293.

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Transformation satisfied by the weight-2 Eisenstein series $E_2(z)$

The limiting case $a \to 1$ of our Theorem gives an equivalent form of the transformation satisfied by the weight-2 Eisenstein series $E_2(z)$ on $\mathrm{SL}_2(\mathbb{Z})$, namely, $E_2\left(-\frac{1}{z}\right) = z^2 E_2(z) + \frac{6z}{\pi i}$.

Corollary

Let α, β be such that $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ and $\alpha\beta = \pi^2$. Then

$$\alpha \sum_{n=1}^{\infty} \frac{n}{e^{2n\alpha} - 1} + \beta \sum_{n=1}^{\infty} \frac{n}{e^{2n\beta} - 1} = \frac{\alpha + \beta}{24} - \frac{1}{4}. \tag{1.7}$$

Proof

- Note that

$$\lim_{a\to 1} \frac{1}{\Gamma(1-a)} {}_1F_2\left(1; 1-\frac{a}{2}, \frac{1-a}{2}, \frac{4\pi^4 n^2}{y^2}\right) = \frac{4\pi^2 n}{y} \sinh\left(\frac{4\pi^2 n}{y}\right). \tag{1.8}$$

- Let $a = 1$ in the main result. Using the well-known special values $\zeta(-1) = -1/12$, $\zeta(0) = -1/2$ and invoking (1.8), we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sigma(n) e^{-ny} - \frac{1}{24}\left(1 + \frac{4\pi^2}{y^2}\right) + \frac{1}{2y} \\
&= \frac{4\pi^2}{y^2} \sum_{n=1}^{\infty} \sigma(n) \left(\sinh\left(\frac{4\pi^2 n}{y}\right) - \cosh\left(\frac{4\pi^2 n}{y}\right)\right) \\
&= -\frac{4\pi^2}{y^2} \sum_{n=1}^{\infty} \sigma(n) e^{-\frac{4\pi^2 n}{y}}.
\end{aligned}$$

- Letting $y = 2\alpha$ with $\alpha\beta = \pi^2$ then leads to (1.7).

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$$\begin{aligned}&\sum_{n=1}^{\infty}\sigma(n)e^{-ny}-\frac{1}{24}\left(1+\frac{4\pi^2}{y^2}\right)+\frac{1}{2y}\\&=\frac{4\pi^2}{y^2}\sum_{n=1}^{\infty}\sigma(n)\left(\sinh\left(\frac{4\pi^2n}{y}\right)-\cosh\left(\frac{4\pi^2n}{y}\right)\right)\\&=-\frac{4\pi^2}{y^2}\sum_{n=1}^{\infty}\sigma(n)e^{-\frac{4\pi^2n}{y}}.\end{aligned}$$

- Letting $y=2\alpha$ with $\alpha\beta=\pi^2$ then leads to (1.7).

Transformation formula satisfied by the logarithm of the Dedekind eta function as a special case

Next, we give a corollary which is equivalent to the transformation formula satisfied by the logarithm of the Dedekind eta function.

Corollary (Dixit-Kesarwani-K.)

If α, β are such that $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ and $\alpha\beta = \pi^2$, then

$$\sum_{n=1}^{\infty} \sigma_{-1}(n) e^{-2n\alpha} - \sum_{n=1}^{\infty} \sigma_{-1}(n) e^{-2n\beta} = \frac{\beta - \alpha}{12} + \frac{1}{4} \log\left(\frac{\alpha}{\beta}\right),$$

Wigert-Bellman result as a special case

Let $\mathrm{Re}(y) > 0$,

$$\sum_{n=1}^{\infty} d(n)e^{-ny} - \frac{1}{4} - \frac{\gamma - \log(y)}{y}$$
$$= \frac{2}{y}\sum_{n=1}^{\infty} d(n)\left\{U\left(1, 1, \frac{4\pi^2 n}{y}\right) + U\left(1, 1, -\frac{4\pi^2 n}{y}\right)\right\}.$$

equivalently,

$$\sum_{n=1}^{\infty} d(n)e^{-ny} - \frac{1}{4} - \frac{\gamma - \log(y)}{y}$$
$$= \frac{2}{y}\sum_{n=1}^{\infty}\left\{\log\left(\frac{2\pi n}{y}\right) - \frac{1}{2}\left(\psi\left(\frac{2\pi i n}{y}\right) + \psi\left(-\frac{2\pi i n}{y}\right)\right)\right\}.$$

- The series on the right-hand converges as we have, as $z \to \infty$,

$$\psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots$$

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Koshliakov's kernels

As we discussed earlier that Koshliakov studied the following two kernels

$$\cos(\pi\nu)M_{2\nu}(2\sqrt{xt}) - \sin(\pi\nu)J_{2\nu}(2\sqrt{xt}), \tag{1.9}$$

$$\sin(\pi\nu)J_{2\nu}(2\sqrt{xt}) - \cos(\pi\nu)L_{2\nu}(2\sqrt{xt}), \tag{1.10}$$

where

$$M_\nu(x) := \frac{2}{\pi}K_\nu(x) - Y_\nu(x), \qquad L_\nu(x) := -\frac{2}{\pi}K_\nu(x) - Y_\nu(x).$$

Let $x > 0$, $\nu \in \mathbb{C}\setminus(\mathbb{Z}\setminus\{0\})$ and $w \in \mathbb{C}$. We define a generalization of the two Koshliakov kernels in (1.9) and (1.10) by

$$\mathscr{G}_\nu(x,w) := \frac{\pi}{\sin(\nu\pi)}\left(\frac{x}{4}\right)^w\left[\left(\frac{x}{4}\right)^{-\nu}\frac{{}_0F_3\left(\begin{matrix}-\\1-\nu, w+1/2, w+1/2-\nu\end{matrix}\middle|\frac{x^2}{16}\right)}{\Gamma(1-\nu)\Gamma(w+1/2)\Gamma(w+1/2-\nu)}\right.$$
$$\left.-\left(\frac{x}{4}\right)^{\nu}\frac{{}_0F_3\left(\begin{matrix}-\\1+\nu, w+1/2, w+1/2+\nu\end{matrix}\middle|\frac{x^2}{16}\right)}{\Gamma(1+\nu)\Gamma(w+1/2)\Gamma(w+1/2+\nu)}\right], \quad (1.11)$$

where ${}_0F_3$ are the hyper-Bessel functions whose theory, in the general case, that is for ${}_0F_n$, was initiated by Delerue.

- The hyper-Bessel functions have been found useful in many applications, for example, they are used to understand the wavefields and the elements of the non-adiabatic transition matrix and the tunnelling loss matrix[12].
- The two expressions inside the square brackets in (1.11) are entire functions of ν and w. As a function of ν, $\mathscr{G}_\nu(x,w)$ has a pole at every non-zero integer but a removable singularity at 0.

[12] N. S. Witte, *Exact solution for the reflection and diffraction of atomic de*

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Generalization of Koshliakov's kernels

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- The kernel $\mathscr{G}_\nu(x, w)$ is not new. In fact, it is a special case of the well-known kernel introduced by Watson[13], namely,

$$\varpi_{\mu,\nu}(xy) = A(xy)^{1/2} \int_0^\infty J_\nu(xt) J_\mu\left(\frac{Ay}{t}\right) \frac{dt}{t}.$$

- Also, for $|\mathrm{Re}(\nu)| < \mathrm{Re}(w) + 1$, we have

$$\mathscr{G}_\nu(x, w) = \varpi_{w-\nu-\frac{1}{2}, w+\nu-\frac{1}{2}}(x).$$

- Watson's kernel and its many other generalizations have been studied by many authors like Bhatnagar[14], Olkha and Rathie[15], Dahiya[16] etc.

[13] G. N. Watson, *Some self-reciprocal functions*, Quart. J. Math. (Oxford) **2** (1931) 298–309.

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In spite of so much work done on Watson's kernel, its importance from the point of view of number theory has not been recognized before. For example, while it is known [17] that for $x > 0$,

$$\mathscr{G}_{\frac{1}{2}}(x, w) = J_{2w-1}(2\sqrt{x}), \tag{1.12}$$

it has not been noticed before that when $w = 0$ and 1, the kernel $\varpi_{w-\nu-\frac{1}{2}, w+\nu-\frac{1}{2}}(x)$, or equivalently $\mathscr{G}_{\nu}(x, w)$, reduces respectively to the first and second Koshliakov kernels:

Theorem (Dixit-Kesarwani-K.)

Let $x > 0$. Then

$$\mathscr{G}_{\nu}(x, 0) = \cos(\pi\nu) M_{2\nu}(2\sqrt{x}) - \sin(\pi\nu) J_{2\nu}(2\sqrt{x}),$$
$$\mathscr{G}_{\nu}(x, 1) = \sin(\pi\nu) J_{2\nu}(2\sqrt{x}) - \cos(\pi\nu) L_{2\nu}(2\sqrt{x}).$$

- These kernels are of prime importance in number theory, for example see[18], [19].

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[18] B. C. Berndt, A. Dixit, A. Roy and A. Zaharescu, *New pathways and connections in number theory and analysis motivated by two incorrect claims of Ramanujan*, Adv. Math. **304** (2017), 809–929.

[19] A. Dixit and R. Kumar, *Superimposing theta structure on a generalized modular relation*, submitted for publication; arXiv:2005.08316.

In spite of so much work done on Watson's kernel, its importance from the point of view of number theory has not been recognized before. For example, while it is known [17] that for $x > 0$,

$$\mathscr{G}_{\frac{1}{2}}(x, w) = J_{2w-1}(2\sqrt{x}), \tag{1.12}$$

it has not been noticed before that when $w = 0$ and 1, the kernel $\varpi_{w-\nu-\frac{1}{2}, w+\nu-\frac{1}{2}}(x)$, or equivalently $\mathscr{G}_\nu(x, w)$, reduces respectively to the first and second Koshliakov kernels:

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Let $x > 0$. Then

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Let $x>0$, $\mathrm{Re}(w)>-1/2$ and $\nu\in\mathbb{C}\backslash(\mathbb{Z}\backslash\{0\})$. Let $\mathscr{G}_\nu(x,w)$ be defined in (1.11). If $\mu\neq-\nu$, then for $\mathrm{Re}(\mu),\mathrm{Re}(\nu),\mathrm{Re}(\mu+\nu)>-\mathrm{Re}(w)-\frac{1}{2}$, we have

$$\int_0^\infty K_\mu(t)t^{\mu+\nu+w}\mathscr{G}_\nu(xt,w)\,dt={}_\mu K_\nu(x,w), \tag{1.13}$$

otherwise, for $-\mathrm{Re}(w)-\frac{1}{2}<\mathrm{Re}(\nu)<\mathrm{Re}(w)+\frac{1}{2}$, we have

$$\int_0^\infty t^w K_\nu(t)\mathscr{G}_\nu(xt,w)\,dt=x^w K_\nu(x). \tag{1.14}$$

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A beautiful result involving $\mathscr{G}_\nu(x,w)$ and ${}_\mu K_\nu(x,w)$

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- Let $r_k(n)$ denote the number of representations of a positive integer n as the sum of k squares, where different signs and different orders of the summands give distinct representations. For example $r_2(5) = 8$.

Theorem (Dixit-Kesarwani-K.)

Let $k \in \mathbb{N}, k \geq 2$ and $Re(z) > 0$. Define $\mathfrak{R}(\mu, k)$ by

$$\mathfrak{R}(\mu, k, z) := \begin{cases} 0, & \textit{if } \operatorname{Re}(\mu) > -\frac{1}{2}, \\ \frac{1}{\sqrt{2z}} \pi^{\frac{1-k}{2}} \Gamma\left(\frac{k}{2}\right), & \textit{if } \mu = -\frac{1}{2}. \end{cases}$$

Then for $\operatorname{Re}(\mu) > -\frac{1}{2}$ or $\mu = -1/2$, the following transformation holds:

$$\sum_{n=1}^{\infty} r_k(n) n^{\mu+1} K_\mu(nz) - \frac{\pi^{\frac{k+1}{2}} 2^\mu \Gamma\left(\mu + \frac{k}{4} + \frac{1}{2}\right)}{z^{\mu+\frac{k}{2}+1} \Gamma\left(\frac{k}{4}\right)}$$

$$= \frac{\pi}{z^{\mu+\frac{k}{4}+\frac{3}{2}}} \sum_{n=1}^{\infty} r_k(n) n^{\frac{1}{2}-\frac{k}{4}} {}_\mu K_{\frac{1}{2}}\left(\frac{\pi^2 n}{z}, \frac{k}{4}\right) - \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \mathfrak{R}(\mu, k, z).$$

A well-known modular transformation

Corollary

Let k be an integer such that $k \geq 2$ and let $\mathrm{Re}(z) > 0$. Then

$$\sum_{n=0}^{\infty} r_k(n) e^{-nz} = \left(\frac{\pi}{z}\right)^{\frac{k}{2}} \sum_{n=0}^{\infty} r_k(n) e^{-\frac{\pi^2 n}{z}}.$$

Outline of the proof

We employ the following transformation of Guinand[20] (also known to Popov[21]) in a rigorous formulation given by Berndt, Dixit, Kim and Zaharescu[22].

Theorem: Let k be a positive integer greater than 3 and let $m = \left\lfloor \frac{1}{2}k \right\rfloor - 1$. Let $F(x), F'(x), F''(x), \ldots, F^{(2m-1)}(x)$ are integrals, and $F(x), xF'(x), x^2F''(x), \ldots, x^{2m}F^{(2m)}(x)$ belong to $L^2(0,\infty)$. Moreover, as $x \to \infty$, let

$$F(x) = O_k\left(x^{-\frac{k}{4}-\frac{1}{2}-\tau}\right), \tag{1.15}$$

for some fixed $\tau > 0$. Let the function G be defined by

$$G(y) = \pi \int_0^\infty F(t) J_{\frac{k}{2}-1}(2\pi\sqrt{yt})\, dt, \tag{1.16}$$

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For $k = 2$ and 3, (1.18) holds if F is continuous on $[0, \infty)$, $F(x), xF'(x) \in L^2(0, \infty)$, and F satisfies (1.15), and if G is defined in (1.16) and satisfies (1.17).

- Replace x by $\pi^2 x/z$, then let $\nu = \frac{1}{2}$ and $w = \frac{k}{4}$ in the resulting equation, and use $\mathscr{G}_{\frac{1}{2}}(x, w) = J_{2w-1}(2\sqrt{x})$, so that for $\mathrm{Re}(\mu) > -\frac{k}{4} - \frac{1}{2}$,

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Proof continued

- Let $F(x) = x^{\mu+\frac{k}{4}+\frac{1}{2}} K_\mu(xz)$ and $G(x) = \frac{\pi}{z^{\mu+\frac{k}{4}+\frac{3}{2}}} {}_\mu K_{\frac{1}{2}}\left(\frac{\pi^2 x}{z}, \frac{k}{4}\right)$ and use

Lemma (Dixit-Kesarwani-K.)

Let $k \in \mathbb{N}$. Then

$$\int_0^\infty x^{\frac{k}{4}-\frac{1}{2}} G(x)\ dx = \begin{cases} 0, & \textit{if } \mathrm{Re}(\mu) > -\frac{1}{2}, \\ \frac{1}{\sqrt{2}} \pi^{\frac{1-k}{2}} \Gamma\left(\frac{k}{2}\right), & \textit{if } \mu = -\frac{1}{2}. \end{cases}$$

to complete the proof.

Further questions

- Are there any direct proofs of the transformations which we get special case of our master identity?
- The two-variable generalization of the modified Bessel function, namely, ${}_\mu K_\nu(z, w)$, begs for a systematic study of its properties similar to those for the modified Bessel function $K_\nu(z)$. In this paper, we have restricted ourselves to obtaining only those properties of ${}_\mu K_\nu(z, w)$ relevant to deriving various transformations.
- Right now we are considering applications of our explicit transformation of $\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny}$ in the theory of Riemann zeta function[23].

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- Are there any applications of results such as

$$\sum_{n=1}^{\infty} \frac{1}{e^{ny}-1} = \frac{1}{4} + \frac{(\gamma - \log(y))}{y} + \frac{2}{y} \sum_{n=1}^{\infty} \left\{ \log\left(\frac{2\pi n}{y}\right) - \frac{1}{2}\left(\psi\left(\frac{2\pi i n}{y}\right) + \psi\left(-\frac{2\pi i n}{y}\right)\right)\right\},$$

from the point of view of transcendental number theory? Note that Erdös [24] has shown that the series $\sum_{n=1}^{\infty} d(n)q^{-n}$ is irrational for any integer q with $|q| \geq 2$. Thus, for example, letting y equal to $\log(2)$ in the above formula would yield an irrational number $\sum_{n=1}^{\infty} \frac{1}{e^{n\log(2)}-1}$ on the left-hand side of the above equation.

- For $\mathrm{Re}(y) > 0$, we have

$$\sum_{n=1}^{\infty} \frac{n^{-2}}{e^{ny}-1} + \frac{\pi^2 - \gamma y}{12} - \frac{1}{y}\zeta(3) = \frac{y}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\psi\left(\frac{2\pi i n}{y}\right) + \psi\left(-\frac{2\pi i n}{y}\right.\right.$$

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Thank You!