

Voting Rules: An Introduction

Ritu Dutta

Department of Mathematics
Dibrugarh University, Assam, India

February 19, 2021

Outline

In this talk, we introduce the different voting methods. Their merits and demerits, how voting rule(s) shaped our democracy, what are the possibilities to improve our democratic institutions? If time permits, we will discuss some recent voting reforms across the globe.

Introduction & Motivation

Often, we encounter situations in which high-quality movies or songs fall badly to the competition and have to satisfy themselves with lower ratings and no acknowledgments. We are familiar with situations like elections where a popular candidate loses and comparatively unpopular or undeserving candidates win. The 2016 presidential election in the United States is a good example, where the elected president (Donald J. Trump) received less popular votes than the runner-up candidate (Hillary Clinton). This problem occurs due to two main reasons namely, the [behavior of the voters](#) and the [drawbacks in the method](#) that are used to select a winner.

Historical Development

Social choice theory or more narrowly voting theory was first studied by two French mathematicians [Marquis de Condorcet](#) and [Jean Charles de Borda](#) in the 18th century during the time of the French Revolution. Condorcet and Borda are the two earlier [criticizers](#) of the plurality rule. Almost 100 years later Dodgson studied voting rules. But it was [Arrow](#) who gives the social choice theory a more general way of discussing [group decision problems](#) and their difficulties by introducing his famous impossibility theorems.

What is Social Choice Theory?

Social choice theory studies the **aggregation of individual preferences into a social preference**. Many real-world situations are eligible for an analysis using tools from social choice theory, such as elections, passing a bill in parliament, choosing from job candidates, searching the internet, and many more.

Sen in his 1970 book, "**Collective Choice and Social Welfare**" states that: "A society consists of a group of people with a variety of preferences and priorities. To make fitting social decisions on behalf of the group as a whole the society must take serious note of the peoples' (possibly diverse) views and interests. Aggregate assessment is central to social choice theory, but to determine how such aggregate social decisions should be made is not an easy task."

Plurality or FPTP rule

This is a voting procedure that is common in general elections. This procedure is mainly used in the [single-member constituency](#). It elects the candidate who gets the most votes. No other conditions are needed to fulfill. Some of the most [well-established democracies](#) such as India, the USA, the UK, Japan, and others used the plurality rule in their general elections.

What's wrong with Plurality rule?

Inexpressiveness: Voters have very little scope regarding their opinion. High chances of vote splitting. Hence, a candidate can win with very low support.

Borda's Count (Borda 1784; Black 1958)

Each voter gives no point to the candidate in the last place, one point to the candidate who is second to last, two points to the candidate third from the bottom, and so on. All the points of all the voters added and the candidate with the most points is the elected.

Voting Rules (Borda Score)

With 3 candidates and 3 Voters.

Points	1 voter	1 voter	1 voter
2	a	a	b
1	b	c	a
0	c	b	c

$\beta(a)=2+2+1=5$; $\beta(b)=2+1+0=3$; $\beta(c)=1+0+0=1$. So, **a** is the winner.

Drawbacks of Borda's Method

Let us consider a profile of 5 voters and 3 candidates. Let the underlying voting rule is Borda.

Points	2 voters	2 voters	1 voter
2	a	b	c
1	b	a	a
0	c	c	b

Drawbacks of Borda's Method

Let us consider a profile of 5 voters and 3 candidates. Let the underlying voting rule is Borda.

Points	2 voters	2 voters	1 voter
2	a	b	c
1	b	a	a
0	c	c	b

Here Borda winner is candidate **a**.

Drawbacks of Borda's Method

Let us consider a profile of 5 voters and 3 candidates. Let the underlying voting rule is Borda.

Points	2 voters	2 voters	1 voter
2	a	b	c
1	b	a	a
0	c	c	b

Here Borda winner is candidate **a**.

Points	2 voters	2 voters	1 voter
2	a	b	c
1	c	c	a
0	b	a	b

Drawbacks of Borda's Method

Let us consider a profile of 5 voters and 3 candidates. Let the underlying voting rule is Borda.

Points	2 voters	2 voters	1 voter
2	a	b	c
1	b	a	a
0	c	c	b

Here Borda winner is candidate **a**.

Points	2 voters	2 voters	1 voter
2	a	b	c
1	c	c	a
0	b	a	b

In this case, the winner is candidate **c**.

Approval Voting (Brams & Fishburn 1978, 1983)

Under this procedure, every voter has a number of votes which is equal to the number of competing candidates, and every voter can cast one vote or no vote for every candidate. The candidate obtaining the highest number of votes is elected.

Scoring Rules

- ▶ A scoring vector is a vector $(s_1, s_2, s_3, \dots, s_n)$, where $s_1 \geq s_2 \geq s_3 \geq \dots \geq s_n \geq 0$, with $s_1 > s_n$.
- ▶ The plurality and Borda are both special cases of scoring rule.
- ▶ For plurality, the scoring vector is $(1, 0, 0, \dots, 0, 0, 0)$ and Borda, the scoring vector is $(n - 1, n - 2, n - 3, \dots, 2, 1, 0)$.

Scoring Rules

k-approval rule:

Scoring rule with $s = (1, 1, 1, \dots, 1, 0, 0, \dots, 0)$. The voters choose a predetermined number (k) of candidates. For example, plurality is a 1-approval rule.

Anti-plurality rule:

Scoring rule with $s = (1, 1, 1, \dots, 1, 0)$. Anti-plurality is a $n - 1$ approval rule.

Veto-rule:

Selects the candidate who was ranked last by the least number of voters.

Majority Voting- Condorcet Approach

Absolute Majority: If a candidate receives more than half of the total first-place votes then that candidate is known as the absolute majority winner.

Condorcet Winner: The Condorcet winner for a given preference profile is the candidate who beats every other candidate in a pairwise comparison.

Condorcet Consistent: A voting rule is said to satisfy the Condorcet consistent if it chooses the Condorcet winner whenever one exists.

Example

Consider a profile with 3 candidates and 9 voters.

4 voters	3 voters	2 voters
a	b	c
b	c	b
c	a	a

Here, candidate b beats both a and c in pairwise comparison.
Hence candidate b is Condorcet or majority winner.

Condorcet Cycle

Consider a profile with 3 candidates and 3 voters.

1 voter	1 voter	1 voter
a	b	c
b	c	a
c	a	b

Here a beats b , b beats c and c beats a . Hence, no Condorcet winner exists.

Dodgson Method

The Dodgson method is a Condorcet consistent voting procedure that means if there is a Condorcet candidate in a profile then Dodgson method elects that candidate as a winner. Otherwise, it elects a candidate which needed minimum steps that make a candidate Condorcet winner.

Black's Suggestions

Another popular name in the field of the voting theory is – Duncan Black. Like many other contemporaries in the field, he also criticizes the plurality rule. Black suggests- if there is a Condorcet winner in a profile then make that candidate as a winner otherwise choose a winner according to Borda's procedure. Black developed some important results on voting theory. Particularly, his ideas on single-peaked preferences and single-peaked domains change the whole ideas of voting and related areas.

Plurality with Runoff (Multi-stage Voting Rules)

In this procedure up to **two voting rounds** are conducted. In the first round, each voter casts one vote for a single candidate. To win in the first round a candidate must obtain either a special plurality or an absolute majority of the votes. If no candidate is declared the winner in the first round then a second round is conducted. This procedure is a very common procedure for electing a single candidate and it is currently used for electing the President in 40 countries, Argentina, Austria, Brazil, Finland, France, India, Portugal, Romania, Russia, Turkey, and Ukraine, etc.

The Hare Rule (Multi-stage Voting Rules)

This procedure was independently proposed by [Carl Andrae in Denmark](#) at 1855 and by [Thomas Hare in England](#) in 1857 for electing a single candidate. It works as follows. In the first step one verifies whether there exists a candidate who is ranked first by an absolute majority of the voters. If such a candidate exists s/he is declared the winner. If no such candidate exists then, in the second step, the candidate who is ranked first by the smallest number of voters is deleted from all ballots and thereafter one again verifies whether there is now a candidate who is ranked first by an absolute majority of the voters. The elimination process continues in this way until a candidate who is ranked first by an absolute majority of the voters is found.

Coombs Method (Multi-stage Voting Rules)

This procedure was proposed by Clyde H. Coombs in 1964. It is almost similar to the Hare rule except that the elimination in a given round under the Coombs method involves the candidate who is ranked last by the largest number of voters.

Voting Paradoxes

In voting literature, paradoxes are defined as an undesirable outcome that a voting procedure may produce. Several voting paradoxes are studied in the literature. They are divided into two different categories viz., simple paradox and conditional paradox. Here we briefly discuss some of them.

Simple Paradoxes:

1. The Condorcet Winner (Condorcet 1785, Black 1958): A candidate a is not elected even though it constitutes a Condorcet winner.
2. The Condorcet Loser Paradox or Borda Paradox (Borda 1784, Black 1958): A candidate a is selected even though it constitutes a Condorcet Loser.

Condorcet winner and loser Paradox

Consider a profile with 3 candidates and 9 voters.

4 voters	3 voters	2 voters
a	b	c
b	c	b
c	a	a

Here, candidate b is the Condorcet winner (majority winner). But if the rule is plurality, then the winner will be candidate a , which is a Condorcet loser candidate.

Voting Paradoxes

Conditional Paradoxes:

1. The Additional Support: If candidate a is elected under a given distribution of voters preferences among the competing candidates, it is possible that, a may not be elected if some voters increases his support for a by moving a to a higher position in his preference ordering.
2. Reinforcement (Young 1974): If a is elected in each of several disjoint electorates, it is possible that a will not be elected if all electorates are combined into a single electorate.

Reinforcement Paradox in Plurality with Runoff

Suppose there are two districts I and II. In District I there are 17 voters whose preference orderings among three candidates a, b, and c, are as follows:

4 voters	1 voter	5 voters	6 voters	1 voter
a	b	b	c	c
b	a	c	a	b
c	c	a	b	a

Reinforcement Paradox

and in district II there are 15 voters whose preference orderings among the three candidates are as follows:

6 voters	8 voters	1 voter
a	b	c
c	c	a
b	a	b

Reinforcement Paradox

But when combined these two districts into a single district the winner will change from candidate *b* to candidate *a*.

4 voters	6 voters	1 voter	13 voters	7 voters	1 voter
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

Here, total voters are $17 + 15 = 32$. In this new profile, candidate *a* has 10 votes, *b* 14 votes and *c* 8 votes.

Voting Properties

1. Unrestricted Domain(U): A social ordering should be determined for any logically possible specifications of individual preferences.
2. Anonymity(A): Voters are equal.
3. Neutrality(N): Alternatives are equal.
4. Homogeneity(H): Proportional changing of preference profile does not change a collective ranking.
5. Unanimity(U): If all the voters place an alternative first, that alternative will also place first in the collective ranking.
6. Pareto Principle(P): If everyone prefers x to y , then society should prefer x to y .

Plurality or Condorcet winner? (Sen, SCW 2020)

Voting groups	A	B	C
Numerical strength	40 %	35 %	25 %
First preference	x	y	z
Second preference	z	z	y
Third preference	y	x	x

Here candidate x is a plurality winner where the Condorcet winner is candidate z . Which candidate do you like to choose? Condorcet or plurality winner?

Plurality or Condorcet winner? (Sen, SCW 2020)

Voting groups	A	B	C
Numerical strength	49 %	48 %	3 %
First preference	s	t	r
Second preference	r	r	s
Third preference	t	s	t

In this case, plurality winner is socialist government (s) but Condorcet winner is Liberal government (r) as r beats s in and r beats t in margin. Is Condorcet winner a good candidate? Liberal government receives just 3% first-place votes.





Some Recent Reforms

1. Different non-profit organizations in the US and other countries working on voting reforms. The demand was very high when Donald Trump won the election! as he receives less popular votes.
2. Sen and Maskin suggest majority voting as an election rule for the US and the rest of the world.
3. Recently, Fargo a city of North Dakota, USA used approval voting in their municipality election.
4. Recently, many cities in the US used Hare rule in their party's presidential primaries.






How Plurality rule affect our Politics: Indian context

1. In the 2014 Lok Sabha Election, BJP came to power just receiving 31% of total votes with National Democratic Alliance (NDA) together its total vote share was just 39%. The question arises of how fair it is to form a government with just 31 percent of votes and utilizing or accessing 100 percent powers!
2. In the 2020 Delhi Legislative Assembly Election, AAP's vote share was 53.4 percent and BJP and Congress vote share were 39.9 and 4.3 percent respectively. If seats were allocated according to the parties' total vote share then AAP will not get more than 37 seats. And BJP and Congress will receive 28 and 3 seats respectively. But due to the plurality rule, AAP got 62 out of 70 seats.
3. Recently, in the 2020 Boroland Territorial Council Election in Assam, voters were betrayed by all political parties. If we see the statistics of the 2020 Boroland election, the BPF is the single largest party that gets 17 seats alone. But in reality, BPF has no power to form a government!

References I

-  Arrow, K. J.
Social Choice and Individual Values.
John Wiley & Sons, 1951 New York.
-  Black, D.
The Theory of Committees and Elections.
Cambridge University Press, 1958.
-  Brams, S. J., & Fishburn, P.C.
Approval Voting.
American Political Science Review, 1978.
-  Felsenthal, D. S. & Nurmi, H.
Voting Procedures for Electing a Single Candidate Proving
Their(In)Vulnerability to Various Voting Paradoxes.
Springer Briefs in Economics, 2018.

References II

-  Maskin, E.
A Modified Version of Arrow's IIA Condition.
Social Choice and Welfare, 2020.
-  Nurmi, H.
Voting Theory: Cui Bono?
-  Peters, H.
Game Theory A Multi-Levelled Approach.
Springer-Verlag Berlin Heidelberg, 2008.
-  Sen, A. K.
Collective Choice and Social Welfare.
Holden-Day, North-Holland, 1970.
-  Sen, A. K.
Majority Decision and Condorcet Winners.
Social Choice and Welfare, 2020.

Thank you for your attention!