μ-Statistically Convergent Multiple Sequences in Probabilistic Normed Spaces

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Outline

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Abstract Metric Space

Definition (Fréchet, 1906)

A metric space is an ordered pair (S,d), where S is an abstract set and d a mapping of $S \times S$ into the real numbers satisfying the following conditions for all $p,q \in S$:

- $d(p,q) \geq 0$
- $d(p,q) = 0$ if and only if $p = q$
- $d(p,q) = d(q,p)$
- $d(p,r) \leq d(p,q) + d(q,r)$.

Definition

A function $f : \mathbb{R}^+ \to [0,1]$ is called a distribution function if it is non-decreasing, left-continuous with $\inf_{t \in \mathbb{R}^+} f(t) = 0$ and $\sup_{t \in \mathbb{R}^+} f(t) = 1$. Let D denotes the set of all distribution functions.

Statistical Metric Space

Definition (Menger, 1942)

A statistical metric space is an ordered pair $(S, \mathscr{F})$. We denote the distribution function $\mathscr{F}(p,q)$ by F_{pq}. The functions F_{pq} satisfy the following $\forall\, p,q \in S$:

- $F_{pq}(0) = 0$
- $F_{pq}(x) = 1$, for all $x > 0$ if and only if $p = q$
- $F_{pq} = F_{qp}$
- If $F_{pq}(x) = 1$ and $F_{qr}(y) = 1$, then $F_{pr}(x+y) = 1$.

Definition (Schweizer and Sklar, 1960)

A continuous t-norm is $* : [0,1] \times [0,1] \to [0,1]$ satisfies the following:

- $a * 1 = a$,
- $a * b = b * a$,
- $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$,
- $(a * b) * c = a * (b * c)$ for all $a,b,c,d \in [0,1]$.

Probabilistic Normed Space

Definition (Šerstnev, 1962)

A triplet $(X, N, *)$ is called a probabilistic normed space (in short a PN-space) if X is a real vector space, N a mapping from X into D (for $x \in X$, the distribution function $N(x)$ is denoted by N_x and $N_x(t)$ is the value of N_x at $t \in \mathbb{R}^+$) and $*$ a t-norm satisfying the following conditions:

(PN-1) $N_x(0) = 0$,

(PN-2) $N_x(t) = 1$, for all $t > 0$ if and only if $x = 0$,

(PN-3) $N_{\alpha x}(t) = N_x\left(\frac{t}{|\alpha|}\right)$, for all $\alpha \in \mathbb{R} \setminus \{0\}$,

(PN-4) $N_{x+y}(s+t) \geq N_x(s) * N_y(t)$, for all $x, y \in X$ and $s, t \in \mathbb{R}^+$.

Example

Let $(X, ||.||)$ be a normed linear space. Let $a * b = \min\{a, b\}$, for all $a, b \in [0, 1]$ and $N_x(t) = \frac{t}{t + ||x||}$, $x \in X$ and $t \geq 0$. Then $(X, N, *)$ is a PN-space.

Outline

Convergence vs Statistical Convergence

Convergence of a sequence

Let $x = \{x_k\}$ be a sequence of real numbers. Then we say that x is convergent to $x_0 \in \mathbb{R}$, if for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < \varepsilon$, for all $k \geq k_0$. We write it as $\lim_{k \to \infty} x_k = x_0$.

Asymptotic Density

Let $K \subseteq \mathbb{N}$, then the asymptotic density of K, denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_{n} \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit exists, where $|A|$ denotes the cardinality of the set A.

Convergence vs Statistical Convergence

Statistical Convergence (Steinhaus, 1951)

A sequence $x = \{x_k\}$ of real number is said to statistically convergent to $x_0 \in \mathbb{R}$, if for every $\varepsilon > 0$ we have

$$\delta(\{k \in \mathbb{N} : |x_k - x_0| \geq \varepsilon\}) = 0.$$

We write it as $stat - \lim x = x_0$.

Multiple Sequences

A multiple sequence is a mapping from $\mathbb{N}^k$ into the set X, where $\mathbb{N}^k$ is the k-th power of the set of natural number $\mathbb{N}$. A term of a multiple sequence $f : \mathbb{N}^k \to X$ is an ordered set of $k+1$ elements $(n_1, n_2, \ldots, n_k, x)$, where $x = f(n_1, n_2, \ldots, n_k) \in X$ and $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$, $n_i \in \mathbb{N}$, for $i = 1, 2, \ldots, k$. The term is also denoted by $x_{n_1 n_2 \ldots n_k}$.

Convergence of Multiple Sequences

Convergent Multiple Sequence

A multiple sequence $x = (x_{n_1 n_2 \ldots n_k})$ is said to be convergent to a number L, if for every $\varepsilon > 0$, there exists a positive integer m_0 such that $|x_{n_1 n_2 \ldots n_k} - L| < \varepsilon$, for all $n_i \geq m_0$, for $i = 1, 2, \ldots, k$. It is denoted by $\lim x_{n_1 n_2 \ldots n_k} = L$.

Cauchy Multiple Sequence

A multiple sequence $x = (x_{n_1 n_2 \ldots n_k})$ is said to be Cauchy, if for every $\varepsilon > 0$, there exists a positive integer m_0 such that $|x_{n_1 n_2 \ldots n_k} - x_{l_1 l_2 \ldots l_k}| < \varepsilon$, for all $n_i \geq m_0$ and $l_i \geq m_0$, for $i = 1, 2, \ldots, k$.

Convergence of Multiple Sequences in PN-Spaces

Convergent Multiple Sequence (Tripathy and Goswami, 2015)

Let $(X, N, *)$ be a PN-space. Then, a multiple sequence $x = (x_{n_1 n_2 \ldots n_k})$ is said to be convergent to $L \in X$ with respect to the probabilistic norm N, if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer m_0 such that $N_{x_{n_1 n_2 \ldots n_k} - L}(\varepsilon) > 1 - \lambda$, for all $n_i \geq m_0$, for $i = 1, 2, \ldots, k$. It is denoted by $N - \lim x_{n_1 n_2 \ldots n_k} = L$.

Cauchy Multiple Sequence (Tripathy and Goswami, 2015)

Let $(X, N, *)$ be a PN-space. Then, a multiple sequence $x = (x_{n_1 n_2 \ldots n_k})$ is said to be Cauchy, if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer m_0 such that $N_{x_{n_1 n_2 \ldots n_k} - x_{l_1 l_2 \ldots l_k}}(\varepsilon) > 1 - \lambda$, for all $n_i \geq m_0$ and $l_i \geq m_0$, for $i = 1, 2, \ldots, k$.

Two valued measure μ

Definition (Connor, 1990)

Let μ denotes a complete $\{0,1\}$-valued finitely additive measure defined on a field Γ of all finite subsets of $\mathbb{N}$ and suppose that $\mu(A) = 0$, if $|A| < \infty$; if $A \subset B$ and $\mu(B) = 0$, then $\mu(A) = 0$; and $\mu(\mathbb{N}) = 1$.

Outline

μ-Statistically Convergent Multiple Sequence

Definition

A multiple sequence $x = (x_{n_1 n_2 \ldots n_k})$ in a probabilistic normed space $(X, N, *)$ is said to be μ-statistically convergent to $L \in X$ with respect to the probabilistic norm N, if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have

$$\mu\left(\left\{(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \ldots, n_k} - L}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0.$$

In this case, we write $\mu - stat_N - \lim x_{n_1 n_2 \ldots n_k} = L$.

μ-Statistically Cauchy Multiple Sequence

Definition

A multiple sequence $x = (x_{n_1 n_2 \ldots n_k})$ in a probabilistic normed space $(X, N, *)$ is said to be μ-statistically Cauchy with respect to the probabilistic norm N, if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there is a positive integer m_0 such that

$$\mu\left(\left\{(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \ldots, n_k} - x_{l_1 l_2 \ldots, l_k}}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0.$$

μ-Statistically Bounded Multiple Sequence

Definition

A multiple sequence $x = (x_{n_1 n_2 \dots n_k})$ in a probabilistic normed space $(X, N, *)$ is said to be μ-statistically bounded with respect to the probabilistic norm N, if there exists an $\varepsilon > 0$ such that

$$\mu\left(\left\{(n_1, n_2, \dots, n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \dots, n_k}}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0,$$

for every $\lambda \in (0, 1)$.

Results

Lemma

Let $(X,N,*)$ *be a PN-space. Then, for every* $\varepsilon > 0$ *and* $\lambda \in (0,1)$*, the following are equivalent:*

(i) $\mu - stat_N - \lim x_{n_1 n_2 \ldots n_k} = L$.

(ii) $\mu\left(\left\{(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \ldots n_k} - L}(\varepsilon) \leq 1 - \lambda\right\}\right) = 0$.

(iii) $\mu\left(\left\{(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \ldots n_k} - L}(\varepsilon) > 1 - \lambda\right\}\right) = 1$.

(iv) $\mu - stat - \lim N_{x_{n_1 n_2 \ldots n_k} - L}(\varepsilon) = 1$.

Theorem

Let $(X,N,*)$ *be a PN-space. If a multiple sequence* $x = (x_{n_1 n_2 \ldots n_k})$ *in* $(X,N,*)$ *is* μ*-statistically convergent with respect to the probabilistic norm* N*, then* $\mu - stat_N - \lim x$ *is unique.*

Results

Theorem

Let $(X, N, *)$ be a PN-space. If $N - \lim x_{n_1 n_2 \ldots n_k} = L$, then $\mu - stat_N - \lim x_{n_1 n_2 \ldots n_k} = L$.

Example

Let us consider the space $(\mathbb{R}, ||\cdot||)$ of real numbers with the usual norm. Let $a * b = ab$ and $N_x(t) = \frac{t}{t+||x||}$, where $x \in X$ and $t \geq 0$. Then, we observe that $(\mathbb{R}, N, *)$ is a PN-space. Let $K \subset \mathbb{N}^k$ be such that $\mu(K) = 0$. We define a sequence $x = (x_{n_1 n_2 \ldots n_k})$ as follows:

$$x_{n_1 n_2 \ldots n_k} = \begin{cases} n_1 n_2 \ldots n_k, & \text{if } (n_1, n_2, \ldots, n_k) \in K \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Results

Example (Contd...)

Then, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, let

$$A_N(\lambda,\varepsilon) = \left\{(n_1,n_2,\ldots,n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \ldots n_k}}(\varepsilon) \leq 1-\lambda\right\}.$$

Now, since

$$\begin{aligned} A_N(\lambda,\varepsilon) &= \left\{(n_1,n_2,\ldots,n_k) \in \mathbb{N}^k : \frac{\varepsilon}{\varepsilon + ||x_{n_1 n_2 \ldots n_k}||} \leq 1-\lambda\right\} \\ &= \left\{(n_1,n_2,\ldots,n_k) \in \mathbb{N}^k : ||x_{n_1 n_2 \ldots n_k}|| \geq \frac{\lambda\varepsilon}{1-\lambda} > 0\right\} \\ &= \left\{(n_1,n_2,\ldots,n_k) \in \mathbb{N}^k : x_{n_1 n_2 \ldots n_k} = n_1 n_2 \ldots n_k\right\} \\ &= \left\{(n_1,n_2,\ldots,n_k) \in K\right\}. \end{aligned}$$

Results

Example (Contd...)

Thus, we have $\mu(A_N(\lambda,\varepsilon)) = 0$, and consequently $x = (x_{n_1 n_2 \ldots n_k})$ is μ-statistically convergent with respect to the probabilistic norm N. However, the sequence $x = (x_{n_1 n_2 \ldots n_k})$ defined by (1) is not convergent in the space $(\mathbb{R}, ||\cdot||)$, thus we conclude that x is also not convergent with respect to the probabilistic norm N.

Theorem

Let $(X, N, *)$ be a PN-space and $x = (x_{n_1 n_2 \ldots n_k})$ be a multiple sequence. Then $\mu - stat_N - \lim x_{n_1 n_2 \ldots n_k} = L$ if and only if there is an index subset $A = \{(m_{n_1}, m_{n_2}, \ldots, m_{n_k}) : m_{n_i} \in \mathbb{N}\}$ of $\mathbb{N}^k$ such that $\mu(A) = 1$ and

$$N - \lim_{(n_1, n_2, \ldots, n_k) \in A} x_{n_1 n_2 \ldots n_k} = L.$$

Results

Theorem

Let $(X, N, *)$ be a PN-space and $x = (x_{n_1 n_2 \ldots n_k})$ be a multiple sequence, whose terms are in the vector space X. Then the following statements are equivalent:

(a) *x is a μ-statistically Cauchy sequence with respect to the probabilistic norm N.*

(b) *There is an index subset $A = \{(m_{n_1}, m_{n_2}, \ldots, m_{n_k}) \in \mathbb{N}^k : m_{n_i} \in \mathbb{N}\} \subset \mathbb{N}^k$ such that $\mu(A) = 1$ and the subsequence $\left\{x_{m_{n_1} m_{n_2} \ldots m_{n_k}}\right\}_{(m_{n_1}, m_{n_2}, \ldots, m_{n_k}) \in A}$ is a Cauchy sequence with respect to the probabilistic norm N.*

Results

Theorem

Let $(X, N, *)$ be a PN-space. Then

(i) *If* $\mu - stat_N - \lim x_{n_1 n_2 \ldots n_k} = \alpha$ *and* $\mu - stat_N - \lim y_{n_1 n_2 \ldots n_k} = \beta$, *then* $\mu - stat_N - \lim (x_{n_1 n_2 \ldots n_k} + y_{n_1 n_2 \ldots n_k}) = \alpha + \beta$.

(ii) *If* $\mu - stat_N - \lim x_{n_1 n_2 \ldots n_k} = \alpha$ *and* $a \in \mathbb{R}$, *then* $\mu - stat_N - \lim a x_{n_1 n_2 \ldots n_k} = a\alpha$.

(iii) *If* $\mu - stat_N - \lim x_{n_1 n_2 \ldots n_k} = \alpha$ *and* $\mu - stat_N - \lim y_{n_1 n_2 \ldots n_k} = \beta$, *then* $\mu - stat_N - \lim (x_{n_1 n_2 \ldots n_k} - y_{n_1 n_2 \ldots n_k}) = \alpha - \beta$.

Outline

μ-statistical limit points for multiple sequences in PN-spaces

Limit Point of a Multiple Sequence (Tripathy and Goswami, 2015)

Let $(X, N, *)$ be a PN-space and let $x = (x_{n_1 n_2 \ldots n_k})$ be a multiple sequence. We say that $L \in X$ is a limit point of x with respect to the probabilistic norm N, if there exists a subsequence of x that converge to L with respect to the probabilistic norm N.

Let $L_N(x)$ denotes the set of all limit points of the multiple sequence $x = (x_{n_1 n_2 \ldots n_k})$.

Definition

Let $(X, N, *)$ be a PN-space and let $x = (x_{n_1 n_2 \ldots n_k})$ be a multiple sequence. We say that $\xi \in X$ is a μ-statistical limit point of the multiple sequence x with respect to the probabilistic norm N, if there is a set

$$M = \{(n_1(j), n_2(j), \ldots, n_k(j)) : n_i(1) < n_i(2) < n_i(3) < \ldots, \text{ for } i = 1, 2, \ldots, k\} \subset \mathbb{N}^k$$

such that $\mu(M) \neq 0$ and $N - \lim x_{n_1(j) n_2(j) \ldots n_k(j)} = \xi$.

Let $\Lambda_N^{\mu}(x)$ denotes the set of all $\mu - stat_N - limit$ points of the multiple sequence $x = (x_{n_1 n_2 \ldots n_k})$.

Results

Theorem

Let $(X, N, *)$ *be a PN-space and* $x = (x_{n_1 n_2 \ldots n_k})$ *be a multiple sequence. If* $\mu - stat_N - \lim x = M$, *then* $\Lambda_N^{\mu}(x) = \{M\}$.

Outline

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Thank you!