Orbits of zipping maps of surfaces of infinite type

Soumya Dey
IMSc, Chennai

Online Weekly Research Seminar for
Early Career Mathematicians from India
May 07, 2021

Plan of the talk

- What is a mapping class group?
- What is a Teichmüller space?
- Action of $Mod(S)$ on $\mathcal{T}(S)$
- Surfaces of infinite type : big MCGs
- The Fenchel-Nielsen metric d_{FN}
- The zipping map
- Examples of some orbits

Surfaces

Definition

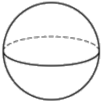
A 2-dimensional manifold, possibly with boundary, is called a *surface*.

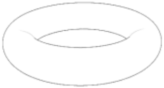
Examples: the plane $\mathbb{R}^2$, the closed unit disk $\mathbb{D}^2$, a punctured disk, a cylinder, a Möbius band, etc.

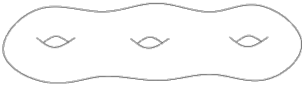
Classification

Theorem

Any closed connected oriented surface is homeomorphic to exactly one of the following surfaces: a sphere, a torus, or a connected sum of finitely many tori.
(i.e. they are classified by the genus)







and so on ...

$Homeo^+(S)$, $Homeo_0(S)$

Definition

$Homeo^+(S) :=$ {orientation-preserving self-homeomorphisms of S}

With composition of maps as binary operation, $Homeo^+(S)$ forms a group.

Definition

$Homeo_0(S) := \{f \in Homeo^+(S) \mid f \text{ is isotopic to } id\}$

In case, the surface S has boundary components, $Homeo^+(S, \partial S)$ is considered, where the homeomorphisms fix ∂S pointwise. Also, the isotopies fix ∂S pointwise.

$Mod(S)$

Let S be any surface.

Definition (Mapping class group)

The *mapping class group* of S is defined as:

$$Mod(S) = Homeo^{+}(S)/Homeo_0(S)$$

Examples:

1) $Mod(\mathbb{S}^2) \cong \langle 1 \rangle \cong Mod(\mathbb{D}^2)$
2) $Mod(\mathbb{T}^2) \cong SL(2, \mathbb{Z})$
3) $Mod(\mathbb{D}^2 - \{p_1, \ldots, p_n\}) \cong B_n$

A generating set

Theorem

$Mod(S_g)$ can be generated by finitely many Dehn twists.

What is a Dehn twist?

Hyperbolic surface

Definition

A hyperbolic structure on a surface S is given by an atlas of charts from S onto open subsets of the hyperbolic plane $\mathbb{H}^2$, such that the transition maps are isometries.

Teichmüller space

Definition

A marked hyperbolic structure on S is a pair (X, f), where $f : S \to X$ is a homeomorphism (called a marking) and X is a hyperbolic surface.

Definition

Two marked hyperbolic structures (X, f) and (Y, g) on S are equivalent, if there is an isometry $\phi : X \to Y$ such that $\phi \circ f$ is homotopic to g.

Definition

The Teichmüller space of S is defined as

$$\mathcal{T}(S) := \{\text{equivalence classes of marked hyperbolic structures on } S\}$$

Building hyperbolic surfaces

How do we endow a surface with a hyperbolic structure?
Assume $\chi(S) < 0$. (To start with, consider S_g with $g \geq 2$.)
1) Cut the surface into pairs of pants.
2) Give hyperbolic structures to the pairs of pants.
3) Glue back to get a hyperbolic surface.

The above algorithm gives Fenchel-Nielsen coordinates of $\mathcal{T}(S)$ and we get a homeomorphism $\mathcal{T}(S_g) \cong \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$.

The action

$Mod(S)$ acts on $\mathcal{T}(S)$ as follows.
Given $\phi \in Mod(S)$, for any $p \in \phi$, define $\phi.(X, f) := (X, f \circ p^{-1})$

This action is well defined, as for $p, q \in \phi$, $(X, f \circ p^{-1})$ and $(X, f \circ q^{-1})$ are equivalent marked hyperbolic structures.

Definition

Given a locally compact space X, an action $G \curvearrowright X$ is properly discontinuous if $\{g \in G \mid gK \cap K \neq \phi\}$ is finite for all compact $K \subset X$.

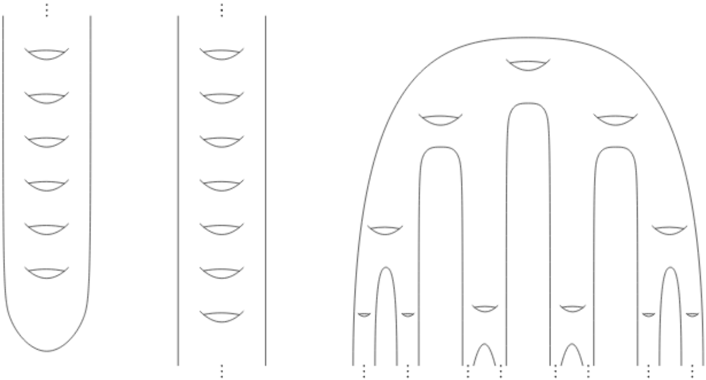
Theorem

The above action $Mod(S_g) \curvearrowright \mathcal{T}(S)$ is properly discontinuous.

Surfaces of infinite type

Definition

A surface S is of *infinite type* if $\pi_1(S)$ is not finitely generated.



How can we classify all surfaces of infinite type?

We need to consider the ends of a surface.

Ends of a surface

Definition

An exiting sequence is a sequence $\{U_n\}_{n\in\mathbb{N}}$ of connected open subsets of S with the following properties:
1) For $m < n$, $U_n \subset \mathrm{U}_m$.
2) U_n is not relatively compact for any $n \in \mathbb{N}$.
3) U_n has compact boundary for all $n \in \mathbb{N}$.
4) Any relatively compact subset of S is disjoint from all but finitely many U_n's.

Definition

Two exiting sequences are *equivalent* if every element of the first is eventually contained in some element of the second, and vice versa. $Ends(S)$ is the set of all equivalence classes of exiting sequences of S.

$Ends(S)$, $Ends_{np}(S)$

Definition

An end is *planar* if it has a neighborhood which can be embedded in a plane. For example, a puncture is a planar end.

An end is *non-planar or accummulated by genus* if it is not planar. In this case, every neighborhood of the end has infinite genus.

Example: Flower surface $\mathcal{F}l(3)$ with 3 non-planar ends.

Topology on $Ends(S)$, $Ends_{np}(S)$

Theorem

For any surface S, the space $Ends(S)$ can be given a topology which is totally disconnected, second countable, and compact. In particular, $Ends(S)$ is homeomorphic to a closed subset of a Cantor set, and $Ends_{np}(S)$ is a closed subset of $Ends(S)$.

Theorem (Richards)

Let X, Y be closed subsets of a Cantor set with $Y \subset X$. Then there exists a surface S such that $Ends(S) \cong X$ and $Ends_{np}(S) \cong Y$.

The classification theorem

Theorem (Kerékjártó-Richards)

Let S_1, S_2 be two surfaces, and let g_i and b_i, respectively, be the genus and the number of boundary components of S_i.
Then $S_1 \cong S_2$ if and only if $g_1 = g_2, b_1 = b_2$, and there is a homeomorphism

$$\phi : Ends(S_1) \to Ends(S_2)$$

that restricts to a homeomorphism

$$\phi_{np} : Ends_{np}(S_1) \to Ends_{np}(S_2)$$

Why ‘big’ ?

For any infinite-type surface S, $Mod(S)$ is uncountable.

In fact, $Mod(S)$ is a (non-Lie) Polish group homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

Note: A Polish group is a topological group homeomorphic to a separable complete metric space.

Why 'big' ?

Theorem (Aougab-Patel-Vlamis, 2020)

1) If S is an orientable infinite-genus surface with no planar end, then $Mod(S)$ contain an isomorphic copy of every finite group.

2) In addition to above, if $Ends(S)$ is self-similar, then $Mod(S)$ also contain an isomorphic copy of every countable group.

Definition

E is *self-similar* if given pairwise-disjoint clopen subsets $E_1, \dots, E_n$ such that $E = E_1 \sqcup \cdots \sqcup E_n$ there exists $i \in \{1, \dots, n\}$ and $A \subset E_i$ such that A is open and homeomorphic to E.

Zipping maps - intuitively

Consider $\mathcal{F}I(3)$.

Zipping maps in terms of shift maps on ladder surface

The problem

Study the action of zipping maps on $\mathcal{T}(S)$.

How do the orbits in $\mathcal{T}(S)$ look like?

Fenchel-Nielsen metric on $\mathcal{T}(S)$

Let $H = (X, f)$ be a given marked hyperbolic structure on S and $\mathcal{P} = \{C_i\}_{i \in I}$ be a set of disjoint simple closed curves on S which gives a geodesic pair of pants decomposition of S with respect to H.

Given two elements $H_1 = (X_1, f_1)$ and $H_2 = (X_2, f_2)$ of $\mathcal{T}(S)$, the Fenchel-Nielsen distance between them, with respect to $(H, \mathcal{P})$, is defined as follows.

$$d_{FN}(H_1, H_2) := \sup_{i \in I} \max \left\{ \left| \log \left(\frac{l_{H_1}(C_i)}{l_{H_2}(C_i)} \right) \right|, \right. \\ \left. \left| l_{H_1}(C_i)\theta_{H_1}(C_i) \; - \; l_{H_2}(C_i)\theta_{H_2}(C_i) \right| \right\}$$

Note that, $d_{FN}(H_1, H_2)$ may not be finite.

Fenchel-Nielsen metric on $\mathcal{T}(S)$

To make d_{FN} a well defined metric, the following subset of $\mathcal{T}(S)$ is considered.

$$\mathcal{T}_{(H,\mathcal{P})}(S) = \{(X, f) \in \mathcal{T}(S) \mid d_{FN}(H, (X, f)) < \infty\}$$

Definition

The metric space $(\mathcal{T}_{(H,\mathcal{P})}(S), d_{FN})$ is called the Fenchel-Nielsen Teichmüller space of S with respect to the pair $(H, \mathcal{P})$.

Action of the simplest zipping map

Consider the hyperbolic structure H on the ladder surface induced by giving length 1 to all the pant curves.

Action of the simplest zipping map

Consider the hyperbolic structure H on the ladder surface induced by giving length 1 to all the pant curves, except one separating curve, for which fix the length to be 2.

Consequences

The sequence of orbit points gives us an infinite set in the closed ball. But the bounded infinite set does not have a limit point. We call this phenomenon 'bounded degeneration'.

Hence, the closed ball is not limit point compact, and hence not compact. This shows that $\mathcal{T}_{FN}(S)$ is not a proper metric space.

Question: Can we generalize the above results in case of other surfaces with finitely many ends?

Conjecture: Yes!

References

1) A primer on mapping class groups by Benson Farb and Dan Margalit

2) Big mapping class groups: an overview by Javier Aramayona and Nicholas G. Vlamis (arXiv:2003.07950v3)

3) On Fenchel-Nielsen coordinates on Teichmüller spaces of surfaces of infinite type by Daniele Alessandrini, Lixin Liu, Athanase Papadopoulos, Weixu Su and Zongliang Sun (arXiv:1003.0980)

Thank you for your time!