# The Stable Marriage problem

Marriages made by algorithms,
guaranteed with stability !

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# Outline of the talk

- Stable Marriage problem
- Gale-Shapley algorithm
- Structure of Stable Matchings
- Application: Hospitals-Residents problem

# Stable Marriage(SM) problem

- Two disjoint sets **M**(men) and **W**(women), each of size **n**
- Each person has a strictly ordered preference list of **all** the members of the other sex
- A Matching is a one-one correspondence between **M** and **W**
- Goal: Find a matching that is "**stable**", if it exists

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  Proof: Algorithmic !

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  Proof: Algorithmic !
- Goal: Find a matching that is “**stable**”

# Matching-partners

- If **m** and **w** are matched(partners) in some matching **M**:

  **m** = $p_M$(**w**) = **M**-partner of **w**

  **w** = $p_M$(**m**) = **M**-partner of **m**

# Blocking Pair

(**m,w**) are said to be a **blocking pair** for a matching **M** if:

- **m** and **w** are not partners in **M**
- **m** prefers **w** to $p_M$(**m**)
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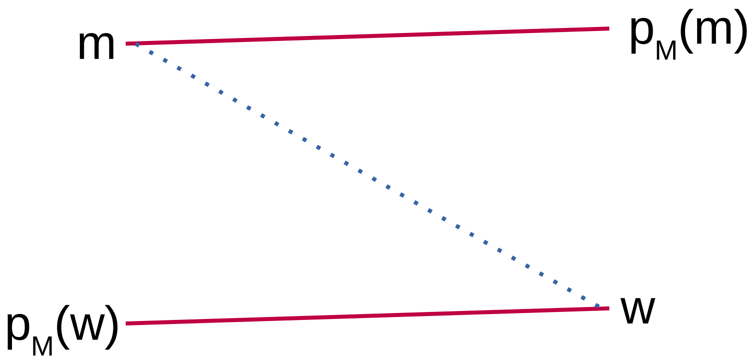



m: .......w......$p_M$(m).......

w: .......m......$p_M$(w).......

# Stability

- A Matching is **unstable** if it has at least one blocking pair
- Absence of blocking pair <=> Matching is **stable**

# Gale-Shapley algorithm

Basic idea:

- Men propose, Women accept/reject
- Repeat until all men(and women) are engaged

# Gale-Shapley algorithm

Assign each person to be free

While some man **m** is free:

  **w** = first woman on **m**’s list to whom **m** hasn’t proposed yet

  if **w** is free:

    assign **m** and **w** as engaged [to each other]

  else

    if **w** prefers **m** to her fiance **m'**:

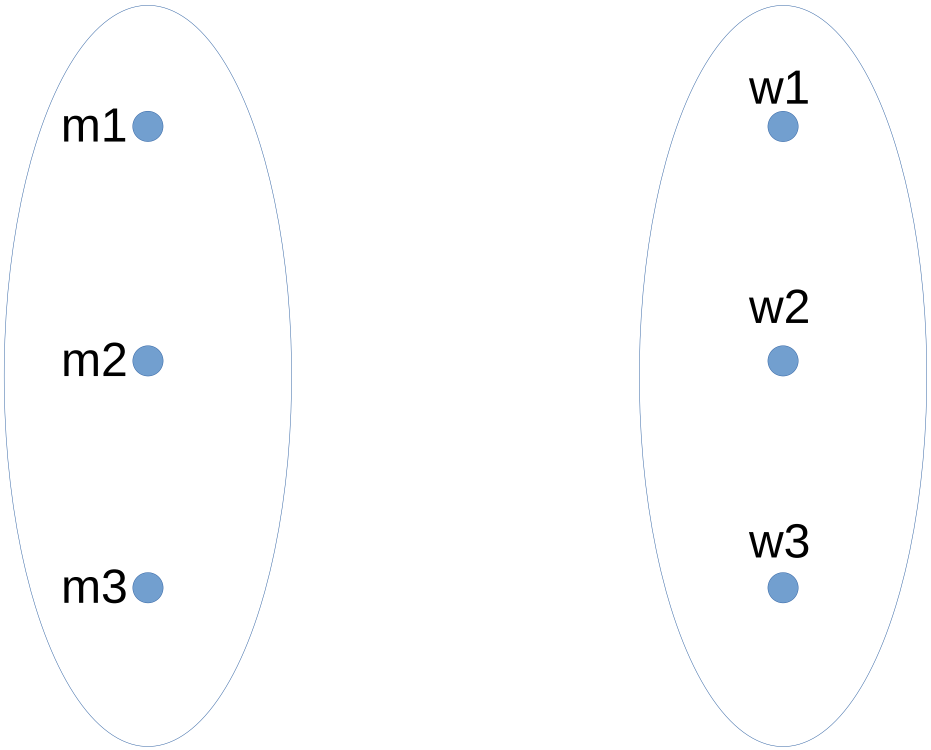
      assign **m** and **w** as engaged and **m'** as free

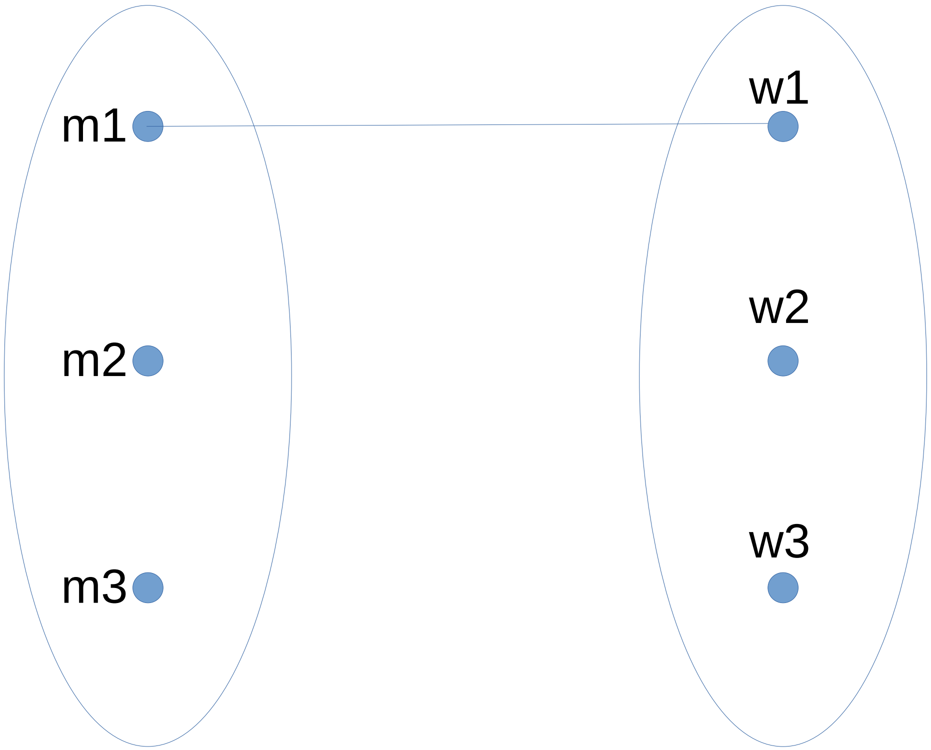
    else

      **w** rejects **m** [**m** remains free]

m1: w1 > w3 > w2
m2: w3 > w2 > w1
m3: w1 > w2 > w3

w1: m2 > m3 > m1
w2: m1 > m2 > m3
w3: m2 > m1 > m3






m1: w1 > w3 > w2
m2: w3 > w2 > w1
m3: w1 > w2 > w3

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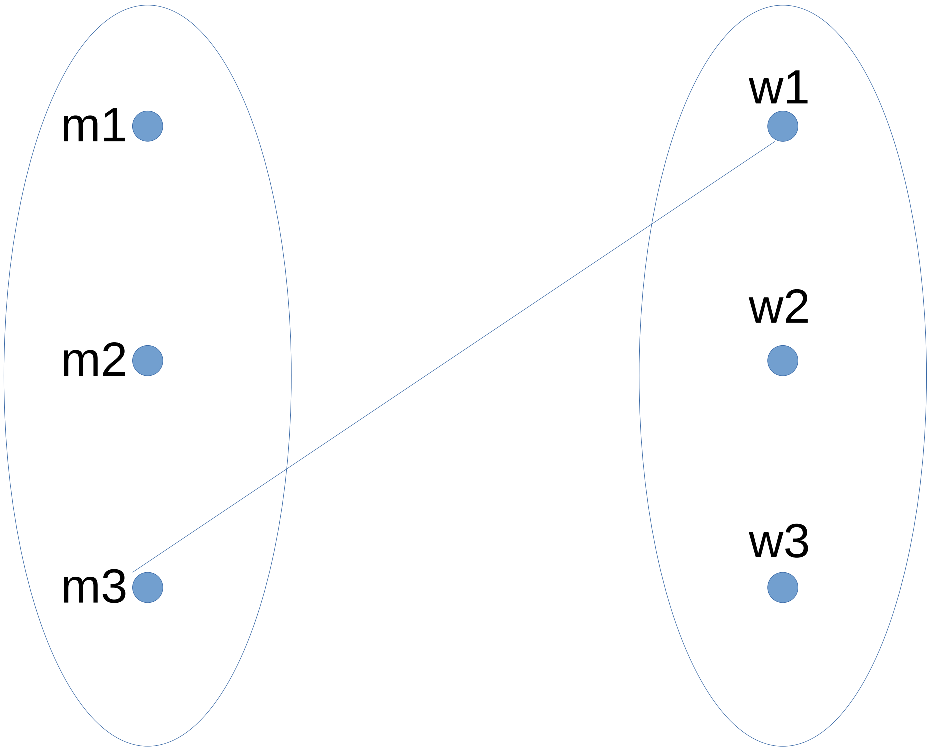
M1 proposes w1, w1 accepts m1




m1: w1 > w3 > w2
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w2: m1 > m2 > m3
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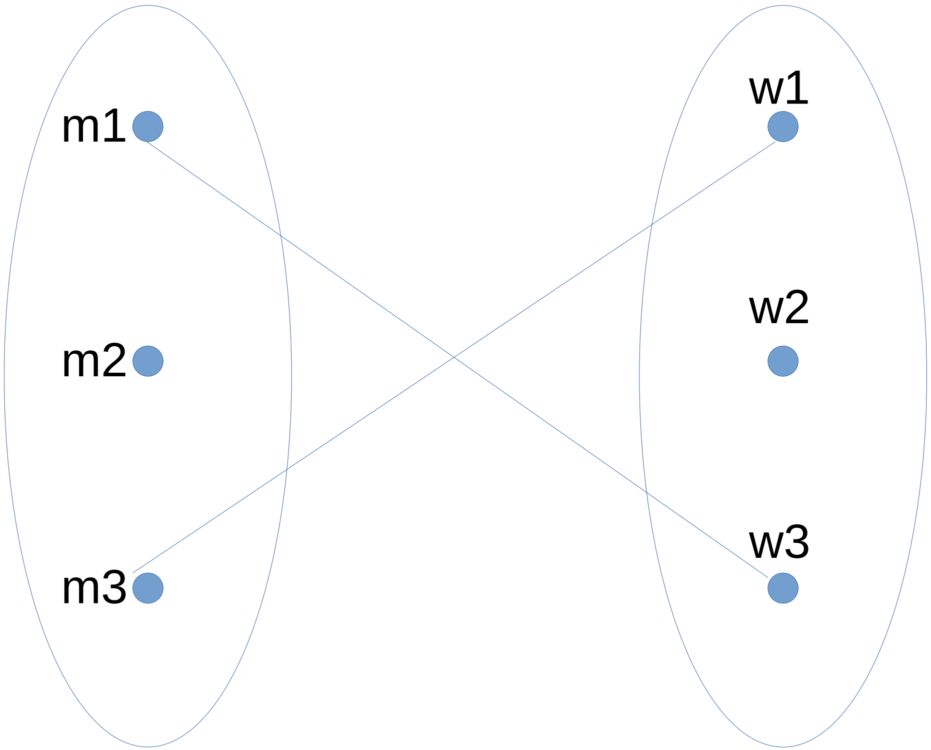
M1 proposes w1, w1 accepts m1
M3 proposes w1, w1 accepts m3, m1 free

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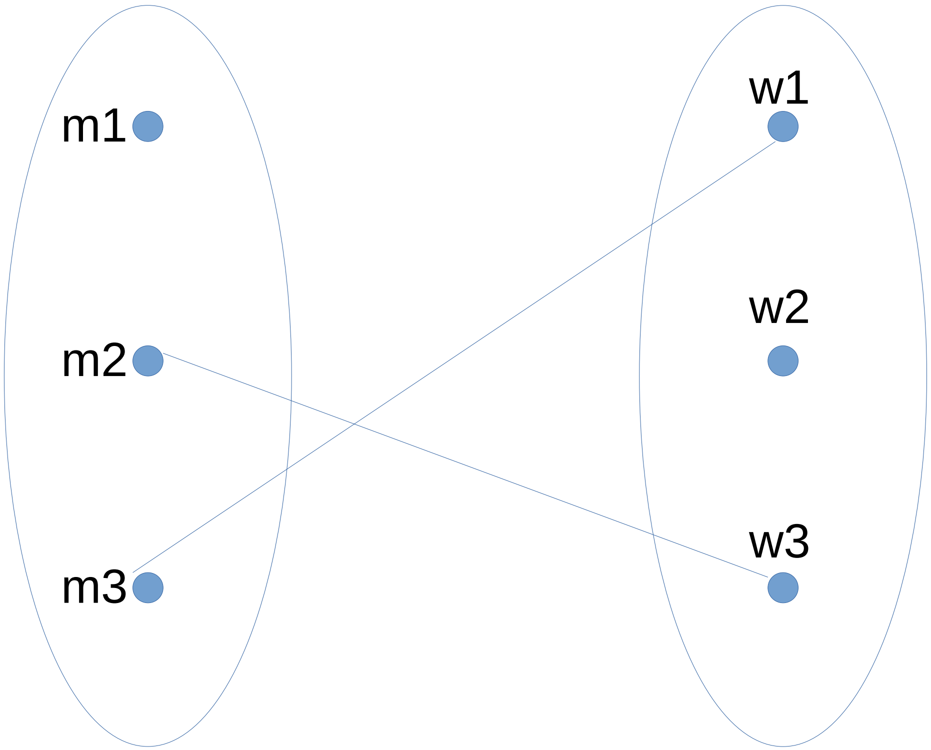


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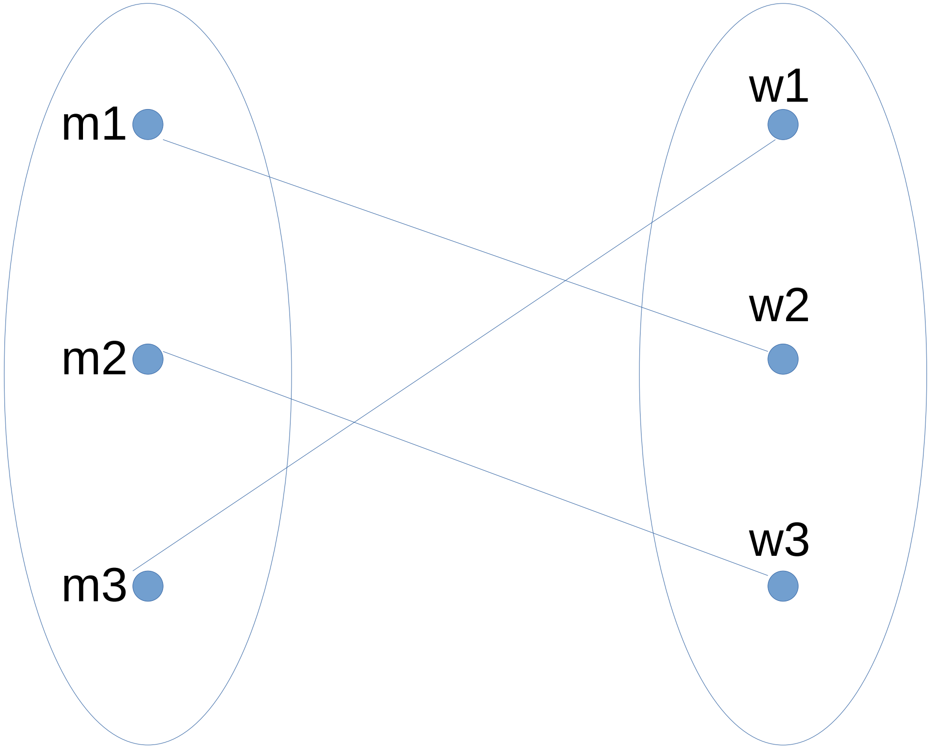


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# Proof of termination

Suffices to show that no man can be rejected by all women.

**Assume that a man has been rejected by all women.**

A woman can reject only when she is engaged.
Once a woman is engaged she never becomes free again.

Therefore, rejection of a man by the last woman on his list =>  All the women were already engaged. But, #men = #women = n, and no has more than one fiance, a contradiction.

# Proof of stability

- Let **M** be the matching generated by the algorithm

  **Assume ∃ a blocking pair (m,w)**

- **m** must have proposed **w** and got rejected
- **w** rejected **m**,

  since it must have had a better partner,

  say **m'** and thus would never accept $\mathbf{p_M(w)}$,

  a contradiction !

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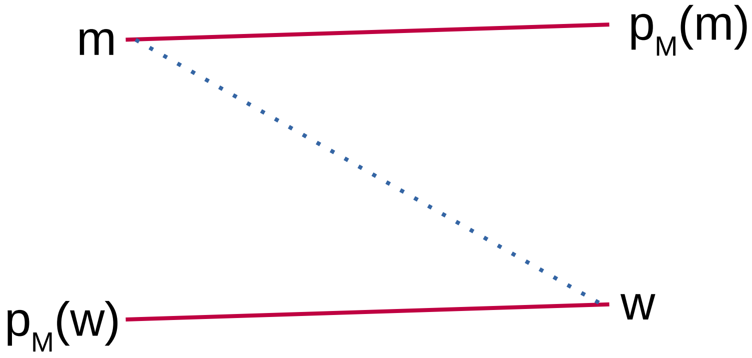


m: .............w......$p_M(m)$.......
w: ......m'....m......$p_M(w)$.......

All possible executions of GS algorithm yield the same stable matching M, and in M, each man gets his best stable partner that he can have in any stable matching

**Proof**

- Let E be an arbitrary excution of the algorithm, that gives stable matching M
- Assume M' is another stable matching s.t:
- m prefers w' = $p_{M'}$(m) to w = $p_M$(m).
- during execution E, w' must have rejected m.
- Suppose, w.l.o.g, this is the first time during E that a woman rejected a stable partner.
- Let m' be the man w' was engaged to (for whom she rejected m)
- m' can have no stable partner whom he prefers to w' (since no woman had previously rejected a stable partner)
- Thus m' prefers w' to $p_{M'}$(m') and (m', w') blocks M', a contradiction !

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Man-optimal stable matching !

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Man-optimal stable matching !
Also woman-pessimal !

# Structure of Stable Matchings

- Let S = set of all possible Stable Matchings of an SM instance

  Dominance relation(≤):

    M dominates M' (M≤M') if every man has atleast as good a partner in M  as he has in M'

- (S, ≤) forms a lattice
- Man-optimal stable matching and woman-optimal stable matching are the greatest and least elements.

# Some common Extensions

- Sets of unequal size
- Unacceptable partners/Incomplete preference lists
- Indifference

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Definition of “stability” changes for each variant !

# Hospitals-Residents(HR) problem

- An <u>asymmetric</u> extension of the SM problem
- Consists of two sets:

  **H**: set of hospitals

  **R**: set of residents
- Each resident can be attached to one hospital
- Each hospital can be attached to *upto* a <u>fixed number(capacity)</u> of residents

- A Matching in the **HR** problem is:

  a (partial) mapping from R to H such that no of residents assigned to a hospital does not exceed the hospital’s capacity.

# Stability

Matching **M** is **unstable** if ∃ (r,h), **r** ϵ **R**, **h** ϵ **H** such that:

- **h** and **r** are acceptable to each other
- Either **r** is unmatched **OR r** prefers **h** to its assigned hospital
- Either **h** doesn't have all its places filled in **M OR h** prefers **r** to atleast one of its residents

# Similarity of HR with SM

- Generalized GS algorithms(hospital-oriented and resident-oriented)
- All possible executions of the algorithms terminate with the same result
- Hospital-optimal and resident-optimal matchings for the corresponing algorithms
- Forms a lattice with a similar dominance relation as in SM

# Hospital oriented GS algorithm

- Assign each **r** ϵ **R** to be free
- Assign each **h** ϵ **H** to be totally unsubscribed
- While [some **h** ϵ **H** is undersubscribed AND **h**’s list contains an **r** ϵ **R** not provisionally assigned to **h**]:

    **r** = first such resident on **h**’s list

    if **r** is already assigned, say to **h'** :

        break the provisional assignment of **r** to **h'**

    provisionally assign **r** to **h**

    for each successor **h'** of **h** on **r**’s list:

        remove **h'** and **r** from each other’s list

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On termination, it generates a hospital-optimal and resident-pessimal matching

# Other Applications

- HR with lower quotas
- Student-Project Allocation

# References


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- Gale, David, and Lloyd S. Shapley. "College admissions and the stability of marriage." The American Mathematical Monthly 69.1 (1962): 9-15.
- Roth, Alvin E. "The evolution of the labor market for medical interns and residents: a case study in game theory." Journal of political Economy 92.6 (1984): 991-1016.
- Stable Marriage and Its Relation to Other Combinatorial Problems: An Introduction to the Mathematical Analysis of Algorithms: Knuth, D.E, American Mathematical Soc

# Other Useful Resources

- https://www.youtube.com/watch?v=pvPtfdE4NSc
- https://www.youtube.com/watch?v=VpDEuindebE
- https://github.com/severus-tux/StableMatching

Thank You !