Finite groups with exactly two conjugacy class sizes and the analogous study in Lie algebra

Tushar Kanta Naik
Indian Institute of Science Education and Research Mohali, India

Weekly online research seminar, Gonit Sora

05:00 pm, 19th March, 2021

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- Motivation
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Motivation

Classification of objects - a natural problem

- The "classification" of objects in a family is a very natural problem in any branch of science.
- In mathematics, the classification problem has been considered for many families, such as topological spaces, knots, surfaces, graphs, groups etc.
- To mention specifically, the classification of the finite simple groups (CFSG) is one of the most celebrated achievements of the last century in the area of mathematics.

Greatest intellectual achievement

Is classification of finite simple groups the greatest intellectual achievement ?

Please follow the link

https://profhugodegaris.wordpress.com/youtube-lecture-course-humanitys-greatest-intellectual-achievement-the-classification-theorem-of-the-finite-simple-groups/

OR

Visit the homepage of Prof. Hugo de Garis.

Classification of finite p-groups

- It is well documented that finite p-groups play important role in describing the structure of finite groups.

- These groups, up to order p^4, were classified early in the history of group theory, and modern work has extended these classifications up to the order p^7.

- For more details, please follow the

https://www.math.auckland.ac.nz/~obrien/recent_work.htm

Or

visit the homepage of Prof. Eamonn O'Brien.

Classification of finite p-groups - difficulty and a new approach

- The number of p-groups grows so quickly that further classifications along these lines seem a near-impossible task.
- To reduce the difficulty, the classification problem in p-groups is, nowadays, done by considering a specific condition on them, such as isoclinism, coclass, exponent, derived length, etc.
- One such condition, in which we are interested here, is the sizes of conjugacy classes in a group.

Definitions and Examples

Conjugacy class

All groups, we are going to discuss here, are finite.

For a group G and element $x \in G$,

- The conjugacy class of x in G is defined as
$$x^G = \{x^y = y^{-1}xy \mid y \in G\}.$$
- The centraliser of x in G is defined as
$$C_G(x) = \{y \mid xy = yx\}.$$

It is easy to see that $|C_G(x)| \times |x^G| = |G|$.

Conjugate type / rank

- A group G is said to be of conjugate type $(1 = m_0, m_1, \ldots, m_r)$, where $1 = m_o < m_1 < \ldots < m_r$, if m_i's are precisely the different sizes of conjugacy classes of G. Here, we also say that the group G is of conjugate rank r.
- Immediate that a group G is abelian if and only if G is of conjugate type (1).

Example - S_3 is conjugate type (1, 2, 3)

Consider the symmetric group $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. There are 3 conjugacy classes, namely, $\{e\}$,

$$(1\ 2\ 3)^{S_3} = \{(1\ 2\ 3), (1\ 3\ 2)\} = (1\ 3\ 2)^{S_3}, \text{ and}$$

$$(1\ 2)^{S_3} = \{(1\ 2), (1\ 3), (2\ 3)\} = (1\ 3)^{S_3} = (2\ 3)^{S_3}.$$

Thus S_3 is of conjugate type (1, 2, 3).

Isoclinism

Isoclinism of groups - P.Hall, 1940

- Roughly speaking, two groups are isoclinic, if their commutator maps are essentially the same.

- Let X be a finite group and $\overline{X} = X/Z(X)$. Then the commutator map $C_X : \overline{X} \times \overline{X} \mapsto X'$, given by $C_X\big(xZ(X), yZ(X)\big) = [x, y]$ is well defined.

Two Finite groups G and H are said to be isoclinic if there exists an isomorphism $\phi : \overline{G} \to \overline{H}$ and an isomorphism $\theta : G' \to H'$ such that the following diagram is commutative;

$$\begin{tikzcd}
\overline{G} \times \overline{G} \arrow[r, "C_G"] \arrow[d, "\phi\times\phi"'] \arrow[dr, phantom, "\circlearrowright"] & G' \arrow[d, "\theta"] \\
\overline{H} \times \overline{H} \arrow[r, "C_H"] & H'.
\end{tikzcd}$$

Some facts

- Let A be an abelian group and G be an arbitrary group. Then both the groups G and $G \times A$ are isoclinic.
- A group G is isoclinic to the trivial group 1 if and only if G is abelian.

Example

Consider the two extra-special p-groups, p>2,

$$G = \langle a, b, c \mid a^p = b^p = c^p = 1 = [a,b] = [a,c],\ c^{-1}bc = ab = ba \rangle$$
$$\cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p,$$

and

$$H = \langle x, y \mid x^{p^2} = y^p = 1, x^{-1}yx = x^{1+p} \rangle$$
$$\cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p\,.$$

Stem groups

- Isoclinism is an equivalence relation, and each equivalence class is called an isoclinic family.
- Let G be a finite p-group. Then there exists a p-group H in the isoclinic family of G such that $Z(H) \leq H'$.
- Such a group H is called a stem group in its isoclinic family.
- Stem groups are not unique in a isoclinic family.

Example - continue

Note that $Z(G) = \langle a \rangle = G' \cong Z_p \cong Z(H) = \langle x^p \rangle = H'$.

Thus, $G/Z(G) = \langle \bar{b}, \bar{c} \rangle \cong Z_p \times Z_p \cong \langle \bar{x}, \bar{y} \rangle = H/Z(H)$.

Now, it is easy to check that the maps ϕ and θ defined as

$\phi(\bar{b}) = \bar{x}$, $\phi(\bar{c}) = \bar{y}$, and $\theta(a) = x^p$

extend to an Isoclinism.

Easy exercise

Check that

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k \mid i^2 = j^2 = k^2 = -1 = ijk\}$$

and

$$D_4 = \langle x, y \mid x^2 = y^2 = 1 = (xy)^4 \rangle$$

are isoclinic.

Relation between isoclinism and conjugate type

Two isoclinic groups are of same conjugate type.

Conjugacy class sizes in groups

Initiated by N. Ito, 1953

- In a series of paper, Ito studied finite groups

  - of conjugate rank 1 (i.e., 2 distinct conjugacy class sizes)

    On finite groups with given conjugate type I, Nayoga Math. Journal, 1953
  - of conjugate rank 2 (i.e., 3 distinct conjugacy class sizes)

    On finite groups with given conjugate type II, Osaka J. Math, 1970
  - of conjugate rank 3 (i.e., 4 distinct conjugacy class sizes)

    On finite groups with given conjugate type III, Mathematische Zeitschrift, 1970.

Ito's findings

- All finite groups of conjugate rank 1 (i.e., exactly two conjugacy class sizes) are necessarily nilpotent.
- All finite groups of conjugate rank 2 (i.e., exactly three conjugacy class sizes) are necessarily solvable.
- If G is a finite simple group of conjugate rank 3 (i.e., exactly four conjugacy class sizes), then G is isomorphic to $SL(2,\ 2^m), m \geq 2$.

Groups with two conjugacy class sizes

Theorem (N. Ito, 1953):

Let G be a finite group with exactly two conjugacy class sizes, namely 1 and $m > 1$. Then $m = p^k$, for some prime p and integer $k \geq 1$.

Moreover $G = P \times A$, where A is an abelian p'-subgroup of G and P is a non-abelian slow p-subgroup of G.

Thus, the investigation boils down to the study of finite p-groups of conjugate type $(1,\ p^n), n \geq 1$.

Some more results

- Let G be a p-group of conjugate type $(1, p^n)$. Then the number of elements in any generating set is at least n.
- Moreover, the number of elements of order p in the center $Z(G)$ is at least p^n.

Example: nilpotency class 3 and conjugate type $(1,\ p^2), p \geq 3$

$$H = \langle a_1,\ a_2,\ b,\ z_1,\ z_2 \mid [a_1,\ a_2] = b, [a_i,\ b] = z_i, a_i^p = 1, i = 1, 2 \rangle .$$

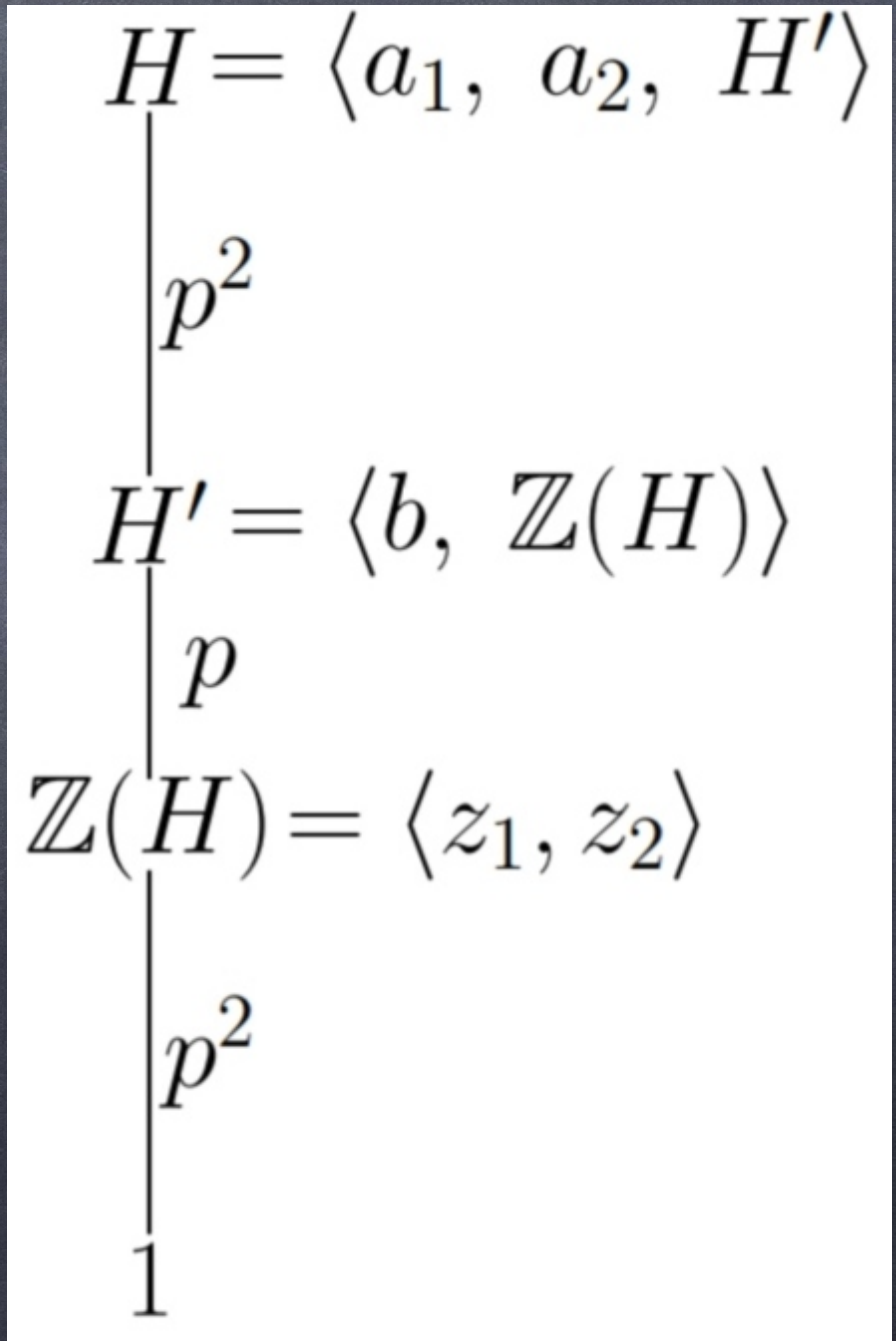
The group H is of order p^5 and exponent p.

The commutator subgroup $H' = \langle b, z_1, z_2 \rangle$ is of order p^3.

The centre $Z(H) = \langle z_1, z_2 \rangle$ is of order p^2 and lies inside commutator subgroup. Hence H is a stem group.

For each $x \in H \backslash Z(H)$, the centraliser $C_H(x) = \langle x, Z(H) \rangle$ is of order p^3.

Thus, the conjugacy class sizes of all non-central element is p^2, and consequently H is of conjugate type $(1,\ p^2)$.



Example: nilpotency class 2 and conjugate type$(1,\ p^r)$, $p \geq 3$, $r \geq 1$
Special p-group

$$G_r = \langle a_1, a_2, \ldots, a_{r+1} \mid a_i^p = [a_j, a_k]^p = 1 = [[a_i, a_j], a_k], 1 \leq i, j, k \leq r+1 \rangle$$

- The group G_r is a special p-group generated by r+1 elements and of order $p^{r(r+1)/2} p^{r+1} = p^{(r+1)(r+2)/2}$.
- The commutator subgroup G_r' = the center $Z(G_r)$ is generated by $[a_i, a_j], 1 \leq i < j \leq r+1$, and thus elementary abelian p-group of order $p^{r(r+1)/2}$. Thus G_r is a stem group.
- For each $x \in G_r \backslash Z(G_r)$, the centraliser $C_{G_r}(x) = \langle x, Z(G_r) \rangle$ is of order $p^{r(r+1)/2} p$.
- Thus the conjugacy class sizes of all non-central element is p^r, and consequently G_r is of conjugate type $(1,\ p^r)$.

Camina group

- Definition: A group G is said to be Camina, if $[x, G] = G'$, for all $x \in G \backslash G'$.
- If G is a non-abelian Camina group, then $Z(G) \leq G'$. In particular, G is a stem group.
- If G is a Camina group of nilpotncy class 2, then $Z(G) = G'$.
- If G is a Camina group of nilpotency class 2, then G is of conjugate type $(1, \ |Z(G)|)$.
- It follows that if G is a Camina group of nilpotency class 2, then G is necessarily a p-group.

Example: Camina p-group of nilpotency class 2

Let F_{p^m} stands for a finite field of p^m elements. Consider the group

$$\mathrm{U}_3(p^m) = \left\{ \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_3 \\ 0 & 0 & 1 \end{bmatrix} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_{p^m} \right\}.$$

- $U_3(p^m)$ is a Camina p-group of nilpotency class 2 generated by 2m elements.
- $Z\big(U_3(p^m)\big) = U_3(p^m)' = \left\{ \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \alpha \in \mathbb{F}_{p^m} \right\}$ is of order p^m.
- $U_3(p^m)$ is of conjugate type $(1,\ p^m)$.

Comparison between the two examples of nilpotency class 2

- Let G be a Camina p-group of nilpotency class 2 with $|Z(G)| = p^{m+n}$, $n \geq 1$, and $A \leq Z(G)$ be a central subgroup of order p^n. Then G/A is a Camina p-group with $|Z(G/A)| = p^m$.
- G/A is stem group of conjugate type $(1,\ p^m)$ and generated by $2(m+n)$ elements.
- Recall that G_m is a stem group of conjugate type $(1,\ p^m)$ and generated by $m+1$ elements.

An open problem

Does there exists a stem group G of conjugate type $(1, p^n), n \geq 3$, and minimally generated by $k, n+1 < k < 2n$ elements?

1953 – 1999 : Not much progress

I. M. Isaacs, Groups with many equal classes, Duke Mathematical Journal, 37(3), 501-506, 1970

Let G be a finite group, which contains a proper normal subgroup N such that all of the conjugacy classes of G which lie outside N have the same sizes. Then either G/N is cyclic or every non-identity element of G/N has prime order.

- If G is of conjugate type $(1, p^n)$, then $exp(G/Z(G)) = p$.
- If G is of conjugate type $(1, 2^n)$, then nilpotency class of G is 2.

A major breakthrough

K. Ishikawa, On finite p-groups which have only two conjugacy lengths, Israel J. Math, 129, 119–123, 2002

If G is a finite p-group of conjugate type $(1, p^n)$, $p > 2$, then nilpotency class of G is either 2 or 3.

Finite p-groups of conjugate type (1, p)

K. Ishikawa, Finite p-groups up to isoclinism, which have only two conjugacy lengths, J. Algebra, 220, 333-345, 1999

A finite p-group G is of conjugate type $(1, p)$, if and only if G is isoclinic to a extra-special p-group.

Finite p-groups of conjugate type $(1, p^2)$ and nilpotency class 2

K. Ishikawa, Finite p-groups up to isoclinism, which have only two conjugacy lengths, J. Algebra, 220, 333-345, 1999

A finite p-group G is of conjugate type $(1,\ p^2)$ and nilpotency class 2 if and only if G is isoclinic to one of the following:

- a Camina p-group with commutator subgroup of order p^2.
- $G_2 = \langle a_1, a_2, a_3 \mid \ a_i^p = [a_j, a_k]^p = 1 = [[a_i, a_j], a_k], 1 \leq i, j, k \leq 3 \rangle$.

Finite p-groups of conjugate type $(1, p^2)$ and nilpotency class 3

K. Ishikawa, Finite p-groups up to isoclinism, which have only two conjugacy lengths, J. Algebra, 220, 333-345, 1999

A finite p-group G is of conjugate type $(1,\ p^2)$ and nilpotency class 3 if and only if G is isoclinic H defined as follow,

$$H = \langle a_1,\ a_2,\ b,\ z_1,\ z_2 \mid [a_1,\ a_2] = b, [a_i,\ b] = z_i, a_i^p = 1, i = 1, 2\rangle .$$

Our Contribution

Tushar Kanta Naik,
and Manoj Kumar Yadav,
Finite p-groups of conjugate type $\{1, p^3\}$,
J. Group Theory,
21 (1), 2018, 65-82.

Finite p-groups of conjugate type $(1, p^3)$, $p > 2$

If G is a finite p-group, $p > 2$, of conjugate type $(1, p^3)$, then nilpotency class of G can not be 3.

Continue

Moreover G is isoclinic to one of the following:

- A Camina p-group of class 2 and center of order p^3.
- $G_3 = \langle a_1, a_2, a_3, a_4 \mid a_i^p = [a_j, a_k]^p = 1 = [[a_i, a_j], a_k], 1 \leq i, j, k \leq 4 \rangle$.
- $G_3/\langle x \rangle$, where $x = [a_1, a_2][a_3, a_4] \in Z(G_3) = G_3'$.
- $G_3/\langle x, y \rangle$, where $y = [a_1, a_3][a_2, a_4]^t \in Z(G_3) = G_3'$, where t is a non-square integer modulo p.

Continue...

We also proved analogous result for the prime $p=2$.

Results till now

- If G is of conjugate type $(1, p)$ or $(1, p^3)$, then nilpotency class of G is **2**.
- If G is of conjugate type $(1, p^2)$, then nilpotency class of G can be both **2** and **3**.

A natural question

Does there exist a finite p-group
of
nilpotency class 3
and
conjugate type $(1, p^n)$,
$n \geq 5$ an odd integer, and
p an odd prime?

Tushar Kanta Naik,
Rahul Dattatraya Kitture,
and Manoj Kumar Yadav,
Finite p-groups of nilpotency class 3 with two conjugacy class sizes,
Israel J. Math,
236 (2), 2020, 899–930.

Finite p-groups of nilpotency class 3 and conjugate type $(1, p^n), p > 2$

- Let $p > 2$ be a prime and $n \geq 1$ an integer. Then there exist finite p-groups of nilpotency class 3 and conjugate type $(1, p^n)$, if and only if n is an even integer.
- Moreover, for each even integer $n = 2m$, every finite p-group of nilpotency class 3 and conjugate type $(1, p^{2m})$ is isoclinic to the group $H_m/Z(H_m)$, where H_m is defined as follows...

Continue..

$$H_m = \left\{ \begin{bmatrix} 1 & & & & \\ a & 1 & & & \\ c & b & 1 & & \\ d & ab-c & a & 1 & \\ f & e & c & b & 1 \end{bmatrix} \right\},$$

where a, b, c, d, e, f $\in \mathbb{F}_{p^m}$,

field of p^m elements.

Remaining challenge (Open problem)

Classify finite p-groups

of

nilpotency class 2

and

conjugate type $(1, p^n), n \geq 4$.

Analogous study
in
Lie algebra

Definition

A Lie algebra L is a vector space over some field F together with a binary operation $[.,.] : L \times L \to L$ called the Lie bracket satisfying the following conditions:

- **Bilinearity**: $[ax + by, z] = a[x, z] + b[y, z]$, and $[z, ax + by] = a[z, x] + b[z, y]$, for all scalars $a, b \in F$, and all elements $x, y, z \in L$.
- **Alternativity**: $[x, x] = 0$, for all $x \in L$.
- **Jacobi identity**: $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$, for all $x, y, z \in L$.

Examples

- Any vector space V can be made into a Lie algebra with the trivial Lie bracket defined as $[x, y] = 0$, for all $x, y \in L$.

- Any associate algebra A can be made into a Lie algebra with the Lie bracket defined as $[x, y] = xy - yx$, for all $x, y \in L$.

- Let V be an F-vector space. Then $End(V)$, the set of F-linear transformation of V, is a Lie algebra with the Lie bracket defined as $[f, g] = f \circ g - g \circ f$, for all $f, g \in End(V)$.

(nilpotent) Lie algebra $\Longleftrightarrow$ (p)-groups

- It is widely believed that results of finite groups that make sense for Lie algebras are often valid.
- More precisely, results of finite p-groups that make sense for finite dimensional nilpotent Lie algebras are often true.
- Recall that a p-group has non-trivial center. Similarly, the center of a nilpotent Lie algebra is non-trivial.
- Classification of nilpotent Lie algebras is done up to the dimension 7 over the field of real and complex numbers.
- Much like p-groups, classification of nilpotent Lie algebras, up to isomorphism, is a very difficult problem.

The analogous study in Lie algebra

- There is no direct analogous of conjugacy classes in Lie algebra.
- But, the number of distinct conjugacy class sizes in a finite group G is equal to the number of different orders of centralizers of elements of elements in G.
- This allows us to investigate the analogous case in Lie algebras.
- Here, we are interested in the finite-dimensional Lie algebras L with exactly two different dimensions of centralisers $C_L(x)$, as x runs over L.

Not necessarily nilpotent

Y. Barena, and I. M. Isaacs, Lie algebra with few centraliser dimensions, J. Algebra, 259, 284-299, 2003.

Suppose that L is is a non-nilpotent finite dimensional Lie algebra over $\mathbb{C}$ with just two distinct centraliser dimensions. Then $dim(L/Z(L)) \leq 3$, and one of following holds.

- $L/Z(L)$ is isomorphic to the 2-dimensional non-nilpotent Lie algebra, i.e., $\{x, y \mid [x, y] = x\}$.
- $L/Z(L)$ is isomorphic to $\{a, x, y \mid [a, x] = x, [a, y] = -y, [x, y] = 0\}$.
- $L/Z(L)$ is isomorphic to $sl_2(\mathbb{C})$, the Lie algebra of 2×2 complex matrices with trace zero and Lie bracket $[X, Y] = XY - YX$.

Bound on nilpotency class

Y. Barena, and I. M. Isaacs, Lie algebra with few centraliser dimensions, J. Algebra, 259, 284-299, 2003.

Suppose that L is a finite dimensional nilpotent Lie algebra over any arbitrary field with just two distinct centraliser dimensions. Then nilpotency class of L is either 2 or 3.

Remaining challenge (Open problem)

Classify finite dimensional nilpotent Lie algebras of c.c.d. (o, m), $m \geq 1$.

We say a Lie algebra L is of c.c.d. (o, m) (stands for centraliser co-dimensions), if co-dimensions of $C_L(x)$ is m, for all non-central element $x \in L \backslash Z(L)$.

Tentative possibility

It seems all the results which are true in finite p-groups of conjugate type $(1, p^n)$ will hold true for finite dimensional Lie algebras of c.c.d. (o, n) over finite field.

The problem remains over arbitrary field, where the computational techniques used in finite groups can not be applied directly.

Thank you