MAT165: PROBLEMS ON INVARIANTS & COLOURING

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(1) The boxes of an $m \times n$ table are filled with numbers such that the rows and columns all add up to the same sum. Prove that $m = n$.

(2) Draw all the tile shapes that can be constructed using 4 unit squares.

(3) Among the above drawn shapes, which can be used to tile a 4×4 board, if you are allowed to use one tile per tiling?

(4) Can you tile a 4×5 board using each tetromino once?

(5) Consider a 7×7 board, which squares can be removed to allow a tiling with a 3×1 tromino for the resulting board?

(6) An alphabet has only two letters – O and A. The language has the rule that if you delete a neighbouring instance of AO from any word, then the meaning of the word doesn't change. Similarly, meanings do not change by inserting an OA or an $AAOO$. Are the words AOO and OAA same or different in meaning?

(7) In an 8×8 table one of the boxes is black and the rest are all white. Prove that one cannot make all boxes white by colouring the rows and columns such that we change the colour of all boxes in a row or a column at once. What about a 3×3 table?

(8) A prince has 2 magic swords, one of which can cut off 21 heads of an evil dragon at one go, while the other can cut off 4, but then the dragon grows another 1985 new heads. Can the prince cut off all the heads of a dragon which originally had 100 heads?

(9) We start with the number 8^n, then we calculate the sum of its digits, and then the sum of the digits of the resulting number, and so on until we get a one digit number. What is this one digit number if $n = 1989$?

(10) There are 1001 stones in a heap. You can do the following: choose any heap having more than 1 stone, remove 1 stone from that heap and divide the heap into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps contain exactly 3 stone?

(11) We start with three numbers, we are allowed to change any two of these, say a and b to $\dfrac{a+b}{\sqrt{2}}$ and $\dfrac{a-b}{\sqrt{2}}$. If we start with $(2, \sqrt{2}, 1/\sqrt{2})$, can we obtain $(1, \sqrt{2}, 1+\sqrt{2})$ by doing this process?

Date: 25 January 2026.

(12) In a classroom each student has at most three enemies. Prove that the class can be divided into two parts so that each student has at most one enemy in their own part.

(13) Each of the numbers $a_1, a_2, \ldots, a_n$ are either 1 or -1, such that

$$S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0.$$

Prove that $4|n$.

(14) Many handshakes are exchanged at an event. A person is called an odd/even person if they have exchanged an odd/even number of handshakes. Show that at any moment, there is an even number of odd persons.

(15) Start with the numbers 18 and 19, and in each step you add another number equal to the sum of any two previous numbers. Can you reach the number 1994 using this process?

(16) There are several $+$ and -1 signs written on a board. You may erase two signs and write a $+$ sign if you erased two equal signs, or a $-$ sign if you erased two unequal signs. Prove that the last remaining sign on the board doesn;t depend on the order of the erasures.