MAT165: PROBLEMS ON NUMBER THEORY

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(1) Prove or disprove: if $a|(b+c)$, then either $a|b$ or $a|c$.

(2) Give an example to show that $a^2 \equiv b^2 \pmod{n}$ doesn't always imply $a \equiv b \pmod{n}$.

(3) How many zeroes does 100! end in?

(4) Prove that if a number has an odd number of divisors, then it must be a perfect square.

(5) Which primes are of the form $n^3 - 1$?

(6) Show that $n^4 + 4$ is never a prime, when $n > 1$.

(7) If $p \neq 1$ is an odd prime, then prove that either $p^2 - 1$ or $p^2 + 1$ is divisible by 10.

(8) Show that there exists k consecutive composite numbers, for every integer k.

(9) If p is a prime such that $n < p < 2n$, then show that $\binom{2n}{n} \equiv 0 \pmod{p}$.

(10) Show that $89|(2^{44} - 1)$ and $97|(2^{48} - 1)$.

(11) Find the remainder when 4444^{4444} is divided by 9.

(12) Find the last three digits of 7^{999}.

(13) What is the last digit of $12345678910111213141516171819^5$?

(14) Given a number n of the form $p^a q^b$ for some primes p, q and integers a, b, how many divisors does n have?

(15) A number consists of 100 zeroes, 100 ones, and 100 twos. Can this number be a perfect square?

(16) Find the remainder when $1989 \cdot 1990 \cdot 1991 + 1992^3$ is divided by 7.

(17) Prove that, for all natural numbers n, we have $5|n^5 + 4n$.

(18) Given natural numbers a and b such that $21|a^2 + b^2$, then show that $441|a^2 + b^2$ as well.

(19) Find the last digit of the number 1989^{1989}.

(20) Prove that $7|2222^{5555} + 5555^{2222}$.

(21) What is the remainder when 7 divides $2^{50} + 41^{65}$?

(22) Prove that $1835^{1910} + 1986^{2061} \equiv 0 \pmod{7}$.

(23) For all $n \geq 1$, if we have

$$\binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \times 2^x.$$

Then, what is the value of x?

(24) Given prime numbers p and $8p^2 + 1$, what is p?

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(25) What is the last digit of $1^2 + 2^2 + \ldots 99^2$?

(26) Prove that $31 | 30^{99} + 61^{100}$.

(27) Prove that $133 | 11^{n+2} + 12^{2n+1}$, for any natural n.

(28) Show that the cube of an integer is of the form $7k$ or $7k \pm 1$, for some $k \in \mathbb{Z}$.

(29) Prove that if both a and b are odd integers, then $16 \mid a^4 + b^4 - 2$.

(30) Prove or disprove: the sum of the squares of two odd numbers cannot be a perfect square.

(31) Prove or disprove: the product of four consecutive integers is 1 less than a perfect square.

(32) If $\gcd(a, b) = 1$, then prove that $\gcd(a^2, b^2) = 1$.

(33) Verify that $2^{35} - 1$ is divisible by 31 and 127.

(34) Show that for all $n \geq 1$, the integer $5^{2n} - 1$ is divisible by 24.