MAT165: PROBLEMS ON PIGEON-HOLE PRINCIPLE (WITH SOLUTIONS)

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Note: In the solutions below, PHP refers to the Pigeon-hole Principle.

(1) Show that among any 13 people, at least two were born in the same month.
Solution. There are $n = 13$ people (pigeons) and $k = 12$ months (holes). By the PHP, there exists at least one month containing $\lceil 13/12 \rceil = 2$ people.

(2) Prove that in any set of $n + 1$ integers, there exist two integers whose difference is divisible by n.
Solution. Let the integers be $a_1, \ldots, a_{n+1}$. Consider their remainders modulo n. There are n possible remainders $\{0, 1, \ldots, n-1\}$. Since there are $n + 1$ integers, by PHP, at least two integers, say a_i and a_j $(i \neq j)$, must have the same remainder. Thus, $a_i \equiv a_j \pmod{n}$, which implies $n \mid (a_i - a_j)$.

(3) Show that among any 6 integers, there are two whose difference is divisible by 5.
Solution. This is a specific case of Problem 2 with $n = 5$.

(4) In a group of 10 people, prove that at least two have the same number of acquaintances within the group (assume acquaintance is mutual).
Solution. Let $n = 10$. The possible number of acquaintances for any person ranges from 0 to 9. However, 0 and 9 cannot coexist: if someone knows 0 people, no one can know all 9 other people. Thus, the set of possible acquaintance counts is either $\{0, 1, \ldots, 8\}$ or $\{1, 2, \ldots, 9\}$. In either case, there are 9 possible values (holes) for 10 people (pigeons). By PHP, two people must have the same number of acquaintances.

(5) Show that any sequence of 5 real numbers in $[0, 1]$ contains two numbers whose difference is at most $\frac{1}{4}$.
Solution. Divide the interval $[0, 1]$ into 4 sub-intervals of length $1/4$ by subdividing equally into 4 parts. We have 5 numbers (pigeons) and 4 sub-intervals (holes). By PHP, two numbers must lie in the same sub-interval. The distance between any two points in an interval of length $1/4$ is at most $1/4$.

(6) Prove that from any 51 integers chosen from $\{1, 2, \ldots, 100\}$, one can find two integers such that one divides the other.
Solution. Any integer x can be written as $x = 2^k \cdot m$, where m is the odd part of x. For numbers in $\{1, \ldots, 100\}$, the possible values for m are the odd numbers $\{1, 3, \ldots, 99\}$.

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There are exactly 50 such odd numbers (holes). Since we select 51 integers (pigeons), two integers must share the same odd part m. Let these be $x = 2^a m$ and $y = 2^b m$. If $a < b$, then $x \mid y$; if $b < a$, then $y \mid x$.

(7) Show that among any 8 distinct integers, there exist two whose sum is divisible by 7.
Solution. Note: As stated for arbitrary distinct integers, this is technically false, e.g., the set $\{1, 8, 15, \dots\}$ *has all terms* $\equiv 1 \pmod 7$.

(8) Let $a_1, a_2, \dots, a_{n+1}$ be integers. Prove that there exist indices $i < j$ such that $a_i + a_{i+1} + \cdots + a_j$ is divisible by n.
Solution. Consider the prefix sums $S_k = \sum_{m=1}^{k} a_m$ for $k = 1, \dots, n+1$, and let $S_0 = 0$. Consider the set $\{S_0, S_1, \dots, S_n\}$ modulo n. There are $n+1$ values (pigeons) and n residues (holes). By PHP, two sums, say S_j and S_i with $j > i$, must be congruent modulo n. Thus $S_j - S_i = a_{i+1} + \cdots + a_j$ is divisible by n.

(9) Show that in any set of 7 points placed inside an equilateral triangle of side length 1, there exist two points whose distance is at most $\frac{1}{2}$.
Solution. Divide the equilateral triangle into 4 smaller equilateral triangles of side length $1/2$ by connecting the midpoints of the sides. We have 7 points (pigeons) and 4 regions (holes). By PHP $\lceil 7/4 \rceil = 2$, so at least two points lie in the same small triangle. The maximum distance between two points in an equilateral triangle of side $1/2$ is $1/2$.

(10) Prove that any integer sequence of length 10 contains a subsequence whose sum is divisible by 10.
Solution. This is an application of Problem 8 with $n = 10$.

(11) Prove that in any set of n integers, there exists a nonempty subset whose sum is divisible by n.
Solution. Let the integers be $a_1, \dots, a_n$. Consider the partial sums $S_k = a_1 + \cdots + a_k$ for $k = 1, \dots, n$. If any $S_k \equiv 0 \pmod n$, we are done. If not, the n sums map to the $n-1$ residues $\{1, \dots, n-1\}$. By PHP, two sums S_j and S_i ($j > i$) have the same residue. Their difference $S_j - S_i = a_{i+1} + \cdots + a_j$ is divisible by n.

(12) Let S be a set of 10 points in the unit square $[0,1] \times [0,1]$. Show that there exist two points whose distance is at most $\sqrt{2}/3$.
Solution. Divide the unit square into a 3×3 grid of 9 identical smaller squares, each with side length $1/3$. We have 10 points (pigeons) and 9 squares (holes). By PHP, two points lie in the same small square. The maximum distance in a square of side s is the diagonal $s\sqrt{2}$. Here, the max distance is $\frac{1}{3}\sqrt{2} = \frac{\sqrt{2}}{3}$.

(13) Let $a_1, \dots, a_{20}$ be integers in the set $\{1, 2, \dots, 100\}$. Prove that there exist two disjoint nonempty subsets with the same sum.

Solution. The number of nonempty subsets is $2^{20}-1$, which is approximately 10^6. The maximum possible sum of any subset is the sum of the 20 largest integers: roughly $20 \times 100 = 2000$. Since $2^{20}-1 > 2000$, by PHP there exist two distinct subsets A and B with the same sum $S_A = S_B$. If A and B are not disjoint, we can remove their intersection $C = A \cap B$ from both. The sets $A' = A \setminus C$ and $B' = B \setminus C$ are disjoint, nonempty (since $A \neq B$), and have the same sum.

(14) Prove that any sequence of n^2+1 distinct real numbers contains an increasing or decreasing subsequence of length $n+1$.
Solution. (This is a famous result, called the Erdős-Szekeres Theorem.) Associate with each term x_k a pair (i_k, d_k), where i_k is the length of the longest increasing subsequence ending at x_k, and d_k is the length of the longest decreasing subsequence ending at x_k. If no such subsequence of length $n+1$ exists, then $i_k, d_k \in \{1, \ldots, n\}$. There are only n^2 possible pairs. Since there are n^2+1 terms, two terms must have the same pair, which is impossible for distinct numbers. Thus, a subsequence of length $n+1$ must exist.

(15) Show that in any coloring of the integers $\{1, 2, \ldots, 13\}$ with 3 colors, there exist two integers of the same color whose difference is at most 4.
Solution. Consider just the subset $\{1, 2, 3, 4\}$. There are 4 integers (pigeons) and 3 colors (holes). By PHP, two integers in this set, say x, y, share the same color. Since $x, y \in \{1, \ldots, 4\}$, their maximum difference is $|4-1| = 3$. Since $3 \leq 4$, the condition is satisfied even within the first 4 integers.

(16) Prove that among any 6 lattice points in the plane, no three collinear, there exist two whose midpoint is also a lattice point.
Solution. A midpoint of (x_1, y_1) and (x_2, y_2) is a lattice point if and only if $x_1 + x_2$ and $y_1 + y_2$ are even. This requires $x_1 \equiv x_2 \pmod 2$ and $y_1 \equiv y_2 \pmod 2$. There are 4 parity classes for coordinates: (Even, Even), (Even, Odd), (Odd, Even), (Odd, Odd). With 6 points (pigeons) and 4 parity classes (holes), at least two points share the same parity class. Their midpoint is a lattice point.

(17) Show that among any 8 real numbers in the interval $[0, 1]$, there exist two numbers x, y such that $|x - y| \leq \frac{1}{7}$.
Solution. Divide $[0, 1]$ into 7 intervals of length $1/7$: $[0, 1/7), \ldots, [6/7, 1]$. With 8 numbers (pigeons) and 7 intervals (holes), two numbers must fall in the same interval. Their distance is at most $1/7$.

(18) Let n be a positive integer. Prove that among any $n+1$ multiples of n, there exist two whose difference is divisible by n^2.

Solution. Let the numbers be $k_1n, k_2n, \ldots, k_{n+1}n$. Consider the integers $k_1, \ldots, k_{n+1}$ modulo n. There are $n+1$ values and n residues. By PHP, two indices i, j satisfy $k_i \equiv k_j \pmod{n}$. Thus $n \mid (k_i - k_j)$, so $k_i - k_j = mn$. The difference of the original numbers is $k_in - k_jn = (k_i - k_j)n = (mn)n = mn^2$, which is divisible by n^2.

(19) Let $a_1, a_2, \ldots, a_{11}$ be integers. Show that there exist indices $i < j$ such that

$$a_i + a_{i+1} + \cdots + a_j \equiv 0 \pmod{11}.$$

Solution. This is Problem 8 with $n = 11$.

(20) Show that in any coloring of the plane with 4 colors, there exist two points of the same color at distance exactly 1.

Solution. This is a difficult problem for which I don't have an easy solution. If someone has a solution for this then please send it to me.

(21) Prove that among any 9 lattice points in the plane, there exist two whose midpoint is also a lattice point.

Solution. As shown in Problem 16, to guarantee a pair with the same coordinate parity, we need 5 points. Since $9 \geq 5$, the property holds. (In fact, with 9 points, there are at least $\lceil 9/4 \rceil = 3$ points with the same parity, guaranteeing multiple pairs).

(22) Place 9 points in an equilateral triangle of side length 1. Show that there exist two points whose distance is at most $\frac{1}{2}$.

Solution. As in Problem 9, divide the triangle into 4 smaller equilateral triangles of side $1/2$. We have 9 points and 4 regions. By PHP, at least $\lceil 9/4 \rceil = 3$ points lie in one small triangle. The maximum distance in a triangle of side $1/2$ is $1/2$, so any two of these points satisfy the condition.

(23) Show that among any 16 integers, there exist two whose sum or difference is divisible by 8.

Solution. We define buckets based on behavior modulo 8. We want to pair x with x (difference divisible) or x with $-x$ (sum divisible). Buckets: $B_0 = \{0\}$, $B_1 = \{1, 7\}$, $B_2 = \{2, 6\}$, $B_3 = \{3, 5\}$, $B_4 = \{4\}$. There are 5 buckets. We have 16 integers. By PHP, at least $\lceil 16/5 \rceil = 4$ integers fall into one bucket.

- If in B_0 or B_4: any pair has diff divisible by 8 (and sum divisible by 8 for B_0, B_4).
- If in B_1, B_2, B_3: The integers are congruent to r or $-r$. If we have at least 2 integers in a bucket, say x, y, then either $x \equiv y$ (diff div by 8) or $x \not\equiv y \implies x \equiv -y$ (sum div by 8).

Thus, the condition holds (actually 6 integers suffice).

(24) Let $a_1, a_2, \ldots, a_{10}$ be integers from the set $\{1, 2, \ldots, 100\}$. Show that there exist two different nonempty subsets whose sums differ by at most 1.

Solution. There are $2^{10} - 1 = 1023$ nonempty subsets. The maximum possible sum is roughly 1000 (precisely the sum of the 10 largest integers is ≤ 1000). The sums are integers in $[1, 1000]$. We have 1023 subsets (pigeons) and about 1000 possible sums (holes). By PHP, there must be two distinct subsets with the *same* sum (difference 0). Since $0 \leq 1$, the condition holds. If disjointness is required, we remove the intersection as in Problem 13.